**1 997 583** 11J68 11R06

**Borwein, Peter** (3-SFR-MS); **Hare, Kevin G.** (3-SFR-MS)

**Non-trivial quadratic approximations to zero of a family of cubic Pisot numbers. (English. English summary)**

*Trans. Amer. Math. Soc.* **355** (2003), *no.* 12, 4767–4779 (*electronic*).

**1 986 822** 11D72 11P05 11Y50

**Borwein, Peter** (3-SFR); **Lisoněk, Petr** (3-SFR);
**Percival, Colin** (3-SFR)

**Computational investigations of the Prouhet-Tarry-Escott problem. (English. English summary)**

*Math. Comp.* **72** (2003), *no.* 244, 2063–2070 (*electronic*).

The Prouhet-Tarry-Escott problem asks for two distinct multisets of integers $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ such that

$$\sum_{i=1}^{n} x_i^e = \sum_{i=1}^{n} y_i^e$$

for $e = 1, 2, \ldots, k$ for some integer $k < n$. If $k = n - 1$, such a solution is called ideal.

The authors describe a method for searching for ideal symmetric solutions. They make an extensive search for solutions up to $n = 12$. They find two new solutions for $n = 10$, which are much smaller than the solutions found by A. Letac in the 1940s.

*Maurice Mignotte* (F-STRAS)

**1 986 811** 11C08 30C10

**Borwein, P. B.** (3-SFR-MS); **Pinner, C. G.** (1-KSS);
**Pritsker, I. E.** (1-OKS)

**Monic integer Chebyshev problem. (English. English summary)**

*Math. Comp.* **72** (2003), *no.* 244, 1901–1916 (*electronic*).

**2004a:11069** 11J72 11J82 33D15 41A21

**Borwein, Peter B.** (3-SFR-MS);
**Zhou, Ping [Zhou, Ping²]** (3-SFX-MSC)

**On the irrationality of a certain multivariate $q$ series. (English. English summary)**

*Proc. Amer. Math. Soc.* **131** (2003), *no.* 7, 1989–1998 (*electronic*).

Results on irrationality are given for the function of several variables: $q > 1,\ x_1, \dots, x_m \neq q^i,\ j = 1, 2, \dots,$

$$L(x_1, \dots, x_m) := \sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j} x_1) \cdots (1 - q^{-j} x_m)}.$$

Using a suitably constructed Padé approximation it is for example shown that

$$\sum_{j_1, \dots, j_m = 0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is irrational and not a Liouville number if $r_1, \dots, r_m$ are positive rationals less than $q$. *R. Wallisser* (D-FRBG)

**2003k:11036** 11C08 11B75 11Y55

**Borwein, Peter** (3-SFR-MS); **Mossinghoff, Michael J.** (1-UCLA)

**Newman polynomials with prescribed vanishing and integer sets with distinct subset sums. (English. English summary)**

*Math. Comp.* **72** (2003), *no.* 242, 787–800 (*electronic*).

Let $d(m)$ be the minimal degree of a polynomial that has all coefficients in $\{0, 1\}$ and a zero of multiplicity $m$ at $-1$. A greedy solution can be defined as follows. Let $J_1 = 1$ and $J_k$ be the least odd integer greater than $J_1 + \cdots + J_{k-1}$ $(k > 1)$. It is easy to see that the polynomial $g_m(x) = \prod_{k=1}^{m} (1 + z^{J_k})$ has the above-defined properties and that

$$\deg g_m = \frac{4}{3} \cdot 2^m - \frac{3}{2} + \frac{(-1)^m}{6}.$$

Therefore,

$$d(m) \leq \frac{4}{3} \cdot 2^m - \frac{3}{2} + \frac{(-1)^m}{6}.$$

The authors show that the equality holds if and only if $m \leq 5$. In the general case, they prove that

$$2^m + c_1 m \leq d(m) \leq \frac{103}{96} \cdot 2^m + c_2$$

and they conjecture that for any $\varepsilon > 0$ the inequality $d(m) < (1 + \varepsilon) 2^m$

holds for sufficiently large $m$. Also, they consider the related problem of finding a set of $m$ positive integers with distinct subset sums and minimal largest element and show that the well-known Conway–Guy sequence yields the optimal solution for $m \leq 9$.

*Sergeĭ V. Konyagin* (RS-MOSC-MM)

**2003i:11154** 11R06

**Borwein, Peter** (3-SFR-MS); **Hare, Kevin G.** (3-SFR-MS)

**General forms for minimal spectral values for a class of quadratic Pisot numbers. (English. English summary)**

*Bull. London Math. Soc.* **35** (2003), *no.* 1, 47–54.

Let

$$l^m(q) = \inf\left\{|y|\colon y = \varepsilon_0 + \varepsilon_1 q^1 + \cdots + \varepsilon_n q^n, \varepsilon_i \in \mathbb{Z}, |\varepsilon_i| \leqslant m, y \neq 0\right\}$$

and

$$a(q) = \inf\left\{|y|\colon y = \varepsilon_0 + \varepsilon_1 q^1 + \cdots + \varepsilon_n q^n, \varepsilon_i = \pm 1, y \neq 0\right\}.$$

A recent result of V. Komornik, P. Loreti and M. Pedicini [J. Number Theory **80** (2000), no. 2, 218–237; MR 2000k:11116] gives a complete description of $l^m(q)$ when $q$ is the golden ratio. This paper extends this result to all unit quadratic Pisot numbers. More precisely the following theorem is proved:

Let $q$ be a quadratic Pisot number that satisfies a polynomial of the form $p(x) = x^2 - ax \pm 1$ with conjugate $r$. Let $q$ have convergents $\{C_k/D_k\}$ and let $k$ be the maximal integer such that $|D_k r - C_k| \leqslant m\frac{1}{1-|r|}$; then $l^m(q) = |D_k q - C_k|$.

When $q$ is a quadratic Pisot number, not necessarily a unit, computations reveal that $l^m(q) = |D_k q - C_k|$ or $|D_{k-1} q - C_{k-1}|$, but no proof of this is known.

The authors also describe an algorithm to find the particular polynomial of height $m$ to which $l^m(q)$ relates.

P. Erdős, M. I. Joó and I. Joó [Bull. Soc. Math. France **120** (1992), no. 4, 507–521; MR 93m:11076] showed that if $q^n - q^{n-1} - \cdots 1 = 0$ then $l^1(q) = q^{n-1} - q^{n-2} - \cdots - 1$. So the authors point out that there are infinitely many Pisot numbers $q$ such that $l^1(q) = a(q)$, answering in the negative a question they raised in a previous article.

At the end of the paper possible extensions of this work to other unit Pisot numbers are discussed. *Christophe Doche* (F-BORD-AR)

**2003j:11023** 11C08 11-03 30C10

**Borwein, P.** (3-SFR-MS)

**Paul Erdős and polynomials. (English. English summary)**

*Paul Erdős and his mathematics*, *I* (*Budapest*, 1999), 161–174, *Bolyai Soc. Math. Stud.*, 11, *János Bolyai Math. Soc.*, *Budapest*, 2002.

The author surveys several problems of Paul Erdős regarding polynomials in analysis and number theory, in each case describing some theorems and conjectures of Erdős, and providing an overview of some recent progress. The problems treated are (i) a conjecture on the minimal $l_1$-norm of a polynomial of the form $\prod_{k=1}^{n}(1-x^{a_k})$, with $n$ fixed and each $a_k$ a positive integer, and its connection with the Diophantine problem of Prouhet, Tarry, and Escott; (ii) problems regarding polynomials with $\pm 1$ coefficients that are "flat," remaining close in some sense to their $L_2$ norm on the unit circle; (iii) a question on the maximal length of a lemniscate obtained as the set of points in the complex plane where a polynomial of prescribed degree has constant modulus; (iv) inequalities relating the values of a polynomial on an interval with the values of its derivative; (v) questions on the distribution of the zeros of a polynomial; and (vi) problems regarding the size of the coefficients of cyclotomic polynomials.

{For the entire collection see 2003h:00022}

*Michael J. Mossinghoff* (1-DVD)

**2003m:11045** 11C08 11Yxx 12D05 42A05

**Borwein, Peter** (3-SFR-MS)

★**Computational excursions in analysis and number theory. (English. English summary)**

CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 10.

*Springer-Verlag*, *New York*, 2002. $x$+220 *pp.* \$69.95.

*ISBN* 0-387-95444-9

The author gives a wonderful overview of one of the most beautiful and active areas of current computational number theory. Let $n \in \mathbb{N}$. Let

$$\mathcal{Z}_n = \left\{ \sum_{i=0}^{n} a_i z^i \colon \ a_i \in \mathbb{Z} \right\},$$

$$\mathcal{F}_n = \left\{ \sum_{i=0}^{n} a_i z^i \colon \ a_i \in \{-1,0,1\} \right\},$$

$$\mathcal{L}_n = \left\{ \sum_{i=0}^{n} a_i z^i \colon\ a_i \in \{-1, 1\} \right\}.$$

The supremum norm, or $L_\infty$ norm, of a polynomial $p$ on a set $A$ is defined as $\sup_{z\in A} |p(z)|$. Let

$$\|p\|_\infty := \sup_{|z|=1} |p(z)|,$$

$$M(p) := \exp\left( \int_0^{2\pi} \log |p(e^{i\theta})| d\theta \right).$$

The latter quantity is called the Mahler measure.

The height of a polynomial $p$, denoted by $H(p)$, is the size of the highest coefficient of $p$. The length is denoted by $L(p)$ and is the sum of the absolute values of the coefficients of $p$. In the Introduction the author gives a list of problems discussed in the book. In the review we formulate some of the problems.

(P1) The integer Chebyshev problem. Find a nonzero polynomial in $\mathcal{Z}_n$ that has smallest possible supremum norm on the unit interval. Analyze the asymptotic behavior as $n$ tends to infinity.

(P2) The Prouhet-Tarry-Escott problem. Find a polynomial with integer coefficients that is divisible by $(z-1)^n$ and has smallest positive length.

(P3) The Erdős-Szekeres problem. For each $n$, minimize

$$\|(1-z^{\alpha_1})(1-z^{\alpha_2})\cdots(1-z^{\alpha_n})\|_\infty ,$$

where the $\alpha_i$ are positive integers. In particular, show that these minima grow faster than $n^\beta$ for any positive constant $\beta$.

