A Sequence related to that of Thue-Morse

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Abstract

We study a sequence, **c**, which encodes the lengths of blocks in the Thue-Morse sequence. In particular, we show that the generating function for **c** is a simple product.

Consider the sequence

$$\mathbf{c} : c_0, c_1, c_2, c_3, \ldots = 1, 3, 4, 5, 7, 9, 11, 12, 13, \ldots$$

defined to be the lexicographically least sequence of positive integers satisfying $n \in \mathbf{c}$ implies $2n \notin \mathbf{c}$. In fact, the lexicographic minimality of $\mathbf{c}$ makes it possible to replace the previous "implies" with "if and only if." Equivalently, $\mathbf{c}$ is defined inductively by $c_0 = 1$ and

$$c_{k+1} = \begin{cases} c_k + 1 & \text{if } (c_k + 1)/2 \notin \mathbf{c} \\ c_k + 2 & \text{otherwise} \end{cases} \tag{1}$$

for $k \geq 0$. This sequence was the focus of a problem of C. Kimberling in the *American Mathematical Monthly* [?]. (In fact, he looked at the sequence $4c_0, 4c_1, 4c_2, \ldots$) The solution was given by D. M. Bloom [?]. Our Corollary **??** answers essentially the same question.

At the 4è Colloque Séries Formelles et Combinatoire Algébrique (Montréal, June 1992) Simon Plouffe and Paul Zimmermann [?] posed the following problem. Show that the generating function for $\mathbf{c}$ is

$$\sum_{k \geq 0} c_k x^k = \frac{1}{1-x} \prod_{j \geq 1} \frac{1 - x^{2e_j}}{1 - x^{e_j}} \tag{2}$$

the sequence of exponents being

$$\mathbf{e} : e_1, e_2, e_3, e_4, \ldots = 1, 1, 3, 5, 11, 21, 43, \ldots$$

where $e_1 = 1$ and

$$e_{j+1} = \begin{cases} 2e_j + 1 & \text{if } j \text{ is even} \\ 2e_j - 1 & \text{if } j \text{ is odd} \end{cases} \tag{3}$$

for $j \geq 1$. They came up with this conjecture by using a method that goes back to Euler. First they assumed that the generating function was of the form

$$\prod_{j \geq 0} \frac{1 - x^{a_j}}{1 - x^{b_j}}$$

for a certain pair of sequences a_j, b_j. Then they took the logarithm to convert the product into a sum. Finally they used Möbius inversion to come up with the candidate sequences. Details of this procedure will be found in the text of Andrews [?].

The purpose of this note is to prove (**??**). Before doing this, however, we will show that $\mathbf{c}$ has a number of other interesting properties. Chief among these is the fact that $\mathbf{c}$ is closely related to the famous Thue-Morse sequence, $\mathbf{t}$. See the survey article of Berstel [**?**] for more information about $\mathbf{t}$.

It is easy to prove the following proposition using the definition (**??**) and induction.

Proposition 1 *Given any positive integer n we have that $n \in \mathbf{c}$ if and only if $n = 2^{2i}(2j+1)$ for some nonnegative integers i and j.* $\blacksquare$

Let χ be the characteristic function of $\mathbf{c}$, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \in \mathbf{c} \\ 0 & \text{otherwise.} \end{cases}$$

Restating the previous proposition in terms of χ yields the next result.

Lemma 2 *The function χ is uniquely determined by the equations*

$$\begin{aligned} \chi(2n+1) &= 1 \\ \chi(4n+2) &= 0 \\ \chi(4n) &= \chi(n). \quad \blacksquare \end{aligned}$$

Another way of obtaining the sequence $\chi(n)$ for $n \geq 1$ is as follows. Starting from the sequence

$$101 \bullet 101 \bullet 101 \bullet 101 \bullet \ldots$$

defined on the alphabet $\{0, 1, \bullet\}$, fill in the holes with the sequence itself, obtaining:

$$101110101011101 \bullet \ldots$$

Iterating this process infinitely many times (by inserting the initial sequence into the holes at each step), one gets a "Toeplitz transform" which is nothing but our sequence χ. The proof of this fact is easily obtained using Lemma **??**. See the article of Allouche and Bacher [**?**] for more information about Toeplitz transformations.

