Identities inspired by Ramanujan Notebooks, part II and III

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Abstract

These are new findings related to the ones found in 1998. I had the idea that maybe the sums in the first note I wrote were looking in the bad direction. They are good yes but maybe there is better. Actually there is, instead of considering powers of $e^{2\pi}$ I should consider powers of $e^{k\pi}$.

After a few experiments with Maple (computer algebra system) and PSLQ (integer relations algorithm) here are the findings and remarks. The results were first announced in April 2006 and later in June 2009. There is a new formula for the Catalan constant (K).

- I could go up to $\zeta(39)$ and powers of $e^{30\pi}$ after that the numbers are too close together for any computation, $e^{-30\pi}$ is of the order of 10^{-40} and even with 2000 decimal digits I get an error.
- The powers of $e^{k\pi}\mp 1$ are related to the divisors of a number.
- The numbers that appear to be expressible are so far : log(A) where A is an integer or simple algebraic like $arccosh(\frac{1}{2})$ or $arctanh(\frac{2}{3})$, $\zeta(2n+1)$, π^n, $\zeta(2n+1)$ and some formulas involving $ln\Gamma(1/4)$ and $ln(\pi)$ but not separately.

Formulas for odd powers of π and ζ(2n+1)

$$\zeta(5)=\frac{694}{204813}\pi^5-\frac{6280}{3251}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}+\frac{296}{3251}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{5\pi n}-1)}-\frac{1073}{6502}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{10\pi n}-1)}+\frac{37}{6502}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{20\pi n}-1)}$$

$$\zeta(5)=\frac{11\pi^5\sqrt{3}}{5670}+2\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{3}\pi n}-1)}-\frac{33}{8}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{12}\pi n}-1)}+\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{48}\pi n}-1)}$$

$$\zeta(3)=\frac{13\pi^3\sqrt{3}}{45}+2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\sqrt{3}\pi n}-1)}-\frac{9}{2}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\sqrt{3}\pi n}-1)}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{4\sqrt{3}\pi n}-1)}$$

$$\zeta(5)=\frac{5\pi^5\sqrt{7}}{3906}+\frac{64}{31}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{7}\pi n}-1)}+\frac{130}{31}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{28}\pi n}-1)}-\frac{4}{31}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{112}\pi n}-1)}$$

That one is the fastest converging series so far with 5.45 digits for each n.

$$\pi = 72\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} - 96\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)} + 24\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n}-1)}$$

$$\pi^3 = 720\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n}-1)} - 900\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} + 180\sum_{n=1}^{\infty}\frac{1}{n^3(e^{4\pi n}-1)}$$

$$\pi^5 = 7056\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - 6993\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} + 63\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}$$

$$\pi^7 = \frac{907200}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{\pi n}-1)} - 70875\sum_{n=1}^{\infty}\frac{1}{n^7(e^{2\pi n}-1)} + \frac{14175}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{4\pi n}-1)}$$

$$\pi^9 = \frac{28226880}{41}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{\pi n}-1)} - \frac{112920885}{164}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{2\pi n}-1)} + \frac{13365}{164}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{4\pi n}-1)}$$

$$\pi^{11} = \frac{27243216000}{4009}\sum_{n=1}^{\infty}\frac{1}{n^{11}(e^{\pi n}-1)} - \frac{218158565625}{32072}\sum_{n=1}^{\infty}\frac{1}{n^{11}(e^{2\pi n}-1)} + \frac{212837625}{32072}\sum_{n=1}^{\infty}\frac{1}{n^{11}(e^{4\pi n}-1)}$$

$$\pi^{13} = \frac{159687158400}{2381}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{\pi n}-1)} - \frac{5109979245525}{76192}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{2\pi n}-1)} + \frac{9823275}{76192}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{4\pi n}-1)}$$

$$\pi^{15} = \frac{9094280832000}{13739}\sum_{n=1}^{\infty}\frac{1}{n^{15}(e^{\pi n}-1)} - \frac{145517374445625}{219824}\sum_{n=1}^{\infty}\frac{1}{n^{15}(e^{2\pi n}-1)} + \frac{8881133625}{219824}\sum_{n=1}^{\infty}\frac{1}{n^{15}(e^{4\pi n}-1)}$$