(P4) Littlewood's problem in $L_\infty$. Find a polynomial in $\mathcal{L}_n$ that has the smallest possible supremum norm on the unit disk. Show that there exist positive constants $c_1$ and $c_2$ such that for any $n$ it is possible to find $p \in \mathcal{L}_n$ with $c_1\sqrt{n+1} \le |p_n(z)| \le c_2\sqrt{n+1}$ for all complex $z$ with $|z| = 1$.

(P5) Erdős's problem in $L_\infty$. Show that there exists a positive constant $c_3$ such that for all sufficiently large $n$ and all $p_n \in \mathcal{L}_n$ we have $\|p_n\|_\infty \ge (1 + c_3)\sqrt{n+1}$.

(P9) Lehmer's problem. Show that any monic polynomial $p$, $p(0) \ne 0$, with integer coefficients that is irreducible and is not a cyclotomic polynomial has Mahler measure at least $1.1762\ldots$. (This latter constant is the Mahler measure of $1 + z - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10}$.)

(P10) Mahler's problem. For each $n$, find the polynomials in $\mathcal{L}_n$ that have largest possible Mahler measure. Analyze the asymptotic behavior as $n$ tends to infinity.

(P12) Closure of measures conjecture of Boyd. The set of all possible values of the Mahler measure of polynomials with integer coefficients in any number of variables is a closed set.

(P13) Multiplicity of zeros of height one polynomials. What is the maximum multiplicity of the vanishing at 1 of a polynomial in $\mathcal{F}_n$?

(P14) Multiplicity of zeros in $\mathcal{L}_n$. What is the maximum multiplicity of the vanishing at 1 of a polynomial in $\mathcal{L}_n$?

(P17) The Schur-Siegel-Smyth trace problem. Fix $\varepsilon > 0$. Suppose

$$p_n(z) := z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in \mathcal{Z}_n$$

has all real, positive roots and is irreducible. Show that, independently on $n$, except for finitely many explicitly computable exceptions, $|a_{n-1}| \geq (2-\varepsilon)n$.

The problems have been investigated by many outstanding mathematicians. Most of the problems have resisted solution for at least fifty years. The author gives an overview of the contemporary progress in the problems and presents the main ideas of the proofs. The book contains necessary background, many exercises and research problems.

In conclusion, I would like to stress that the present book is on the whole very clearly written. The interested reader may also consult the numerous bibliographic references given at the end.

*Sergeĭ V. Konyagin* (RS-MOSC-MM)

**2003a:11135** 11R06 11Y60

**Borwein, Peter** (3-SFR-MS); **Hare, Kevin G.** (3-SFR-MS)

**Some computations on the spectra of Pisot and Salem numbers. (English. English summary)**

*Math. Comp.* **71** (2002), *no.* 238, 767–780 (*electronic*).

P. Erdős, I. Joó and V. Komornik [Bull. Soc. Math. France **118** (1990), no. 3, 377–390; MR 91j:11006] introduced

$$l(q) = \inf\{|y| \colon y \in \Lambda(q),\ y \neq 0\},$$

where

$$\Lambda(q) = \left\{\varepsilon_0 + \varepsilon_1 q + \cdots + \varepsilon_n q^n,\ \varepsilon_i \in \{\pm 1, 0\}\right\}$$

for a Pisot number $q$.

In this paper, $l(q)$ and other related objects such as

$$a(q) = \inf\{|y| \colon y \in A(q),\ y \neq 0\},$$

with

$$A(q) = \left\{\varepsilon_0 + \varepsilon_1 q + \cdots + \varepsilon_n q^n,\ \varepsilon_i \in \{\pm 1\}\right\}$$

for an algebraic number $q$ are extensively investigated.

The study relies on an algorithm to compute $l(q)$ and $a(q)$. In particular, it gives $l(q)$ and $a(q)$ for all the Pisot numbers $q$ in the range $[1, 2]$ of degree less than or equal to 9 and 10, respectively. They coincide in a fairly small number of cases and it would be interesting to know if the set of Pisot numbers $q$ such that $l(q) = a(q)$ is finite or not.

A spectrum $\Lambda$ is said to be discrete if the intersection of $\Lambda$ with any interval $[a, b]$ of the real line has only a finite number of elements. An example of a non-Pisot $q$ such that $A(q)$ is discrete is given for the first time in this paper. An infinite family of Salem numbers sharing this property is also described. This is quite surprising since $A(q) \subset \Lambda(q)$ and it is believed that $\Lambda(q)$, which contains $A(q)$, is discrete if and only if $q$ is Pisot.

Another natural question is explored. Does $0 \in A(q)$ for every Pisot number $q$? In other terms, does a Pisot number $q$ always satisfy a polynomial equation with $\pm 1$ coefficients? This is mostly the case, but the answer is negative in general. The first exception is of degree 6 and the authors notice that all such Pisot numbers found are greater than 1.95.

They conclude with a list of interesting open problems.

*Christophe Doche* (F-BORD-AR)

**2002i:11065** 11J54 11B83 11C08

**Borwein, Peter** (3-SFR-MS);
**Choi, Kwok-Kwong Stephen** (3-SFR-MS)

**Explicit merit factor formulae for Fekete and Turyn polynomials. (English. English summary)**

*Trans. Amer. Math. Soc.* **354** (2002), *no.* 1, 219–234 (*electronic*).

For $q$ a prime number the Fekete polynomial is defined by $f_q(x) = \sum_{k=0}^{q-1} \left(\frac{k}{q}\right) x^k$, where $\left(\frac{\cdot}{q}\right)$ denotes the Legendre symbol. The Turyn polynomials are obtained by cyclically shifting the coefficients of Fekete polynomials: $f_q^t(x) = \sum_{k=0}^{q-1} \left(\frac{k+t}{q}\right) x^k$. The merit factor is a measure of the smoothness of a polynomial over the unit circle and is defined by $\mathrm{MF}(f) = \left(\|f\|_4^4 / \|f\|_2^4 - 1\right)^{-1}$, where $\|\cdot\|_p$ denotes the $L_p$ norm. Golay asked if the merit factor of polynomials with $\{-1, +1\}$ coefficients is bounded, and this problem is related to questions of Littlewood and Erdős. The largest known asymptotic value of the merit factor of such polynomials is 6, and this is attained by Turyn polynomials where the shift is approximately one quarter of the length.

The authors determine explicit formulae for the $L_4$ norm (and hence the merit factor) of Fekete and Turyn polynomials, and for some related classes of polynomials, such as the half-Fekete polynomials, $\sum_{k=0}^{(q-1)/2} \left(\frac{k}{q}\right) x^k$. In particular, the authors show that

$$\left\| f_q^{[q/4]} \right\|_4^4 = \frac{7}{6}q^2 - q - \frac{1}{6} - \gamma_q,$$

where $[\alpha]$ denotes the integer nearest $\alpha$ and $\gamma_q$ depends on the class number of $\mathbf{Q}(\sqrt{-q})$. This refines a result of T. Høholdt and H. E. Jensen [IEEE Trans. Inform. Theory **34** (1988), no. 1, 161–164; Zbl 0652.40006]. *Michael J. Mossinghoff* (1-DVD)

**2003i:11001** 11-01

**Borwein, Peter** (3-SFR); **Jörgenson, Loki** (3-SFR)

**Visible structures in number theory.**

*Amer. Math. Monthly* **108** (2001), *no.* 10, 897–910.

The authors give a number of examples of visual patterns revealing fascinating number-theoretic results. Most of the authors' examples concern digits (of various kinds) of interesting numbers.

*Igor Rivin* (1-PRIN)

**1 862 751** 11-03 01A70

**Borwein, Jonathan M.; Borwein, Peter B.**

**Ramanujan and pi. (English. English summary)**

*Ramanujan*: *essays and surveys*, 187–199, *Hist. Math.*, 22, *Amer. Math. Soc.*, *Providence*, *RI*, 2001.

**2002k:11114** 11J54 11L99

**Borwein, Peter** (3-SFR-MS);
**Choi, Kwok-Kwong Stephen** (PRC-HK)

**Merit factors of polynomials formed by Jacobi symbols. (English. English summary)**

*Canad. J. Math.* **53** (2001), *no.* 1, 33–50.

Let $N = p_1 p_2 \cdots p_r$ for primes $p_1 < p_2 < \cdots < p_r$. For integer $t$ with $1 \le t \le N$, define the polynomial $f_t(z) = \sum_{n=0}^{N-1} \left(\frac{n+t}{N}\right) z^n$, where $\left(\frac{n+t}{N}\right)$ is the Jacobi symbol. The authors give the following estimate for the fourth power of the $L_4$-norm of $f_t$ on the complex unit circle: for any $\varepsilon > 0$,

$$\|f_t\|_4^4 = \frac{5}{3}N^2 - 4Nt + 8t^2 + O\left(\frac{N^{2+\varepsilon}}{p_1}\right),$$

where the implied constant is independent of $N$ and $t$. This generalizes results of T. Høholdt and H. E. Jensen [IEEE Trans. Inform. Theory **34** (1988), no. 1, 161–164] and J. M. Jensen, H. E. Jensen and Høholdt [IEEE Trans. Inform. Theory **37** (1991), no. 3, part 1, 617–626; MR 92j:94009] in the cases $r=1$ and $r=2$, respectively. The authors give an exact formula for $\|f_N\|_4^4$ in the case where $r=2$ and $p_2=p_1+2$.

Of all polynomials of degree $N-1$ with coefficients $\pm 1$, the asymptotically smallest known $L_4$ norm is $(7/6)^{1/4}N^{1/2}$, which is realized for the polynomials $f_{[N/4]}$ as $N$ runs through primes. This relates to a conjecture of P. Erdős [Ann. Polon. Math. **12** (1962), 151–154; MR **25** #5330]. See also the authors' paper [Trans. Amer. Math. Soc. **354** (2002), no. 1, 219–234 (electronic); MR 2002i:11065].