The connection with the Thue-Morse sequence can now be obtained. This sequence is

$$\mathbf{t} : t_0, t_1, t_2, t_3, \ldots = 0, 1, 1, 0, 1, 0, 0, 1, \ldots$$

defined by the conditions

$$\begin{aligned} t_0 &= 0 \\ t_{2n+1} &\equiv t_n + 1 \pmod 2 \\ t_{2n} &= t_n. \end{aligned}$$

We will need a lemma relating $\mathbf{t}$ and χ. All congruences in this and any future results will be modulo 2.

Lemma 3 *For every positive integer, n, we have*

$$\chi(n) = t_n + t_{n-1}.$$

Proof. This is a three case induction based on Lemma **??** and the definitions of χ and $\mathbf{t}$. We will only do one of the cases as the others are similar.

$$\begin{aligned} t_{4n} + t_{4n-1} &\equiv t_{2n} + t_{2n-1} + 1 \\ &\equiv t_n + t_{n-1} + 2 \\ &\equiv \chi(n) \\ &= \chi(4n). \quad \blacksquare \end{aligned}$$

Define d_k to be the first difference sequence of c_k, i.e., $d_k = c_k - c_{k-1}$, for $k \geq 0$ ($c_{-1} = 0$). So $\mathbf{d}$ is the sequence

$$d_0, d_1, d_2, d_3, d_4, \ldots = 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, \ldots$$

Note that from the definition of $\mathbf{c}$ in (**??**), the value of d_k is either 1 or 2. Write the Thue-Morse sequence in term of its blocks

$$\mathbf{t} = 011010011\ldots = 0^{d'_0}1^{d'_1}0^{d'_2}1^{d'_3}\ldots$$

defining a sequence d'_k. It is this sequence that is related to our original one via the difference operator.

Theorem 4 *For all $k \geq 0$ we have $d_k = d'_k$.*

Proof. Since both sequences consist of 1's and 2's, we need only verify that the 1's appear in the same places in both. It will be convenient to let $c'_k = \sum_{i \leq k} d'_i$.

$$\begin{array}{rcll} d_{k+1} = 1 & \Leftrightarrow & \chi(c_k + 1) = 1 & \text{(definitions)} \\ & \Leftrightarrow & \chi(c'_k + 1) = 1 & \text{(induction)} \\ & \Leftrightarrow & t_{c'_k+1} + t_{c'_k} \equiv 1 & \text{(Lemma ??)} \\ & \Leftrightarrow & t_{c'_k+1} \neq t_{c'_k} & \text{(clear)} \\ & \Leftrightarrow & d'_{k+1} = 1 & \text{(definitions).} \quad \blacksquare \end{array}$$

Brlek [**?**] used the sequence $\mathbf{d}$ in calculating the number of factors of $\mathbf{t}$ of given length. The paper of de Luca and Varricchio [**?**] attacks the same problem in a different way.

Now if $n \in \mathbf{c}$ then we will consider its *rank*, $r(n)$, which is the function satisfying $c_{r(n)} = n$. Note that $r(n)$ is not defined for all positive integers n. In order to obtain a formula for $r(n)$, we will need a definition. Let the base 4 expansion of n be

$$n = \sum_{i \geq 0} \epsilon_i 4^i$$

with the $\epsilon_i \in \{0,1,2,3\}$ for all i. Define a function s by

$$s(n) = \sum_i s(\epsilon_i) \quad \text{where} \quad s(0) = s(3) = 0, s(1) = -1, s(2) = 1.$$

In other words, $s(n)$ is the opposite of the sum of the base 4 digits of n, each of them being reduced mod 3.

Theorem 5 *If $n \in \mathbf{c}$ then*

$$r(n) = (2n + 1 + s(2n + 1))/3 - 1.$$

Proof. The proof is an induction breaking up into three cases

1. $n = 2^{2i}(2j + 1)$
2. $n = 2^{2i}(2j + 1) - 1$
3. $n = 2^{2i-1}(2j + 1) - 1$

where $i > 0$ and $j \geq 0$. (We do not need to consider the case $n = 2^{2i-1}(2j + 1)$ because these integers are not in our sequence.) The arguments are similar, so we will only do the first case. By Proposition **??** we have $n + 1 \in \mathbf{c}$. So we need only show that $r'(n + 1) = r'(n) + 1$ where $r'(n)$ is the right side of the equation in the statement of the theorem. Now n is a multiple of 4, so the digits (base 4) of $2n + 1$ and and $2n + 3$ are identical except for the units digits which are 1 and 3, respectively. Thus

$$\begin{aligned} r'(n + 1) &= (2n + 3 + s(2n + 3))/3 - 1 \\ &= (2n + 3 + s(2n + 1) + s(3) - s(1))/3 - 1 \\ &= (2n + 1 + s(2n + 1))/3 \\ &= r'(n) + 1. \quad \blacksquare \end{aligned}$$

As straightforward corollaries we have the next two results.