$$\pi^{17} = \frac{19118853097344000}{2926507}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{\pi n}-1)} - \frac{1223606634625249875}{187296448}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{2\pi n}-1)} + \frac{36395233875}{187296448}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{4\pi n}-1)}$$

$$\pi^{19} = \frac{1688703016492800000}{26190337}\sum_{n=1}^{\infty}\frac{1}{n^{19}(e^{\pi n}-1)} - \frac{216154810673098171875}{3352363136}\sum_{n=1}^{\infty}\frac{1}{n^{19}(e^{2\pi n}-1)} + \frac{824562019771875}{3352363136}\sum_{n=1}^{\infty}\frac{1}{n^{19}(e^{4\pi n}-1)}$$

$$\zeta(3) = 28\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n}-1)} - 37\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} + 7\sum_{n=1}^{\infty}\frac{1}{n^3(e^{4\pi n}-1)}$$

$$\zeta(5) = 24\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - \frac{259}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - \frac{1}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}$$

$$\zeta(7) = \frac{304}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{\pi n}-1)} - \frac{103}{4}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{2\pi n}-1)} + \frac{19}{52}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{4\pi n}-1)}$$

$$\zeta(9) = \frac{20000}{861}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{\pi n}-1)} - \frac{1617613}{1640}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{2\pi n}-1)} + \frac{373}{34440}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{4\pi n}-1)}$$

$$\zeta(11) = \frac{92992}{4009}\sum_{n=1}^{\infty}\frac{1}{n^{11}(e^{\pi n}-1)} - \frac{1617613}{64144}\sum_{n=1}^{\infty}\frac{1}{n^{11}(e^{2\pi n}-1)} + \frac{1453}{64144}\sum_{n=1}^{\infty}\frac{1}{n^{11}(e^{4\pi n}-1)}$$

$$\zeta(13) = \frac{717696}{30953}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{\pi n}-1)} - \frac{124738499}{4952480}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{2\pi n}-1)} + \frac{2179}{4952480}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{4\pi n}-1)}$$

$$\zeta(15) = \frac{3503872}{151129}\sum_{n=1}^{\infty}\frac{1}{n^{15}(e^{\pi n}-1)} - \frac{243606007}{9672256}\sum_{n=1}^{\infty}\frac{1}{n^{15}(e^{2\pi n}-1)} + \frac{13687}{9672256}\sum_{n=1}^{\infty}\frac{1}{n^{15}(e^{4\pi n}-1)}$$

$$\zeta(17) = \frac{203545088}{8779521}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{\pi n}-1)} - \frac{801871973691}{31840396160}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{2\pi n}-1)} + \frac{2986673}{95521188480}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{4\pi n}-1)}$$

$$\zeta(19) = \frac{7893541888}{340474381}\sum_{n=1}^{\infty}\frac{1}{n^{19}(e^{\pi n}-1)} - \frac{168852101149}{6704726272}\sum_{n=1}^{\infty}\frac{1}{n^{19}(e^{2\pi n}-1)} + \frac{7708537}{87161441536}\sum_{n=1}^{\infty}\frac{1}{n^{19}(e^{4\pi n}-1)}$$

Other formulas

$$\pi - 3\log(2) = 12\left(\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} + \sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}+1)}\right)$$

$$\log(2) = 10\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}+1)} + 6\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} - 4\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)}$$

$$\log(3) = -9\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} - \frac{49}{3}\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}+1)} + \frac{14}{3}\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}+1)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{n(e^{3\pi n}-1)} + \frac{7}{3}\sum_{n=1}^{\infty}\frac{1}{n(e^{3\pi n}+1)} - \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n(e^{6\pi n}+1)}$$

$$\log(5) = \frac{57}{4}\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} + \frac{91}{4}\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}+1)} - \frac{13}{2}\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}+1)}$$

$$+\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n(e^{5\pi n}-1)} - \frac{7}{4}\sum_{n=1}^{\infty}\frac{1}{n(e^{5\pi n}+1)} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n(e^{10\pi n}+1)}$$

$$\pi = 72\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} - 96\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)} + 24\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n}-1)}$$

