Conjectures on the Enumeration
of
Tableaux of Bounded Height(1)

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Abstract: We express general conjectures for the explicit form of P-recurrences for the number of Young standard tableaux of height bounded by h. These recurrences are compatible with known results and Regev's asymptotic evaluations.

Résumé en français

Le but de cette note est essentiellement de présenter des conjectures ainsi que des indications sur les raisons qui nous ont portées à énoncer ces conjectures. Les conjectures portent sur la forme explicite d'équations de récurrence que semble satisfaire la suite $t_h(n)$, des nombres de tableaux de Young de hauteur bornée par h, ainsi que la suite $t_h^{(2)}(n)$, des nombres de paires de tels tableaux ayant même forme (voir (2)). La forme des récurrences (1a et 2a) et certains aspects explicites des coefficients (1b, 1c, 1d, 2b, 2c, 2d et 2e) font l'objet de ces conjectures dans le cas général, et pour le cas h impair, on donne encore plus de détails (1e et 1f). Un résultat de Zeilberger [3] assure que de telles récurrences (voir (3)) existent mais sa démonstration ne semble pas permettre de déduire une forme explicite aussi précise que (4) ou (6). On montre aussi que le comportement asymptotique des solutions des recurrences obtenues par le biais de nos conjectures est compatible avec les résultats de Regev [2] sur le comportement asymptotique des nombres $t_h(n)$.

1. 1. INTRODUCTION

Let us first fix some notation. A partition λ of a positive integer n is a sequence of integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ such that $\sum_i \lambda_i = n$. We write $\lambda \vdash n$ to express this fact, and denote $\ell(\lambda) = k$ the number of parts (the λ_i's) of λ. We say that k is the *height* $h(\lambda)$ of λ. The height of the empty partition (of 0) is set to be 0. The (Ferrer's) diagram of a partition is the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$.

A Young standard tableau T is an injective labeling of a Ferrer's diagram by the elements of $\{1, 2, \ldots, n\}$, such that $T(i,j) < T(i+1,j)$, for $1 \leq i < k$, and $T(i,j) < T(i,j+1)$, for $1 \leq j < \lambda_i$. We further say that λ is the *shape* of the tableau T. For a given λ, the number f_λ of tableaux of shape λ is given by the *hook length* formula

$$f_\lambda = \frac{n!}{\prod_c h_c},$$

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where $c = (i, j)$ runs over the set of points in the diagram of λ, and

$$h_c = \lambda_i + \#\{j \mid \lambda_j \geq i\} - i - j + 1.$$

Other classical results in this context are

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!,$$

and

$$\sum_{\lambda \vdash n} f_\lambda = \text{ coeff of } \frac{x^n}{n!} \text{ in } e^{x+x^2/2}.$$

We are interested in the enumeration of tableaux of height bounded by some integer h, this is to say that we want to compute the numbers

$$t_h(n) = \sum_{h(\lambda) \leq h} f_\lambda, \tag{1}$$

as well as

$$t_h^{(2)}(n) = \sum_{h(\lambda) \leq h} f_\lambda^2. \tag{2}$$

For the cases $h = 2, 3, 4, 5$ (see Regev [2] and Gouyou-Beauchamps [1]), nice expressions have been given for the $t_h(n)$'s or their generating function. We also should mention at this point that Zeilberger, in [3], has shown that the $t_h(n)$'s are P-recursive, this is to say that they satisfy a recurrence of the form

$$\sum_{k=0}^{m} p_k(n) t_h(n-k) = 0, \tag{3}$$

for some polynomials $p_k(n)$ and some integer m. Still, his proof gives no clear indication on the bounds for m or the respective degrees of the $p_k(n)$'s. We propose, in this note, explicit values for the degree of the polynomials appearing in (3) as well as for the value of m.

2. 2. CONJECTURES FOR $t_h(n)$

Using the first values of the numbers $t_h(n)$ for small h's, and an undetermined coefficient method, we looked for simple P-recurrences for these numbers. The surprising outcome of these experiments was that these recurrences were of relatively low degree. A careful study of the first of these recurrences led us to the following conjectures. We then predicted the form of the recurrences for larger h's using these conjectures and further computations showed that these conjectured recurrences agreed with those obtained by the previous undetermined coefficient method.