*Ronald J. Evans* (1-UCSD)

**2001m:60124** 60G99 42A05

**Borwein, Peter** (3-SFR-MS); **Lockhart, Richard** (3-SFR-MS)

**The expected $L_p$ norm of random polynomials. (English. English summary)**

*Proc. Amer. Math. Soc.* **129** (2001), *no.* 5, 1463–1472 (*electronic*).

Summary: "The results of this paper concern the expected $L_p$ norm of random polynomials on the boundary of the unit disc (equivalently of random trigonometric polynomials on the interval $[0,2\pi]$). Specifically, for a random polynomial $q_n(\theta)=\sum_0^{n-1} X_k e^{ik\theta}$ let $\|q_n\|_p^p=\int_0^{2\pi}|q_n(\theta)|^p d\theta/(2\pi)$. Assume the random variables $X_k,\ k\geq 0$, are independent and identically distributed, have mean 0, variance equal to 1 and, if $p>2$, a finite $p$th moment $E(|X_k|^p)$. Then $E(\|q_n\|_p^p)/n^{p/2}\to\Gamma(1+p/2)$ and

$$\frac{E(\|q_n^{(r)}\|_p^p)}{n^{(2r+1)p/2}}\to(2r+1)^{-p/2}\Gamma(1+p/2)\quad\text{as } n\to\infty.$$

"In particular, if the polynomials in question have coefficients in the set $\{+1,-1\}$ (a much studied class of polynomials), then we can compute the expected $L_p$ norms of the polynomials and their derivatives: $E(\|q_n\|_p)/n^{1/2}\to(\Gamma(1+p/2))^{1/p}$ and

$$\frac{E(\|q_n^{(r)}\|_p)}{n^{(2r+1)/2}}\to(2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

"This complements results of Fielding in the case $p:=0$, Newman and Byrnes in the case $p:=4$, and Littlewood et al. in the case $p=\infty$."

**2001m:42002** 42A05 41A17

**Borwein, Peter** (3-SFR-MS); **Erdélyi, Tamás** (1-TXAM)

**Trigonometric polynomials with many real zeros and a Littlewood-type problem. (English. English summary)**

*Proc. Amer. Math. Soc.* **129** (2001), *no.* 3, 725–730 (*electronic*).

The authors examine the size of a real trigonometric polynomial of degree at most $n$ having at least $k$ zeros in $\mathbf{T}=\mathbf{R}/(2\pi\mathbf{Z})$. The result is used to give a new proof of a theorem of Littlewood concerning flatness of unimodular trigonometric polynomials with an explicit constant in contrast to Littlewood's approach.

Let $\mathcal{T}_n$ be the set of all real trigonometric polynomials of degree at most $n$. In particular, the following results are proven.

(1) Suppose $p\in\mathcal{T}_n$ has at least $k$ zeros in $\mathbf{T}$ (counting multiplicities). Let $\alpha\in(0,1)$. Then

$$m\{t\in\mathbf{T}\colon\ |p(t)|\le\alpha\|p\|_{L_\infty(\mathbf{T})}\}\ge\frac{\alpha}{e}\frac{k}{n},$$

where $m(A)$ denotes the one-dimensional Lebesgue measures of $A\subset K$.

(2) Assume that $p\in\mathcal{T}_n$ satisfies the inequalities $\|p\|_{L_2(\mathbf{T})}\le An^{1/2}$ and $\|p'\|_{L_2(\mathbf{T})}\ge Bn^{3/2}$. Then $\|p\|^2_{L_\infty(\mathbf{T})}\ge(2\pi-\varepsilon)\|p\|^2_{L_2(\mathbf{T})}$, where $\varepsilon=(\pi^3/1024e)(B^6/A^6)$.

The last assertion implies the following explicit version of Littlewood's theorem:

Let $p\in\mathcal{T}_n$ be of the form

$$p(t)=\sum_{k=1}^n a_k\cos(kt-\gamma_k),\quad a_k=\pm1,\quad \gamma_k\in\mathbf{R},\quad k=1,2,\dots,n.$$

Then $\|p\|^2_{L_\infty(\mathbf{T})}\ge(2\pi-\varepsilon)\|p\|^2_{L_2(\mathbf{T})}$, where $\varepsilon=(\pi^3/1024e)(1/27)$.

*Sergeĭ V. Konyagin* (RS-MOSC-MM)

**2001j:11061** 11J54 11B75

**Borwein, Peter** (3-SFR-MS); 
**Choi, Kwok-Kwong Stephen** (PRC-HK); 
**Yazdani, Soroosh** (3-WTRL-B)

**An extremal property of Fekete polynomials. (English. English summary)**

*Proc. Amer. Math. Soc.* **129** (2001), *no.* 1, 19–27 (*electronic*).

The Fekete polynomials are defined as $F_q(z) := \sum_{k=1}^{q-1} \left(\frac{k}{q}\right) z^k$ where $(\frac{\bullet}{q})$ is the Legendre symbol. After cyclic permutation, these polynomials provide sequences with smallest known $L_4$ norm among the polynomials with $\pm 1$ coefficients. The main result of the paper is the following: Theorem. Let $f(x) = a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1}$ be monic, with odd $N$ and coefficients $\pm 1$. If $\max\{|f(e^{2ik\pi/N})|\colon\ 0 \le k < N\} = \sqrt{N}$, then $N$ must be an odd prime and $f$ is equal to $F_q$.

This result gives a partial answer to a problem of Harvey Cohn on character sums. The proof is elementary but very tricky.

*Maurice Mignotte* (F-STRAS)

**2002a:26019** 26D05 41A17

**Borwein, Peter** (3-SFR-MS); **Erdélyi, Tamás** (1-TXAM)

**Markov-Bernstein type inequalities under Littlewood-type coefficient constraints. (English. English summary)**

*Indag. Math. (N.S.)* **11** (2000), *no.* 2, 159–172.

This paper, which completes earlier work of the authors [Ramanujan J. **1** (1997), no. 3, 309–323; MR 98m:41021], is devoted to establishing the Markov- and Bernstein-type inequalities for various classes of polynomials whose coefficients are restricted by inequality constraints. Let $\mathcal{F}_n := \{p\colon p(x) = \sum_0^n a_j x^j, a_j \in \{-1,0,1\}\}$, i.e. $\mathcal{F}_n$ denotes the set of all polynomials of degree at most $n$ with coefficients lying in the set $\{-1,0,1\}$, and let $\mathcal{L}_n := \{p\colon p(x) = \sum_0^n a_j x^j, a_j \in \{-1,1\}\}$. Let $\mathcal{K}_n$ be the collection of complex polynomials of the form $p(x) = \sum_0^n a_j x^j$, $|a_0| = 1$, $|a_j| \le 1$, and let $\mathcal{G}_n := \{p\colon p(x) = \sum_m^n a_j x^j, |a_m| = 1, |a_j| \le 1\}$, where $m$ is an unspecified nonnegative integer not greater than $n$. Obviously $\mathcal{L}_n \subset \mathcal{F}_n$ and $\mathcal{K}_n \subset \mathcal{G}_n$. Let $\|\cdot\|_A$ denote the supremum norm on $A \subset \mathbf{R}$. The authors establish the right Markov-type inequalities for the classes $\mathcal{F}_n$, $\mathcal{L}_n$, $\mathcal{K}_n$ and $\mathcal{G}_n$ on $[0,1]$. Namely there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n^{3/2} \le \max_{0 \ne p \in \mathcal{G}_n} |p'(1)|/\|p\|_{[0,1]} \le \max_{0 \ne p \in \mathcal{G}_n} \|p'\|_{[0,1]}/\|p\|_{[0,1]} \le c_2 n^{3/2}$$

and

$$c_1 n \log(n+1) \le \max_{0 \ne p \in \mathcal{A}_n} |p'(1)|/\|p\|_{[0,1]} \le \max_{0 \ne p \in \mathcal{A}_n} \|p'\|_{[0,1]}/\|p\|_{[0,1]} \le c_2 n \log(n+1),$$

where $\mathcal{A}_n$ designates any of the classes $\mathcal{F}_n$, $\mathcal{L}_n$ and $\mathcal{K}_n$. For $y \in [0,1)$ the Bernstein-type inequalities

$$c_1(1-y)^{-1}\log(2/(1-y)) \le \max_{p \in \mathcal{B}} \|p'\|_{[0,y]}/\|p\|_{[0,1]} \le c_2(1-y)^{-1}\log(2/(1-y))$$

are also proved with absolute constants $c_1$ and $c_2$, where $\mathcal{B}$ designates any of the classes $\mathcal{F} := \{p\colon p \in \bigcup_0^\infty \mathcal{F}_n, |p(0)| = 1\}$, $\mathcal{L} := \bigcup_0^\infty \mathcal{L}_n$ and $\mathcal{K} := \bigcup_0^\infty \mathcal{K}_n$.

*Saulius Norvidas* (LI-VILN)

**2001k:11036** 11C08

**Borwein, Peter** (3-SFR-MS); **Mossinghoff, Michael J.** (1-UCLA)

**Polynomials with height 1 and prescribed vanishing at 1. (English. English summary)**

*Experiment. Math.* **9** (2000), *no.* 3, 425–433.

The authors study the minimal degree $d(m)$ of a polynomial with all coefficients in $\{-1,0,1\}$ and a zero of order $m$ at 1. They determine $d(m)$ for $m \le 10$ and compute all the extremal polynomials. Each of the extremal polynomials is a pure product polynomial, namely, it has the form $\prod_{k=1}^m (x^{e_k} - 1)$ where the $e_k$ are positive integers. It is known that

$$m^2 \ll d(m) \ll m^2 \log(m+1).$$

The upper bound was proved by A. Bloch and G. Pólya [Proc. Lond. Math. Soc Ser. (2) **33** (1931), 102–114; Zbl 003.10501]; the lower estimate is a recent result of P. B. Borwein, T. Erdélyi and G. Kós [Proc. London Math. Soc. (3) **79** (1999), no. 1, 22–46; MR 2000c:11111]. However, it follows from one conjecture of Erdős and Szekeres that for any $c > 0$ the degree of a pure product polynomial with all coefficients in $\{-1,0,1\}$ and a zero of order $m$ at 1 is $\gg m^c$.