$$\zeta(3) = \frac{\pi^3}{28} + \frac{16}{7}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n}+1)} - \frac{2}{7}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}+1)}$$

$$\zeta(5) = 24\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - \frac{259}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - \frac{1}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}$$

$$\zeta(5) = \frac{-7}{1840}\pi^5 + \frac{328}{115}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - \frac{419}{460}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - \frac{9}{115}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{3\pi n}-1)} + \frac{261}{1840}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{6\pi n}-1)} - \frac{9}{1840}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{12\pi n}-1)}$$

$$\zeta(5) = \frac{-149}{43983}\pi^5 + \frac{785}{4344}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{12\pi n}-1)} - \frac{22765}{4344}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{6\pi n}-1)} + \frac{1570}{543}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{3\pi n}-1)} - \frac{61}{4344}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{20\pi n}-1)} + \frac{1769}{4344}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{10\pi n}-1)} - \frac{122}{543}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{5\pi n}-1)}$$

$$\zeta(7) = \frac{304}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{\pi n}-1)} - \frac{103}{4}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{2\pi n}-1)} - \frac{19}{52}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{4\pi n}-1)}$$

$$\zeta(9) = \frac{64}{3}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{\pi n}-1)} + \frac{441}{20}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{2\pi n}-1)} - 32\sum_{n=1}^{\infty}\frac{1}{n^9(e^{3\pi n}-1)} - \frac{4763}{60}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{4\pi n}-1)} + \frac{529}{8}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{6\pi n}-1)} - \frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{12\pi n}-1)}$$

Results of June 2009

Je présente ici une série d'identités inspirées des Notebooks de S. Ramanujan. C'est le 3ème article de cette série, des résultats avaient été publiés en 1998 et 2006. Une nouvelle série pour la constante de Catalan a été trouvée impliquant la fonction cosh(kπn) avec k variant de 1 à 4, et qui permet de calculer rapidement cette constante.

Un motif général peut être dégagé de ces identités. En effet, le motif (1,2,4) revient constamment. Une autre remarque concerne un changement de variable avec $e^{\pi n}$, G.H. Hardy (collected papers) remarquait que si $F(x) = \prod_{n=1}^{\infty}\frac{1}{1-x^n}$ alors en posant $x=e^{\pi}$ on retrouve plusieurs identités sous forme de produit infini plutôt que de somme, le log d'un produit infini est une somme infinie. La fonction F(x) n'est autre que la fonction partages de n, largement étudiée par Hardy et Ramanujan.

I present here a series of identities inspired from Ramanujan notebooks, third of a series from 1998 and 2006. A new identity for the Catalan constant involving $\cosh(\pi n)$ is given, giving a relatively rapid convergent series for that number. A general motif with exponents 1,2,4 always occur. A new series of approximations is also presented with incredible precision.

$$\frac{1}{\pi} = 8\sum_{n=1}^{\infty}\frac{n}{e^{\pi n}-1} - 40\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n}-1} + 32\sum_{n=1}^{\infty}\frac{n}{e^{4\pi n}-1}$$

$$\pi = 72\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{\pi n}-1} - 96\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{2\pi n}-1} + 24\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{4\pi n}-1}$$

$$\frac{\pi}{24} = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{3}{q^{n}-1} - \frac{4}{q^{2n}-1} + \frac{1}{q^{4n}-1}\right), q = e^{\pi}$$

$$\frac{1}{\pi^2} = 2\sum_{n=1}^{\infty}\frac{n^2}{cosh(\pi n)-1} - 32\sum_{n=1}^{\infty}\frac{n^2}{cosh(2\pi n)-1} + 32\sum_{n=1}^{\infty}\frac{n^2}{cosh(4\pi n)-1}$$

$$\pi^2 = 120\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(\pi n)-1} - 420\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(2\pi n)-1} + 120\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(4\pi n)-1}$$

$$2K = 22\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(\pi n)-1} - 71\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(2\pi n)-1} + 22\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(4\pi n)-1}\ , avec\ K = Cte\ de\ Catalan$$

$$1 = 24\sum_{n=1}^{\infty}\frac{n}{e^{\pi n}-1} - 96\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n}-1} + 96\sum_{n=1}^{\infty}\frac{n}{e^{4\pi n}-1}$$