(1a) *The numbers $t_h(n)$ satisfy a recurrence of the form*

$$\sum_{k=0}^{\lfloor h/2 \rfloor + 1} p_k(n) t_h(n-k) = 0, \tag{4}$$

with polynomials $p_k(n)$ each of degree $\leq \lfloor h/2 \rfloor$.

(1b) *The coefficient of $t_h(n)$ in (4) is*

$$p_0(n) = \prod_{k=1}^{\lfloor h/2 \rfloor} (n + k(h-k)).$$

(1c) *The coefficients of the $t_h(n-k)$'s in (4), $2 \leq k \leq \lfloor h/2 \rfloor + 1$, are of the form*

$$p_k(n) = q_k(n) \prod_{i=1}^{k-1} (n-i), \tag{5}$$

with the $q_k(n)$'s polynomials of respective degrees $\leq (\lfloor h/2 \rfloor - k + 1)$.

(1d) *The polynomials $q_k(n)$ of (5) are such that the recurrence (4) is true with the unique initial condition $t_h(0) = 1$.*

(1e) *For odd h, the coefficient of $t_h(n-1)$ in (4) is*

$$-p_1(n) = n\, p_0(n) - (n-1)\, p_0(n-1)$$

and the leading coefficient of $q_k(n)$ is the coefficient of z^k in the polynomial

$$\prod_{j=0}^{m} (1 - (-1)^{(m-j)}(2j+1)z),$$

where $h = 2m+1$. Also the degree of $q_k(n)$ is exactly $m - k + 1$.

Using these conjectures and an indeterminate coefficients method, we obtain the following recurrences for $h = 1, 3, 5, 7$ (the case $h = 1$ is trivial).

$$t_1(n) = t_1(n-1)$$

$$\begin{aligned}(n+2)\,t_3(n) = &(2\,n+1)\,t_3(n-1)\\ &+ 3\,(n-1)\,t_3(n-2)\end{aligned}$$

$$\begin{aligned}(n+4)\,(n+6)\,t_5(n) = &\big(3\,n^2+17\,n+15\big)\,t_5(n-1)\\ &+ (n-1)\,(13\,n+9)\,t_5(n-2)\\ &- 15\,(n-1)\,(n-2)\,t_5(n-3)\end{aligned}$$

$$\begin{aligned}(n+6)(n+10)(n+12)\,t_7(n) = \; &(4\,n^3+78\,n^2+424\,n+495)\,t_7(n-1)\\ &+ (n-1)(34\,n^2+280\,n+305)\,t_7(n-2)\\ &- (n-1)(n-2)(76\,n+290)\,t_7(n-3)\\ &- 105\,(n-1)(n-2)(n-3)\,t_7(n-4).\end{aligned}$$

And for $h = 2, 4, 6$,

$$\begin{aligned}(n+1)\,t_2(n) = &2\,t_2(n-1)\\ &- 4\,(n-1)\,t_2(n-2)\end{aligned}$$

$$\begin{aligned}(n+3)\,(n+4)\,t_4(n) = &4\,(3+2\,n)\,t_4(n-1)\\ &+ 16\,n\,(n-1)\,t_4(n-2)\end{aligned}$$

$$\begin{aligned}(n+5)(n+8)(n+9)\,t_6(n) = \; &4\,(84+46\,n+5\,n^2)t_6(n-1)\\ &+ 4\,(n-1)(10\,n^2+58\,n+33)\,t_6(n-2)\\ &- 144\,(n-1)(n-2)\,t_6(n-3)\\ &- 144\,(n-1)(n-2)(n-3)\,t_6(n-4).\end{aligned}$$