*Sergeĭ V. Konyagin* (RS-MOSC-MM)

**2001f:11143** 11M06 11Y35

**Borwein, P.** (3-SFR-MS)

**An efficient algorithm for the Riemann zeta function. (English. English summary)**

*Constructive, experimental, and nonlinear analysis* (*Limoges*, 1999), 29–34, *CMS Conf. Proc.*, 27, *Amer. Math. Soc.*, *Providence*, *RI*, 2000.

The author gives formulae for Riemann's zeta-function designed for its high-precision calculation. These are not meant to compete with the Riemann-Siegel formula in applications to large-scale computations of the zeros on the critical line. The shape of each formula depends on an auxiliary polynomial, the Chebyshev polynomials being a good choice. The proofs are simple and elementary. These algorithms are compared with the standard method based on the Euler-Maclaurin summation, and certain advantages are pointed out.

{For the entire collection see 2001d:00056}

*Matti Jutila* (FIN-TURK)

**2002d:11103** 11L10 11L40

**Borwein, Peter** (3-SFR-MS);
**Choi, Kwok-Kwong Stephen** (3-SFR-MS)

**Merit factors of character polynomials. (English. English summary)**

*J. London Math. Soc.* (2) **61** (2000), *no.* 3, 706–720.

Summary: "Let $q$ be a prime and $\chi$ be a non-principal character modulo $q$. Let

$$f_\chi^t(z) := \sum_{\eta=0}^{q-1} \chi(n+t) z^n,$$

where $1 \le t \le q$ is the character polynomial associated to $\chi$ (cyclically permuted $t$ places). The principal result is that for any non-principal and non-real character $\chi$ modulo $q$ and $1 \le t \le q$,

$$\|f_\chi^t(z)\|_4^4 = \tfrac{4}{3} q^2 + O(q^{3/2} \log^2 q),$$

where the implicit constant is independent of $t$ and $q$. Here $\|\cdot\|_4$ denotes the $L_4$ norm on the unit circle.

"It follows from this that all cyclically permuted character polynomials associated with non-principal and non-real characters have merit factors that approach 3. This complements and completes results of Golay, Høholdt and Jensen, and Turyn (and others). These results show that the merit factors of cyclically permuted character polynomials associated with non-principal real characters vary asymptotically between 3/2 and 6.

"The averages of the $L_4$ norms are also computed. Let $q$ be a prime number. Then

$$\sum_{\chi(\bmod q)} \|f_\chi^t\|_4^4 = (2q-3)(q-1)^2,$$

where the summation is over all characters modulo $q$."

**2000k:11001** 11-00 01A05 01A75 11-03

**Berggren, Lennart [Berggren, John L.]** (3-SFR); **Borwein, Jonathan** (3-SFR); **Borwein, Peter** (3-SFR)

★**Pi: a source book. (English. English summary)**

Second edition.
*Springer-Verlag*, *New York*, 2000. *xx*+736 *pp.* $64.95.
*ISBN* 0-387-98946-3

The first edition has been reviewed [1997; MR 98f:01001].

**2001a:41015** 41A17

**Borwein, Peter** (3-SFR-MS); **Erdélyi, Tamás** (1-TXAM)

**Pointwise Remez- and Nikolskii-type inequalities for exponential sums. (English. English summary)**

*Math. Ann.* **316** (2000), *no.* 1, 39–60.

Summary: "Let

$$E_n := \left\{ f\colon\ f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t},\ a_j, \lambda_j \in \mathbf{R} \right\},$$

so $E_n$ is the collection of all $(n+1)$-term exponential sums with constant first term. We prove the following two theorems.

"Theorem 1 (Remez-type inequality for $E_n$ at 0). Let $s \in (0, \frac{1}{2}]$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that $\exp(c_1 ns) \le \sup_f |f(0)| \le \exp(c_2 ns)$, where the supremum is taken for all $f \in E_n$ satisfying $m(\{x \in [-1,1]\colon\ |f(x)| \le 1\}) \ge 2 - s$.

"Theorem 2 (Nikol′skiĭ-type inequality for $E_n$). There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1^{1+1/q} \left( \frac{1+qn}{\min\{y-a, b-y\}} \right)^{1/q} \le \sup_{0 \ne f \in E_n} \frac{|f(y)|}{\|f\|_{L_q[a,b]}} \le \left( \frac{c_2(1+qn)}{\min\{y-a, b-y\}} \right)^{1/q}$$

for every $a < y < b$ and $q > 0$.

"It is quite remarkable that, in the above Remez- and Nikol′skiĭ-

type inequalities, $E_n$ behaves like $\mathcal{P}_n$, where $\mathcal{P}_n$ denotes the collection of all algebraic polynomials of degree at most $n$ with real coefficients."

*Michael Felten* (Hagen)

**2000j:11100** 11J54 11B83

**Borwein, Peter** (3-SFR-MS)**; Mossinghoff, Michael** (1-APLS)

**Rudin-Shapiro-like polynomials in $L_4$. (English. English summary)**

*Math. Comp.* **69** (2000), *no.* 231, 1157–1166.

Let $p$ be a polynomial, $\alpha > 0$. Denote

$$\|p\|_\alpha = \left(\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^\alpha d\theta\right)^{1/\alpha}, \ \|p\|_\infty = \max_\theta |p(e^{i\theta})|.$$

Let $p$ be a Littlewood polynomial of degree $n$, namely, a polynomial with coefficients $\{+1,-1\}$ of degree $n$. A conjecture of P. Erdős [Ann. Polon. Math. **12** (1962), 151–154; MR **25** #5330] states that there exists an absolute constant $c > 1$ such that for any Littlewood polynomial $p$ of degree $n \geq 1$ the inequality $\|p\|_\infty > c\|p\|_2$ holds. The paper is connected with a related open question of whether the inequality $\|p\|_4 > c'\|p\|_2$ holds for some $c' > 1$. The merit factor of a Littlewood polynomial $p$ of degree $n-1$ is defined as

$$Mf(p) = \frac{\|p\|_2^4}{\|p\|_4^4 - \|p\|_2^4}.$$

In the paper the Rudin-Shapiro-like polynomials generated by the iterations $p(x) \to p(x) \pm x^{d+1}p^*(-x)$, where $d$ is the degree of the polynomial $p$, $p^*(x) = x^d p(1/x)$, are studied. The authors prove an explicit formula for the merit factor of the Rudin-Shapiro-like polynomials. This gives that the merit factors of the polynomials generated by these iterations with initial polynomial $p_0$ approach $3/(4\gamma - 3)$, where

$$\gamma = \frac{\|p_0\|_4^4 + \|p_0(z)p_0^*(-z)\|_2^2}{2\|p_0\|_2^4} \geq 1.$$

Also, all the polynomials $p_0$ with $\gamma = 1$ of degree less than 40 are determined.

*Sergeĭ V. Konyagin* (RS-MOSC-MM)

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**2000m:11121** 11T22 11R18

**Borwein, Peter** (3-SFR-MS);
**Choi, Kwok-Kwong Stephen** (3-BC)

**On cyclotomic polynomials with $\pm 1$ coefficients. (English. English summary)**

*Experiment. Math.* **8** (1999), *no.* 4, 399–407.

The polynomials of the title are those of the form

$$P(x) = \sum_{i=0}^{N-1} c_i x^i, \quad c_i \in \{\pm 1\},$$

where $N > 1$ and each zero of $P(x)$ has modulus 1. If we write $N = p_1 \cdots p_r$, where the $p_i$ are (not necessarily distinct) primes, then examples of these polynomials $P(x)$ are clearly given by

$$\pm\varphi_{p_1}(\pm x)\varphi_{p_2}(\pm x^{p_1}) \cdots \varphi_{p_r}(\pm x^{p_1 \cdots p_{r-1}}),$$

where $\varphi_p(x) = 1 + x + \cdots + x^{p-1}$. The authors offer substantial evidence for their conjecture that these are the *only* examples, and they prove this conjecture when $N$ is odd or a power of 2.

*Ronald J. Evans* (1-UCSD)

**2000c:11111** 11J54 12E10

**Borwein, Peter** (3-SFR); **Erdélyi, Tamás** (1-TXAM);
**Kós, Géza** (H-EOTVO-AN)

**Littlewood-type problems on $[0, 1]$.**

*Proc. London Math. Soc.* (3) **79** (1999), *no.* 1, 22–46.

The authors examine a number of problems concerning polynomials with coefficients restricted in various ways. The following results, among others, are proven.

(1) There is an absolute constant $c_1 > 0$ such that every polynomial $p$ of the form $p(x) = \sum_{j=0}^{n} a_j x^j$ with $|a_j| \le 1$, $a_j \in \mathbf{C}$, has at most $c_1(n(1 - \log|a_0|))^{1/2}$ zeros at 1. On the other hand, for any $n \in \mathbf{N}$ and $a_0 \in \mathbf{R}$ with $0 < |a_0| \le 1$, there exists a polynomial $p$ of the form $p(x) = \sum_{j=0}^{n} a_j x^j$, $|a_j| \le 1$, $a_j \in \mathbf{R}$, such that $p$ has a zero at 1 with multiplicity at least $\min(\frac{1}{6}(n(1 - \log|a_0|))^{1/2} - 1, n)$.

(2) There are absolute constants $c_2 > 0$ and $c_3 > 0$ such that $\exp(-c_2\sqrt{n}) \le \inf_p \max_{x \in [0,1]} |p(x)| \le \exp(-c_3\sqrt{n})$ for every $n \ge 2$, where the infimum is taken over all polynomials $p$ of the form $p(x) = \sum_{j=0}^{n} a_j x^j$ with $|a_j| \le 1$, $a_j \in \mathbf{C}$. Moreover, the upper bound is attained for some polynomial with coefficients from $\{-1, 0, 1\}$.

(3) There is an absolute constant $c_4 > 0$ such that every polynomial

$p$ of the form $p(x) = \sum_{j=0}^{n} a_j x^j$ with $|a_j| \le 1$, $|a_0| = |a_n| = 1$, $a_j \in \mathbf{C}$, has at most $c_4\sqrt{n}$ real roots. The estimate is the best possible up to the constant $c_4$.