$$1 = 24\sum_{n=1}^{\infty}\frac{n^3}{e^{\pi n}-1} - 264\sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n}-1}$$

$$1 = 4\sum_{n=1}^{\infty}\frac{n^7}{e^{\pi n}-1} - 484\sum_{n=1}^{\infty}\frac{n^7}{e^{2\pi n}-1}$$

$$1 = \frac{63}{1382}\sum_{n=1}^{\infty}\frac{n^{11}}{e^{\pi n}-1} + \frac{131103}{1382}\sum_{n=1}^{\infty}\frac{n^{11}}{e^{2\pi n}-1}$$

$$1 = \frac{1}{7234}\sum_{n=1}^{\infty}\frac{n^{15}}{e^{\pi n}-1} - \frac{32641}{7234}\sum_{n=1}^{\infty}\frac{n^{15}}{e^{2\pi n}-1}$$

$$\zeta(4) = 14\sum_{n=1}^{\infty}\frac{n^{-4}}{cosh(\pi n)-1} - \frac{259}{4}\sum_{n=1}^{\infty}\frac{n^{-4}}{cosh(2\pi n)-1} + \frac{7}{2}\sum_{n=1}^{\infty}\frac{n^{-4}}{cosh(4\pi n)-1}$$

$$\zeta(5) = 24\sum_{n=1}^{\infty}\frac{n^{-5}}{e^{\pi n}-1} - \frac{259}{10}\sum_{n=1}^{\infty}\frac{n^{-5}}{e^{2\pi n}-1} - \frac{1}{10}\sum_{n=1}^{\infty}\frac{n^{-5}}{e^{4\pi n}-1}$$

$$log(2) = 16\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{\pi n}-1} - 24\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{2\pi n}-1} + 8\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{4\pi n}-1}$$

$$log(3) = \frac{76}{3}\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{\pi n}-1} - \frac{112}{3}\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{2\pi n}-1} - \frac{4}{3}\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{3\pi n}-1} + \frac{28}{3}\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{4\pi n}-1} + \frac{16}{3}\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{6\pi n}-1} - \frac{4}{3}\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{12\pi n}-1}$$

$$log(\emptyset) = 11\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{\pi n}-1} - 14\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{2\pi n}-1} + 3\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{4\pi n}-1} + \sum_{n=1}^{\infty}\frac{n^{-1}}{e^{5\pi n}-1} - 10\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{10\pi n}-1} + 20\sum_{n=1}^{\infty}\frac{n^{-1}}{e^{20\pi n}-1}$$

$$1-\frac{1}{\pi} = 16\sum_{n=1}^{\infty}\frac{n}{e^{\pi n}-1} - 56\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n}-1} + 56\sum_{n=1}^{\infty}\frac{n}{e^{4\pi n}-1}$$

$$0 = [1, 2^{k-1} + 2^{k/2} + 2^2, 2^k\ ] \equiv [a,b,c] \ with\ k = 4,6,8,\ldots$$

$$0 = \sum_{n=1}^{\infty}\frac{n^k}{cosh(\pi n)-1} - (2^{k-1} + 2^{k/2} + 4)\sum_{n=1}^{\infty}\frac{n^k}{cosh(2\pi n)-1} + 2^k\sum_{n=1}^{\infty}\frac{n^k}{cosh(4\pi n)-1}$$

$$\pi^3 = 360\sum_{n=1}^{\infty}\frac{n^{-3}}{sinh(\pi n)} - 90\sum_{n=1}^{\infty}\frac{n^{-3}}{sinh(2\pi n)}$$

$$\pi^5 = 3528\sum_{n=1}^{\infty}\frac{n^{-5}}{sinh(\pi n)} - \frac{63}{2}\sum_{n=1}^{\infty}\frac{n^{-5}}{sinh(2\pi n)}$$

$$2K - \zeta(2) = 2\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(\pi n)-1} - 1\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(2\pi n)-1} + 2\sum_{n=1}^{\infty}\frac{n^{-2}}{cosh(4\pi n)-1} \quad avec\ K = Cte\ de\ Catalan$$