Recurrences for bigger h's are easy to obtain in the same manner. But, the computation time gets to be quite large for $h \simeq 20$. We have checked that these recurrences are consistent with explicit computation (using (1)) of the $t_h(n)$ as far as reasonable computation time allowed ($n \simeq 40$). Moreover, for very large values of n ($n \simeq 2000$), the values of $t_h(n)$, obtained through (4), are strikingly consistent with the asymptotic expressions given by Regev in [2]. It is easy to show that the solution of a recurrence satisfying (1a) through (1d) is asymptotic to h^n, and a little extra work shows that the asymptotic behavior of the solution of the recurrence obtained with these conjectures is (see [2])

$$cte\,\frac{h^n}{n^{h(h-1)/4}}.$$

In fact, a simple translation of (4) in term of a differential equation for the generating function of the numbers $t_h(n)$, gives the following, for $h = 7$,

$$\begin{aligned}
&(1-7\,x)\,(1+5\,x)\,(1-3\,x)\,(1+x)\,x^3\,\frac{d^3}{dx^3}\,y(x)\\
&\quad+\left(-552\,x^2+974\,x^3-102\,x+31+945\,x^4\right)\,x^2\,\frac{d^2}{dx^2}\,y(x)\\
&\quad+\left(1890\,x^4-1901\,x^2+2528\,x^3-686\,x+281\right)\,x\,\frac{d}{dx}\,y(x)\\
&\quad+\left(720-1001\,x-1001\,x^2+630\,x^4+1036\,x^3\right)y(x)=720
\end{aligned}$$

From this differential equation we easily find the (regular) singularity of smallest module of $y(x)$ since it is a root of the dominating polynomial. Using $y(x) \sim (1/7 - x)^r$, we solve for r and find $r = 19/2$ thus

$$cte\,\frac{7^n}{n^{21/2}},$$

since the asymptotic behavior of the coefficients of $(1 - hx)^r$ is $cte\,h^n/n^{r+1}$.

For odd h ($h = 2m + 1$), we have also obtained the following candidate for the generating function $\sum_m c_m z^m$ of coefficients c_m of n^{m-2} in the polynomial $g_2(n)$ of (5) (of degree $m - 1$)

(1f) *One has the generating function*

$$\sum_m c_m z^m = \frac{9\,x^3+217\,x^4+91\,x^5+3\,x^6}{(1-x)^7}$$

Recall that conjecture (1e) implicitly gives the coefficient of n^{m-1} in these polynomials. It appears that similar generating functions can be found for all coefficients of the $q_k(n)$'s. The $g_2(n)$ for $h = 3, 5, 7, 9$ are

$$3$$

$$13\,n+9$$

$$34\,n^2+280\,n+305$$

$$70\,n^3+1862\,n^2+13433\,n+18991$$

3. 3. CONJECTURES FOR $t_h^{(2)}(n)$

(2a) *The numbers* $t_h^{(2)}(n)$ *satisfy a recurrence of the form*

$$\sum_{k=0}^{\lfloor h/2\rfloor+1} p_k(n) t_h(n-k) = 0, \tag{6}$$

with polynomials $p_k(n)$ *each of degree* $\le h$*.*

(2b) *The coefficient of* $t_h^{(2)}(n)$ *in (6) is*

$$p_0(n) = \prod_{k=1}^{\lfloor h/2\rfloor} (n + k(h-k))^2.$$

(2c) *The coefficients of the* $t_h^{(2)}(n-k)$*'s in (6),* $2 \le k \le \lfloor h/2\rfloor + 1$*, are of the form*

$$p_k(n) = q_k(n) \prod_{i=1}^{k-1} (n-i)^2, \tag{7}$$

with the $q_k(n)$*'s polynomials of respective degrees* $\le (h-2k)$*.*

(2d) *The polynomials* $q_k(n)$ *of (7) are such that the recurrence (6) is true with the unique initial condition* $t_h(0) = 1$*.*

(2e) *The leading coefficient of* $q_k(n)$ *is the coefficient of* z^k *in the polynomial*

$$\prod_{j=0}^{m} (1 - (2j+1)^2 z),$$

where $h = 2m+1$*.*

All remarks that we have made about the $t_h(n)$'s also apply to the $t_h^{(2)}(n)$'s with the necessary modifications. For example, for $h = 5$, the recurrence is

$$\begin{aligned}(n+6)^2\,(n+4)^2\,t_5^{(2)}(n) &= \left(375 - 400\,n - 843\,n^2 - 322\,n^3 - 35\,n^4\right)\,t_5^{(2)}(n-1)\\ &\quad + \left(259\,n^2 + 622\,n + 45\right)(n-1)^2\,t_5^{(2)}(n-2)\\ &\quad - 225\,(n-1)^2\,(n-2)^2\,t_5^{(2)}(n-3).\end{aligned}$$

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5. REFERENCES

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