Most of the upper estimates for the number of zeros and lower estimates for the norms of polynomials are based on the key estimate stating that there are absolute constants $c_5 > 0$ and $c_6 > 0$ such that $|f(0)|^{c_5/a} \le \exp(c_2/a) \sup_{x\in[1-a,1]} |f(x)|$ for every $a \in (0,1]$ and function $f$ analytic on the unit disk $D = \{z \in \mathbf{C}\colon\ |z| < 1\}$ and satisfying the inequality $|f(z)| < 1/(1-|z|)$ for $z \in D$.

*Sergeĭ V. Konyagin* (RS-MOSC-MM)

**99i:11057** 11J72 41A21

**Borwein, Peter B.** (3-SFR)**; Zhou, Ping [Zhou, Ping²]** (3-SFR)

**On the irrationality of a certain $q$ series. (English. English summary)**

*Proc. Amer. Math. Soc.* **127** (1999), *no.* 6, 1605–1613.

Let $q \ge 1$ be an integer and let $r$ and $s$ be positive rationals such that $1 + q^m r - q^{2m} s \ne 0$ for all integers $m \ge 0$. The authors prove that $\sum_{j=0}^{\infty} 1/(1 + q^j r - q^{2j} s)$ is irrational but is not a Liouville number. In a previous paper [J. Number Theory **37** (1991), no. 3, 253–259; MR 92b:11046], the first author proved the irrationality of $\sum_{n=1}^{\infty} 1/(q^n + r)$ by examining the Padé approximations to a suitable function. The present paper generalizes these techniques to the two-variable case.

*Lawrence Washington* (1-MD)

**2001g:11114** 11J72

**Borwein, Peter B.** (3-SFR-MS)**;**
**Zhou, Ping [Zhou, Ping²]** (3-SFR-MS)

**On the irrationality of $\sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{i} (1 + q^{-j} r + q^{-2j} s)$. (English. English summary)**

*Approximation theory IX*, *Vol. I.* (*Nashville*, *TN*, 1998), 51–58, *Innov. Appl. Math.*, *Vanderbilt Univ. Press*, *Nashville*, *TN*, 1998.

The authors state that, for any integer $q > 1$ and for any $r, s \in \mathbf{Q}_+$, the sum $\sum_{i\ge 0}\ q^{-i}\ \prod_{j=0}^{i}\ (1 + q^{-j} r + q^{-2i} s)$ is irrational but not a Liouville number. They use a technique similar to earlier papers of Zhou and D. S. Lubinsky [Analysis **17** (1997), no. 2-3, 129–153; MR 99c:11089] and Zhou [Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 3, 387–397; MR 2000b:11087]. This approach consists in examining the Padé approximants to an appropriate function and showing that they provide rational approximations that are too rapid to be consistent with rationality.

{Reviewer's remarks: Unfortunately, the proof contains two serious errors. First, the functional equation (2.19) is incorrect, and therefore the definition (2.7) of $S_k(x,y)$ needs a major modification to make valid the last line of (2.19). As a consequence, the left-hand sides of (2.13), (2.15) need an extra factor $R_n(x,y)$, $R_n(xq^{-l},yq^{-l})$, respectively. Secondly, from the definition of $R_k(x,y)$ in (2.6) one sees that the factor $q^{-l(l+1)}$ in (2.24), and thus in (2.14) has to be replaced by $q^{-2ln}$ implying a modification of (2.15), compare (2.25). To sum up, the denominators forgotten in (2.7) are too small in (2.14), (2.15), and consequently later, overall become too large to get the desired contradiction.}

{For the entire collection see 2000k:41002}

*P. Bundschuh* (D-KOLN)

**2001b:41004** 41A10 11C08 26C05 26C10

**Borwein, Peter** (3-SFR-MS)

**Some old problems on polynomials with integer coefficients. (English. English summary)**

*Approximation theory IX*, *Vol. I.* (*Nashville*, *TN*, 1998), 31–50, *Innov. Appl. Math.*, *Vanderbilt Univ. Press*, *Nashville*, *TN*, 1998.

This paper surveys difficult problems concerning bounds on various norms of polynomials with integer coefficients. There are three essentially independent sections: (1) Integer Chebyshev problems, (2) Prouhet-Tarry-Escott problems, (3) Littlewood-type problems. The basic problem of the first section is to determine $\Omega[\alpha,\beta] \equiv \lim_{n\to\infty}(\min\|p\|)^{1/n}$, where $\|p\|$ is the infimum of the supnorms on the interval $[\alpha,\beta]$ of polynomials $p$ of degree $n$. The author reports on recent work on $\Omega[0,1]$ and draws attention to the Schur-Siegel-Smyth trace problem to bound the next-to-leading coefficient of an irreducible polynomial with all roots positive. The main conjecture of the second section is that for each integer $N$, there is a polynomial with integer coefficients with 1 a root of multiplicity $N$ for which the absolute sum of the coefficients is $2N$. This is equivalent to the problem of finding two distinct $N$-tuples of integers whose corresponding $k$th power sums agree for $1 \le k \le N-1$. The final section treats polynomials, all of whose coefficients are $\pm1$ and 0, or just $\pm1$. The questions here are about bounds on various norms of polynomials of degree $n$ and on the maximum multiplicity of 1 as a zero. A hundred references are provided.

{For the entire collection see 2000k:41002}

*E. J. Barbeau* (3-TRNT)

**2000c:11168** 11P82

**Bell, J. P.** (3-WTRL-PM); **Borwein, P. B.** (3-SFR);
**Richmond, L. B.** (3-WTRL-B)

**Growth of the product $\prod_{j=1}^{n}(1-x^{a_j})$.**

*Acta Arith.* **86** (1998), *no.* 2, 155–170.

Summary: "We estimate the maximum of $\prod_{j=1}^{n}|1-x^{a_j}|$ on the unit circle, where $1 \le a_1 \le a_2 \le \cdots$ is a sequence of integers. We show that when $a_j$ is $j^k$ or when $a_j$ is a quadratic in $j$ that takes on positive integer values, the maximum grows as $\exp(cn)$, where $c$ is a positive constant. This complements results of Sudler and Wright that show exponential growth when $a_j$ is $j$.

"In contrast we show, under fairly general conditions, that the maximum is less than $2^n/n^r$, where $r$ is an arbitrary positive number. One consequence is that the number of partitions of $m$ with an even number of parts chosen from $a_1, \cdots, a_n$ is asymptotically equal to the number of such partitions with an odd number of parts when $a_i$ satisfies these general conditions."

*Mihail N. Kolountzakis* (GR-CRET)

**99h:11147** 11Y60 11B65 68Q25

**Borwein, Jonathan M.** (3-SFR); **Borwein, Peter B.** (3-SFR)

★**Pi and the AGM.**

A study in analytic number theory and computational complexity.
Reprint of the 1987 original.
Canadian Mathematical Society Series of Monographs and Advanced Texts, 4.
A Wiley-Interscience Publication.
*John Wiley & Sons, Inc.*, *New York*, 1998. *xvi*+414 *pp.*
*ISBN* 0-471-31515-X

The original edition has been reviewed [1987; MR 89a:11134].

From the preface: "We aspired in the writing of *Pi and the AGM* to produce a book with a long shelf life and we are gratified by the extent to which this has been true. Nonetheless, in the dozen years since the writing of this book, some events of note have taken place. Foremost has been Bruce Berndt's completion of his four-volume masterpiece *Ramanujan's notebooks* [*Part I*, Springer, New York, 1985; MR 86c:01062; *Part II*, Springer, New York, 1989; MR 90b:01039; *Part III*, Springer, New York, 1991; MR 92j:01069; *Part IV*, Springer, New York, 1994; MR 95e:11028] that has finally made Ramanujan's work truly accessible. Additionally, in a variety of places results have been improved upon (there are now, for example, better irrationality estimates for pi due to Hata). That said, the corpus of material stands reassuringly intact.

"For afficianados of pi we have, in the interim, coauthored an extensive collection of historical and current readings titled *Pi: a source book* [Springer, New York, 1997; MR 98f:01001]. Included therein is a description of the recent algorithms for the computation of individual binary digits of pi.

"Our source book is also a good place to access more recent references and computational records. Kanada and Takahashi's current record of over 50 billion digits should be compared to Kanada's record of 200 million at the time this book was written. While we resist noting many details here, we succumb to the temptation to record that the first occurrence of the sequence 0123456789 has been found in the decimal expansion of pi beginning at the 17 387 594 880th digit after the decimal point. In consequence the status of several famous intuitionistic examples due to Brouwer and Heyting has changed."

**99c:41006** 41A10 41A17

**Borwein, Peter** (3-SFR)**; Erdélyi, Tamás** (1-TXAM)

**Müntz's theorem on compact subsets of positive measure. (English. English summary)**

*Approximation theory*, 115–131, *Monogr. Textbooks Pure Appl. Math.*, 212, *Dekker*, *New York*, 1998.

The central result of the paper is the following theorem: Suppose $(\lambda_i)_{i=-\infty}^{\infty}$ is a sequence of distinct real numbers satisfying

$$\sum_{\substack{i=-\infty \\ \lambda_i\neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

Then there is a constant $c$ depending only on $(\lambda_i)_{i=-\infty}^{\infty}$, $A$, $\alpha$ and $\beta$ (and not on the number of terms in $p$) such that $\|p\|_{[\alpha,\beta]} \le c\|p\|_A$ for every Müntz polynomial $p \in \operatorname{span}\{x^{\lambda_i}\colon i \in \mathbf{Z}\}$, for every set $A \subset (0,\infty)$ of positive Lebesgue measure, and for every $[\alpha,\beta] \subset (\operatorname{ess\,inf} A,\ \operatorname{ess\,sup}\ A)$. The main tool in proving this theorem is a Remez-type inequality established by the authors [J. Amer. Math. Soc. **10** (1997), no. 2, 327–349; MR 97k:41027].

{For the entire collection see 99a:00051}

*Yuri V. Kryakin* (PL-WROC)

**99c:41027** 41A17

**Borwein, Peter** (3-SFR); **Erdélyi, Tamás** (1-TXAM)

**A Remez-type inequality for non-dense Müntz spaces with explicit bound. (English. English summary)**

*J. Approx. Theory* **93** (1998), *no.* 3, 450–457.