Results of April 2006

Here is a table of results with the coefficients :

Results with $\zeta(z), \pi^z, S(z,1),\ S(z,2), S(z,4)$ where $S(k,m)$ is

$$\sum_{n=1}^{\infty} \frac{1}{n^k(e^{m\pi n}-1)} = S(k,m)$$

The display is the integer relation that makes the vector [$\zeta(z)$, $\pi^z, S(z,1),\ S(z,2), S(z,4)$] equal to 0.

```
3, [1, 0, -28, 37, -7]
5, [-10, 0, 240, -259, -1]
7, [-52, 0, 1216, -1339, 19]
9, [27930, -1, 39680, 16401, -221]
11, [-538116, 2, -1108992, 33843, -1083]
13, [-4616430, 2, -27095040, 17859891, 2289]
15, [52761192, -2, 100614144, 4902099, 6141]
17, [-2381720400, 8, 2954035200, -7717548915, 72915]
19, [39005722340, -14, -1613496320, 79624947155, -6155]
21, [-221481750994700, 6732, 850750073733120, -1293713997563111, 421840591]
23, [1226556695374, 9, -84963147907072, 87416281554613, -20256793]
25, [5203911059453925, -1921, -1567010009907200, 11974832749256745, -620441695]
27, [8148451902123610, -306, -1701430826106880, 17998334655707395, -25353295]
29, [-13964212268709884725, -140282, 1170802376844509184000, -
1198730801485834158745, 103905205295]
31, [-54171522182777606620, 20123, 56675100822926786560, -
165018145241264800615, 52782800815]
33, [-58909621176539933053170, 2545893, -131688610996976101621760,
13869368616464330113914, 27431905401506]
```

Results with Zeta(z), S(z,1), S(z,2), S(z,4).
The display is the integer relation that
make the vector [Zeta(z), S(z,1), S(z,2), S(z,4)] equal to 0.

```
3, [1, -28, 37, -7]
5, [-10, 240, -259, -1]
7, [-52, 1216, -1339, 19]
9, [34440, -800000, 869253, -373]
11, [-64144, 1487872, -1617613, 1453]
13, [-4952480, 114831360, -124738499, 2179]
15, [-9672256, 224247808, -243606007, 13687]
17, [-95521188480, 2214570557440, -2405615921073, 2986673]
19, [87161441536, -2020746723328, 2195077314937, -7708537]
21, [-511417392598515200, 11856658470320209920, -12879494224969023401,
969451783081]
23, [-809054768954368, 18757052482453504, -20375166492392041, 4472029801]
25, [-9402626177680619520, 217989612878618951680, -236794866356583343203,
1122603152483]
27, [-433559729330180096, 10051608326929645568, -10918727935370641697,
149780635937]
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29, [7059176577223900993810432‌0, -16365929038649703234797568‌00,
1777776435935195869731249349, -525747526375283909]
31, [-5201675717954103509155‌84, 1205951638122476107674419‌2, -
1309985153604688162335514‌3, 11231299844779783]
33, [-106522448875193678734089029222‌40, 246960650058691978789414796001280, -
268265139838691529135407355836517, 4960814599174753990757]
35, [-27329424568373031691454382‌08, 63360282530610722654381932544, -
68826167447973382833596090501, 368805384092328154‌1]
37, [-142419297361040107458048064649363456‌0,
330183568110664184029463962591218892‌80, -
358667427583286690274455997636231526‌61, 414484753382422115139942‌61]
39, [31536188043819957788458947444736, -73113203658542508855754566192332‌8,
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Results with Pi^z, S(z,1), S(z,2), S(z,4).
The display is the integer relation that
make the vector [Pi^z, S(z,1), S(z,2), S(z,4)] equal to 0.

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```
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791259765625]
```

References

[1] Berndt, Bruce, Ramanujan Notebooks (volumes I to V), Springer Verlag.

[2] Plouffe, Simon, *Identities inspired from Ramanujan Notebooks II*, 1998, unpublished notes.

[3] Finch, Steven R., Mathematical Constants, Encyclopedia of Mathematics and its Applications, 94 (2003).

[4] Flajolet, Philippe. Recent articles from INRIA.

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MC.