Corollary 6 *If $n \in \mathbf{c}$ then*

$$r(n) = 2n/3 + O(\log\ n)$$

and $r(n)$ takes the value $(2n)/3$ infinitely often. $\blacksquare$

Corollary 7 *For any nonnegative integer k*

$$c_k = 3k/2 + O(\log\ k)$$

and $c_k = 3k/2$ infinitely often. $\blacksquare$

We shall now prove the generating function (**??**). First we note a property of the exponents that is a simple consequence of their definition (**??**).

Lemma 8 *Let* $f_k = \sum_{j \le k} e_j$. *Then*

$$f_k = \begin{cases} e_{k+1} - 1 & \text{if } k \text{ is even} \\ e_{k+1} & \text{if } k \text{ is odd.} \end{cases} \blacksquare$$

Also, it is convenient to cancel the denominator of the product into the numerator to obtain the following equivalent statement.

Theorem 9 *The generating function for* $\mathbf{c}$ *is*

$$\sum_{k \ge 0} c_k x^k = \frac{1}{1-x} \prod_{j \ge 1} (1 + x^{e_j}).$$

Proof. It suffices to show that if $k \ge 0$ then

$$g_k(x) = \frac{1}{1-x}(1+x^1)(1+x^1)(1+x^3)\cdots(1+x^{e_k})$$

is the generating function for the sequence

$$1, 3, 4, 5, 7, \ldots, c_{f_k}, 2^k, 2^k, 2^k, \ldots$$

with $c_{f_k} = 2^k - 1$. The proof is an induction, breaking up into two parts depending on the parity of k. We will do the case where k is odd. (Even k is similar.) Now, by Lemma **??**, $g_k(x)(1 - x^{e_{k+1}})$ is the generating function for the sequence

$$1, 3, \ldots, c_{f_k}, 2^k + 1, 2^k + 3, \ldots, 2^k + c_{f_k}, 2^{k+1}, 2^{k+1}, \ldots$$

Using Proposition **??** and the fact that k is odd, we see that $2^k + 1 = c_{f_k+1}$ and $2^k + c_{f_k} = 2^{k+1} - 1 = c_{f_{k+1}}$. So we want to show that

$$c_{f_k+1}, c_{f_k+2}, \ldots, c_{f_{k+1}} = 2^k + c_1, 2^k + c_2, \ldots, 2^k + c_{f_k}.$$

But if $n < 2^k$, then the highest power of 2 dividing n is equal to the highest power dividing $2^k + n$. Thus, by Proposition **??** again, $n \in \mathbf{c}$ if and only if $2^k + n \in \mathbf{c}$. This gives us the desired equality of the two sequences. $\blacksquare$

One possible generalization of $\mathbf{c}$ is the sequence $\mathbf{c}^{(\alpha)}$ defined by $n \in \mathbf{c}^{(\alpha)}$ if and only if $\alpha n \notin \mathbf{c}^{(\alpha)}$. Thus $\mathbf{c}$ is the special case $\alpha = 2$.

The following observation is a direct consequence of our definitions.

Proposition 10 *If $\chi^{(\alpha)}(n)$ is the characteristic function of $\mathbf{c}^{(\alpha)}$, then the sequence $(\chi^{(\alpha)}(n))$ is the unique fixed point of the morphism*

$$\begin{array}{rcl} 1 & \rightarrow & 1^{\alpha-1}0 \\ 0 & \rightarrow & 1^{\alpha-1}1 \end{array}$$

which begins with 1. ■

One can also see that $\mathbf{c}^{(\alpha)}$ satisfies analogs of many of our previous theorems. For example, the following result is a generalization of Theorem **??** and has an analogous proof.

Theorem 11 *The generating function for $\mathbf{c}^{(\alpha)}$ is*

$$\frac{1}{1-x}\prod_{j\geq 1}\frac{1-x^{\alpha e_j}}{1-x^{e_j}}. \quad ■$$

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