Summary: "Let $\Lambda := (\lambda_k)_{k=0}^{\infty}$ be a sequence of distinct nonnegative real numbers with $\lambda_0 := 0$ and $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$. Let $\varrho \in (0,1)$ and $\varepsilon \in (0, 1-\varrho)$ be fixed. An earlier work of the authors shows that $C(\Lambda, \varepsilon, \varrho) := \sup\{\|p\|_{[0,\varrho]}\colon\ p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \cdots\},\ m(\{x \in [\varrho, 1]\colon\ |p(x)| \le 1\}) \ge \varepsilon\}$ is finite. In this paper an explicit upper bound for $C(\Lambda, \varepsilon, \varrho)$ is given. In the special case $\lambda_k = k^\alpha,\ \alpha > 1$, our bounds are essentially sharp."

**99c:30005** 30C15 26C10 30B10

**Beaucoup, Frank** (F-ENSMS-AM); **Borwein, Peter** (3-SFR-EX); **Boyd, David W.** (3-BC); **Pinner, Christopher** (3-SFR-EX)

**Multiple roots of $[-1,1]$ power series. (English. English summary)**

*J. London Math. Soc.* (2) **57** (1998), *no.* 1, 135–147.

The authors consider the family of all power series of the form $1 + \sum_{n=1}^{+\infty} a_n z^n$ with $a_n \in [-1,1]$ for all $n \in \mathbf{N}^*$.

Among all positive roots (of multiplicity $k \in \mathbf{N}^*$) of such series, let $r(k)$ denote the smallest one. The authors provide a description of the power series admitting $r(k)$ as a $k$-fold root.

They prove that the algebraic number $r(k)$ satisfies, for any $k \ge 1$,

$$1 - \frac{1}{k+1} \ge r(k) \ge \sqrt{\frac{k}{(k+1)^{1+1/k}}} \ge 1 - \frac{\log(e\sqrt{k})}{k+1}.$$

They develop an algorithm to compute approximate values of $r(k)$ for $k \in \{1, 2, \cdots, 27\}$. These computations lead them to conjecture that $r(k) \sim 1 - C/(k+1)$ (as $k \to +\infty$) where $C = 1.230\cdots$.

*Raphaële Supper* (F-STRAS-MI)

**98k:30006** 30C15 30B10

**Beaucoup, Franck** (F-ENSMS-AM); **Borwein, Peter** (3-SFR-EX); **Boyd, David W.** (3-BC); **Pinner, Christopher** (3-SFR-EX)

**Power series with restricted coefficients and a root on a given ray. (English. English summary)**

*Math. Comp.* **67** (1998), *no.* 222, 715–736.

Summary: "We consider bounds on the smallest possible root with a specified argument $\varphi$ of a power series $f(z)=1+\sum_{i=1}^\infty a_iz^i$ with coefficients $a_i$ in the interval $[-g,g]$. We describe the form that the extremal power series must take and hence give an algorithm for computing the optimal root when $\varphi/2\pi$ is rational. When $g\geq 2\sqrt{2}+3$, we show that the smallest disc containing two roots has radius $(\sqrt{g}+1)^{-1}$ coinciding with the smallest double real root possible for such a series. It is clear from our computations that the behaviour is more complicated for smaller $g$. We give a similar procedure for computing the smallest circle with a real root and a pair of conjugate roots of a given argument. We conclude by briefly discussing variants of the beta-numbers (where the defining integer sequence is generated by taking the nearest integer rather than the integer part). We show that the conjugates, $\lambda$, of these pseudo-beta-numbers either lie inside the unit circle or their reciprocals must be roots of $[-1/2,1/2)$ power series; in particular we obtain the sharp inequality $|\lambda|<3/2$."

This work is related to the papers of B. Solomyak [Proc. London Math. Soc. (3) **68** (1994), no. 3, 477–498; MR 95c:30010], A. M. Odlyzko and B. Poonen [Enseign. Math. (2) **39** (1993), no. 3-4, 317–348; MR 95b:11026] and O. Yamamoto [J. Symbolic Comput. **18** (1994), no. 5, 403–427; MR 96d:30006]. *Vanja Hadzijski* (BG-AOS)

**99j:30004** 30C10

**Borwein, P.** (3-SFR); **Erdélyi, T.** (1-TXAM)

**Littlewood-type problems on subarcs of the unit circle. (English. English summary)**

*Indiana Univ. Math. J.* **46** (1997), *no.* 4, 1323–1346.

For $M>0$ and $\mu\geq 0$, let $S_M^\mu$ denote the collection of all analytic functions $f$ on the open unit disk $D:=\{z\in \mathbf{C}\colon\ |z|<1\}$ that satisfy $|f(z)|\leq M/(1-|z|)^\mu$, $z\in D$. Define $\mathcal{F}_n:=\{f\colon\ f(x)=\sum_{j=0}^n a_jx^j,\ a_j\in\{-1,0,1\}\}$ and denote the set of all polynomials with coefficients from the set $\{-1,0,1\}$ by $\mathcal{F}:=\bigcup_{n=0}^\infty \mathcal{F}_n$. For $M>0$ and $\mu\geq 0$ let $K_M^\mu:=\{f\colon\ \ f(x)=\sum_{j=0}^n a_jx^j,\ \ a_j\in\mathbf{C},\ \ |a_j|\leq M_j^\mu,\ \ |a_0|=1,\ \ n=0,1,2,\cdots\}$. Also let $S:=S_1^1,\ S_M:=S_M^1$, and $K_M:=K_M^0$. Denote by $P_n$ and $P_n^c$ the set of all polynomials of degree at most $n$ with real coefficients and complex coefficients, respectively. The height of a polynomial $p_n(z) := \sum_{j=0}^n a_j z^j$, $a_j \in \mathbf{C}$, $a_n \neq 0$, is defined by $H(p_n) := \max\{|a_j|/|a_n|\colon\ j = 0, 1, 2, \cdots, n\}$. Also $\|p\|_A := \sup_{z\in A} |p(z)|$ and $\|p\|_{L_p(A)} := (\int_A |p(z)|^q |dz|)^{1/q}$ are used for measurable functions $p$ defined on a measurable subset $A$ of the unit circle or the real line and for $q \in (0, \infty)$. The authors, among other things, show that Konyagin's conjecture [S. V. Konyagin, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 4, 385–388; MR 97k:42002] holds on subarcs of the unit circle $\partial D$. First the authors obtain the following result concerning the lower bounds on subarcs in the supremum norm. Theorem 1: Let $0 < a < 2\pi$ and $M \geq 1$. Let $A$ be a subarc of the unit circle with length $l(A) = a$. Then there is an absolute constant $c_1 > 0$ such that $\|f\|_A \geq \exp(-c_1(1 + \log M)/a)$ for every $f \in S_M$ that is continuous on the closed unit disk and satisfies $|f(z_0)| \geq 1/2$, $z_0 \in \mathbf{C}$; $|z_0| = 1/(4M)$. They show that the above theorem is also true for every $f \in K_M$. These results are shown to be sharp up to a constant. Next, the authors extend their results to the $L_1$ norm (and hence to all $L_p$ norms with $p \geq 1$). Theorem 2: Let $0 < a < 2\pi$, $M \geq 1$, and $\mu = 1, 2, \cdots$. Let $A$ be a subarc of the unit circle with length $l(A) = a$. Then there is an absolute constant $c_1 > 0$ such that $\|f\|_{L_1(A)} \geq \exp(-c_1(\mu + \log M)/a)$ for every $f \in S_M^\mu$ that is continuous on the closed unit disk and satisfies $|f(z_0)| \geq 1/2$, $z_0 \in \mathbf{C}$; $|z_0| = 1/(4M2^\mu)$. Under the same hypothesis, it is shown that

$$\|f\|_{L_1(A)} \geq \exp(-c_1(1 + \mu \log \mu + \log M)/a)$$

for every $f \in K_M^\mu$. Consequently, it is shown that: if $A$ is a subarc of the unit circle with length $l(A) = a$ and if $(p_k)$ is a sequence of monic polynomials that tends to 0 in $L_1(A)$, then the sequence $(H(p_k))$ of heights tends to $\infty$. Finally, in the following theorem, the authors show that the theory does not extend to arbitrary sets of positive measure. Theorem 3: For every $\varepsilon > 0$ there is a polynomial $p \in K_1$ such that $|p(z)| < \varepsilon$ everywhere on the unit circle except possibly in a set of linear measure at most $\varepsilon$. The above results should be compared with earlier results of the authors published recently in a joint paper with G. Kós [Proc. London Math. Soc. (3) **79** (1999), no. 1, 22–46]. These state that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that $\exp(-c_1\sqrt{n}) \leq \inf_{0\neq p\in\mathcal{F}_n} \|p\|_{[0,1]} \leq \exp(-c_2\sqrt{n})$.

*Jay M. Jahangiri* (Burton, OH)

**98m:41021** 41A17

**Borwein, Peter** (3-SFR); **Erdélyi, Tamás** (1-TXAM)

**Markov- and Bernstein-type inequalities for polynomials with restricted coefficients. (English. English summary)**

*Ramanujan J.* **1** (1997), *no.* 3, 309–323.

The Markov-type inequality $\|p'\|_{[0,1]} \leq cn\log(n+1)\ \|p\|_{[0,1]}$ is proved for all polynomials of degree at most $n$ with coefficients from $-1, 0, 1$ with an absolute constant $c$. Here $\|\cdot\|_{[0,1]}$ denotes the supremum norm on $[0,1]$. The Bernstein-type inequality $|p'(y)| \leq c(1-y)^{-2}\|p\|_{[0,1]},\ \ y \in [0,1)$, is shown for every polynomial $p$ of the form $p(x) = \sum_{j=m}^{n} a_j x^j,\ |a_m| = 1,\ |a_j| \leq 1,\ a_j \in \mathbf{C}$. The inequality $|p'(y)| \leq c(1-y)^{-1}\log(2/(1-y))\|p\|_{[0,1]},\ \ y \in [0,1)$, is also proved for every analytic function $p$ on the open unit disk $D$ that satisfies the growth condition $|p(0)| = 1,\ |p(z)| \leq 1/(1-|z|),\ z \in D$. *M. Lachance* (1-MI2)

**98m:11014** 11C08 11R06

**Borwein, Peter** (3-SFR-EX); **Pinner, Christopher** (3-BC)

**Polynomials with $\{0,+1,-1\}$ coefficients and a root close to a given point. (English. English summary)**

*Canad. J. Math.* **49** (1997), *no.* 5, 887–915.

For a given algebraic number $\alpha$, the authors study how small $P(\alpha)$ can be when $P$ is a polynomial with $\{0,+1,-1\}$ coefficients, or, what is essentially the same, how closely $\alpha$ can be approximated by some root of such a $P$.

Let $\mathcal{B}_N$ denote the set of roots of all polynomials with $\{0,+1,-1\}$ coefficients and degree at most $N$. For points $\alpha$ away from the unit circle, the authors show that the distance to the nearest root satisfies

$$-\min_{\beta\in\mathcal{B}_N\setminus\{\alpha\}} \log|\alpha-\beta| \asymp N.$$

For Pisot or Salem numbers, they prove

$$-\min_{\beta\in\mathcal{B}_N\setminus\{\alpha\}} \log|\alpha-\beta| \sim (\log\alpha)\cdot N.$$

And for a $d$-root of unity:

$$-\min_{\beta\in\mathcal{B}_N\setminus\{\alpha\}} \log|\alpha-\beta| \ll \sqrt{N}\cdot\log N.$$

They study in more detail the special very interesting case $\alpha = 1$, and obtain more precise results in this case. Many pictures illustrate computer experiments. In some cases, extreme polynomials are determined explicitly. *Maurice Mignotte* (F-STRAS)

**1 483 913** 11Y60 01A60 11F03 33E20

**Bailey, D. H.** (1-NASA9)**; Borwein, J. M.** (3-SFR)**;**
**Borwein, P. B.** (3-SFR)

**Ramanujan, modular equations, and approximations to pi or How to compute one billion digits of pi [MR 90d:11143].**

*Organic mathematics* (*Burnaby*, *BC*, 1995), 35–71, *CMS Conf. Proc.*, 20, *Amer. Math. Soc.*, *Providence*, *RI*, 1997.

**99h:00014** 00A99 00A30 01A99

**Borwein, Jonathan M.** (3-SFR)**; Borwein, Peter B.** (3-SFR)**;**
**Corless, Robert M.** (3-WON-A)**; Jörgenson, Loki;**
**Sinclair, Nathalie**

**What is organic mathematics?**

*Organic mathematics* (*Burnaby*, *BC*, 1995), 1–18, *CMS Conf. Proc.*, 20, *Amer. Math. Soc.*, *Providence*, *RI*, 1997.

The authors explain—at some length—the purpose of the workshop on organic mathematics.

From the text: "While information technologies are changing at a rate far beyond the capacity of individual users to assimilate, it is evident that current levels of sophistication allow for the creation of a supportive technological environment. What is called for at this stage is exploratory development to provide us with some necessary experience. This is one of the key aspects of the Organic Mathematics Project which produced the *Proceedings*.

"A mathematician working in ideal conditions would be able to look at a fresh problem and easily access any related material, find all the work on simpler but similar problems, and quickly carry out any sub-computations needed for the solution of the fresh problem. Such a person also would be able to consult freely not only with colleagues, but with experts with whom they were not previously familiar. Naturally, a successful mathematician also needs the ability to generate fresh, workable, and appropriate ideas for each new problem, and this spark is to us the essence of good mathematics. Such a mathematician would surely thrive within the conditions outlined above. That said, there are formidable difficulties to providing and maintaining such an ideal environment.

"What we want to accomplish with the Organic Mathematics Project is a more thoughtful use of the technologies available, moving towards the ideal described above. We are especially interested in the benefits of integration or of unity. We want the information in the *Proceedings* of this workshop to form examples of 'living documents', connected to their references, connected to each other, connected to algorithms for live mathematical work on the part of the reader. We want them to be, in a word, 'organic'.

"The Organic Mathematics Project was directed towards the exploration of the emerging network and information technologies within the context of mathematics. Numerous groups around the world are engaged in enhancing the specific aspects of the information highway and its associated processes for transporting data. However, relatively few are actively integrating and adapting the raw technological building blocks to suit particular fields of endeavour such as mathematics. In our case, we have incorporated several different mechanisms into a single coherent environment which supports the contributions and interactions between mathematics researchers inclined towards experimental mathematics."

{For the entire collection see 98f:00027}

**98f:00027** 00B25 11-06

★**Organic mathematics.**

Proceedings of the workshop held in Burnaby, BC, December 12–14, 1995.

Edited by J. Borwein, P. Borwein, L. Jörgenson and R. Corless.

CMS Conference Proceedings, 20.

*Published by the American Mathematical Society, Providence, RI*; *for the Canadian Mathematical Society, Ottawa, ON*, 1997. x+412 *pp.* $79.00. *ISBN* 0-8218-0668-8

Contents: Jonathan M. Borwein, Peter B. Borwein, Robert M. Corless, Loki Jörgenson and Nathalie Sinclair, What is organic mathematics? (1–18); George E. Andrews, Pfaff's method. III. Comparison with the WZ method [MR 98b:33014] (19–34); D. H. Bailey, J. M. Borwein and P. B. Borwein, Ramanujan, modular equations, and approximations to pi or How to compute one billion digits of pi [MR 90d:11143] (35–71); David H. Bailey and Simon Plouffe, Recognizing numerical constants (73–88); J. M. Borwein and F. G. Garvan, Approximations to $\pi$ via the Dedekind eta function (89–115); David W. Boyd, The beta expansion for Salem numbers (117–131); Joe Buhler, David Eisenbud, Ron Graham and Colin Wright [Colin Douglas Wright], Juggling drops and descents [MR 96a:05016] (133–154); Arjeh M. Cohen and David B. Wales, GL(4)-orbits in a 16-dimensional module for characteristic 3 [MR 97k:20068] (155–174); K. Belabas and H. Cohen [Henri Cohen], Binary cubic forms and cubic number fields (175–204); Robert M. Corless, Continued fractions and chaos [MR 94g:58135] (205–238); P. J. Forrester and A. M. Odlyzko, A nonlinear equation and its application to nearest neighbor spacings for zeros of the zeta function and eigenvalues of random matrices (239–251); Andrew Granville, Arithmetic properties of binomial coefficients. I. Binomial coefficients modulo prime powers (253–276); John H. Hubbard, Jean Marie McDill, Anne Noonburg and Beverly H. West, A new look at the Airy equation with fences and funnels (277–303); Jeffrey C. Lagarias, The $3x+1$ problem and its generalizations [MR 86i:11043] (305–334); C. W. H. Lam, The search for a finite projective plane of order 10 [MR 92b:51013] (335–355); Stan Wagon, New visualization ideas for differential equations (357–381); Selected images from the *Proceedings* (383–386); Stephen P. Braham, Internet, executable content, and the future of mathematical science communication (387–405); Wayne Haga and Sinai Robins, On Kruskal's principle (407–412).

{Some of the papers are being reviewed individually.}

**98g:30008** 30C15

**Borwein, Peter** (3-SFR)**; Erdélyi, Tamás** (1-TXAM)

**On the zeros of polynomials with restricted coefficients.**

*Illinois J. Math.* **41** (1997), *no.* 4, 667–675.

This paper deals with polynomials of the form

$$p(x)=\sum_{j=0}^{n} a_j\,x^j,\quad a_j\in\mathbf{C},\quad |a_j|\le 1,\quad |a_0|=1.$$

The authors prove that such polynomials have at most $c_1\sqrt{n}$ zeros inside any polygon with vertices on the unit circle and at most $c_2/\alpha$ zeros inside any polygon with vertices on the circle $\{z\in\mathbf{C}\colon\ |z|=1-\alpha\}$, $\alpha\in(0,1)$. The constant $c_1$ depends only on the polygon and $c_2$ only on the number of vertices of the polygon.

Under the additional hypothesis $|a_n|=1$, the authors also prove that these polynomials have at most $c_3(n\alpha+\sqrt{n})$ zeros in the strip $\{z\in\mathbf{C}\colon\ |\operatorname{Im} z|\le\alpha\}$ and in the sector $\{z\in\mathbf{C}\colon\ |\arg z|\le\alpha\}$. Here, $c_3$ is an absolute constant.

The authors also verify that these majorizations are sharp.

*Raphaële Supper* (F-STRAS-MI)

**98f:01001** 01-00 01A05 01A75 11-00 11-03 26-03 65-03 68-03

★**Pi: a source book.**

Edited by Lennart Berggren, Jonathan Borwein and Peter Borwein.

*Springer-Verlag*, *New York*, 1997. xx+716 *pp.* \$59.95.
*ISBN* 0-387-94924-0

These 70 facsimile papers and excerpts constitute a mixture of historical tracts and expository accounts in the long, fascinating and occasionally whimsical saga of the number pi. Early work of ancient non-European civilizations is sampled along with European research since the seventeenth century (for example, the transcendence papers of Hermite and Lindemann). Several recent contributions record progress in computing pi to an incredible degree of accuracy, reflecting the expertise of two of the editors in this area. In the body of the book, the papers follow one another chronologically without explanation. For context and sources, the reader must check through the table of contents, the bibliography, the list of credits and the first appendix describing the early history of pi. A second appendix dates the progress in computing pi, while the third provides a list of formulae. The judicious representative selection makes this a useful addition to one's library as a reference book, an enjoyable survey of developments and a source of elegant and deep mathematics of different eras. *E. J. Barbeau* (3-TRNT)

**98b:01045** 01A99

**Bailey, D. H.** (1-NASA9); **Borwein, J. M.** (3-SFR-EX);
**Borwein, P. B.** (3-SFR-EX); **Plouffe, S.** (3-SFR-EX)

**The quest for pi.**

*Math. Intelligencer* **19** (1997), *no.* 1, 50–57.

Introduction: "This article gives a brief history of the analysis and computation of the mathematical constant $\pi = 3.14159\ldots$, including a number of formulas that have been used to compute $\pi$ through the ages. Some exciting recent developments are then discussed in some detail, including the recent computation of $\pi$ to over six billion decimal digits using high-order convergent algorithms, and a newly discovered scheme that permits arbitrary individual hexadecimal digits of $\pi$ to be computed.

"For further details of the history of $\pi$ up to about 1970, the reader is referred to Petr Beckmann's readable and entertaining book [*A history of $\pi$ (pi)*, Second edition, The Golem Press, Boulder, Colo., 1971; MR **56** #8261]. A listing of milestones in the history of the computation of $\pi$ is given in Tables 1 and 2, which we believe to be more complete than other readily accessible sources."

**98d:11165** 11Y60

**Bailey, David** (1-NASA9); **Borwein, Peter** (3-SFR);
**Plouffe, Simon** (3-SFR)

**On the rapid computation of various polylogarithmic constants. (English. English summary)**

*Math. Comp.* **66** (1997), *no.* 218, 903–913.

The authors give algorithms for the computation of the $n$th digit of certain transcendental constants in (essentially) linear time and logarithmic space. The complexity class considered is denoted by $\mathrm{SC}^*$, which means space $=\log^{O(1)}(n)$ and time $=O(n\log^{O(1)}(n))$. As a typical example, the authors show how to compute, say, just the billionth binary digit of $\log(2)$, using single precision, within a few hours.

The existence of such an algorithm, which appears to be quite surprising at first, is based on the following idea. Suppose a constant $C$ can be represented as $C=\sum_{k=0}^\infty 1/(b^{ck}q(k))$, where $b\geq 2$ and $c$ are positive integers, and $q$ is a polynomial with integer coefficients ($q(k)\neq 0$). The task is to compute the $n$th digit of $C$ in base $b$. First observe that it is sufficient to compute $b^nC$ modulo 1. Clearly,

$$b^nC \bmod 1=\sum_{k=0}^\infty \frac{b^{n-ck}}{q(k)} \bmod 1 = \\ \sum_{k=0}^{[n/c]}\frac{b^{n-ck} \bmod q(k)}{q(k)} \bmod 1+\sum_{k=1+[n/c]}^\infty \frac{b^{n-ck}}{q(k)} \bmod 1.$$

In each term of the first sum, $b^{n-ck} \bmod q(k)$ is computed using the well-known fast exponentiation algorithm modulo the integer $q(k)$. Division by $q(k)$ and summation are performed using ordinary floating-point arithmetic. Concerning the infinite sum, note that the exponent in the numerator is negative. Thus, floating-point arithmetic can again be used to compute its value with sufficient accuracy. The final result, a fraction between 0 and 1, is then converted to the desired base $b$. With certain minor modifications, this scheme can be extended to numbers of the form $C=\sum_{k=0}^\infty p(k)/(b^{ck}q(k))$, where $p$ is a polynomial with integer coefficients.

It now happens that a large number of interesting transcendentals are of the form described. Many of the formulas depend on various polylogarithmic identities. Thus, define the $m$th polylogarithm $L_m$ by $L_m(z)=\sum_{j=1}^\infty z^j/j^m,\ |z|<1$. Then, for example, $-\log(1-2^{-n})=$

$L_1(1/2^n)$, or $\pi^2 = 36L_2(\frac{1}{2}) - 36L_2(\frac{1}{4}) - 12L_2(\frac{1}{8}) + 6L_2(\frac{1}{64})$. One of the most striking identities is, however,

$$\pi = \sum_{j=0}^{\infty} \frac{1}{16^j}\left(\frac{4}{8j+1} - \frac{2}{8j+4} - \frac{1}{8j+5} - \frac{1}{8j+6}\right).$$

Using these formulas, it is easily shown that $-\log(1-2^{-n})$, $\pi$ or $\pi^2$ are in SC$^*$. The authors demonstrate their technique by computing the ten billionth hexadecimal digit of $\pi$, as well as the billionth hexadecimal digit of $\pi^2$ and $\log(2)$. *Andreas Guthmann* (Pirmasens)

**97k:41027** 41A17

**Borwein, Peter** (3-SFR)**; Erdélyi, Tamás** (1-TXAM)

**Generalizations of Müntz's theorem via a Remez-type inequality for Müntz spaces. (English. English summary)**

*J. Amer. Math. Soc.* **10** (1997), *no.* 2, 327–349.

The authors investigate uniform approximation by Müntz polynomials of the form $p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}$, where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ are given. A fundamental theorem in the paper is a Remez-type inequality for such polynomials. The result says that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then there exists a constant $c$ such that $\|p\|_{[0,\rho]} \leq c\|p\|_A$ for every Müntz polynomial $p$ and for every set $A \subset [\rho, 1]$ of positive Lebesgue measure. By applying this inequality, the authors prove two long-standing conjectures.

It is well known that the Müntz polynomials are dense in $C[0,1]$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$, while the Müntz rational functions (which are quotients of Müntz polynomials) are always dense in $C[0,1]$. It was conjectured by D. J. Newman in 1978 that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then the product of Müntz polynomials is not dense in $C[0,1]$. The authors prove that this conjecture is true. A further open problem was whether the classical theorem of Müntz for $C[0,1]$ holds for $C(A)$, where $A$ is an arbitrary compact subset of $[0,\infty)$ with positive Lebesgue measure. By using the Remez-type inequality above, the authors are also able to verify this basic result. *Günther Nürnberger* (Mannheim)

**97k:26014** 26D15

**Borwein, Peter** (3-SFR); **Erdélyi, Tamás** (1-TXAM)

**Sharp extensions of Bernstein's inequality to rational spaces. (English. English summary)**

*Mathematika* **43** (1996), *no.* 2, 413–423 (1997).

Summary: "Sharp extensions of some classical polynomial inequalities of Bernstein are established for rational function spaces on the unit circle, on $K = \mathbf{R} \pmod{2\pi}$, on $[-1,1]$ and on $\mathbf{R}$. The key result is the establishment of the inequality

$$|f'(z_0)| \le \max\left\{ \sum_{\substack{j=1\\|a_j|>1}}^{n} \frac{|a_j|^2-1}{|a_j-z_0|^2}, \sum_{\substack{j=1\\|a_j|<1}}^{n} \frac{1-|a_j|^2}{|a_j-z_0|^2} \right\} \|f\|_{\partial D}$$

for every rational function $f = p_n/q_n$, where $p_n$ is a polynomial of degree at most $n$ with complex coefficients and $q_n(z) = \prod_{j=1}^{n}(z-a_j)$ with $|a_j| \ne 1$ for each $j$, and for every $z_0 \in \partial D$, where $\partial D = \{z \in \mathbf{C}\colon\ |z| = 1\}$. The above inequality is sharp at every $z_0 \in \partial D$." *L. Leindler* (H-SZEG-B)

**1 413 248** 00A35

**Borwein, J.** (3-SFR-EX); **Borwein, P.** (3-SFR); **Girgensohn, R.** (D-ULUB-MI); **Parnes, S.**

**Making sense of experimental mathematics.**

*Math. Intelligencer* **18** (1996), *no.* 4, 12–18.

**97d:30005** 30C15 41A17

**Borwein, P.** (3-SFR); **Erdélyi, T.** (1-TXAM)

**Questions about polynomials with $\{0,-1,+1\}$ coefficients: Research Problems 96-3.**

*Constr. Approx.* **12** (1996), *no.* 3, 439–442.

The authors raise 14 problems in connection with the location and multiplicity of zeros of polynomials whose coefficients are restricted. Approximation-theoretic properties of such polynomials are also treated. All the problems are well-founded and sufficient background and history are also given. *J. Szabados* (H-AOS)

**97e:41026** 41A17 41A44

**Borwein, Peter** (3-SFR); **Erdélyi, Tamás** (3-SFR)

**A sharp Bernstein-type inequality for exponential sums.**

*J. Reine Angew. Math.* **476** (1996), 127–141.

This paper is concerned with the right Bernstein-type inequalities for exponential sums. When E. Schmidt proved that there exists always a best uniform approximation in the set of exponential sums of order $n$, he established an inequality of the form $|f(0)| \leq c_n \cdot \delta^{-1} \|f\|_{[-\delta,+\delta]}$, with $c_n$ depending only on the degree $n$ of the exponential sum. Lorentz and the authors had found some bounds for $c_n$. In the paper under review the optimal bound is derived by an interesting comparison argument from total positivity. The exponential sums of order $n$ contain the polynomials of degree $n-1$. The function with the maximal derivative at 0 turns out to be a polynomial, and so $c_n$ can be obtained from a Bernstein inequality for polynomials. Estimates for unsymmetrical intervals and $L_p$-norms are also presented.

*Dietrich Braess* (D-BCHMM)

**97h:41038** 41A30 41A17

**Borwein, Peter** (3-SFR); **Erdélyi, Tamás** (1-TXAM)

**The full Müntz theorem in $C[0,1]$ and $L_1[0,1]$. (English. English summary)**

*J. London Math. Soc.* (2) **54** (1996), *no.* 1, 102–110.

The authors give necessary and sufficient conditions for the density of the Müntz system $\{1, x^{\lambda_1}, \cdots, x^{\lambda_n}, \cdots\}$ in $C[0,1]$, $L_2[0,1]$ and $L_1[0,1]$. This leads to the conjecture that the Müntz system $\{1, x^{\lambda_1}, \cdots, x^{\lambda_n}, \cdots\}$ with $-1/p < \lambda_i < \lambda_{i+1}$ is dense in $L_p[0,1]$, $1 \leq p < \infty$, and $C[0,1]$ (when $p=\infty$) if and only if $\sum_{i=1}^{\infty} (\lambda_i + 1/p)/((\lambda_i + 1/p)^2 + 1) = \infty$. This result is proved here for $p=1$ and $p=\infty$. For $p=2$, the equivalent condition $\sum_{i=1}^{\infty} (2\lambda_i + 1)/((2\lambda_i + 1)^2 + 1) = \infty$ is given. For general $p$, $1 \leq p \leq \infty$, the result has now been settled by Opperstein (using the special cases $p=1$ and $p=\infty$). The proof of the present results and outline of the proof for $1 < p < \infty$ are given by the authors in [*Polynomials and polynomial inequalities*, Springer, New York, 1995; MR 97e:41001 (see pp. 111–114, 204–205)].

*Z. Ditzian* (3-AB)