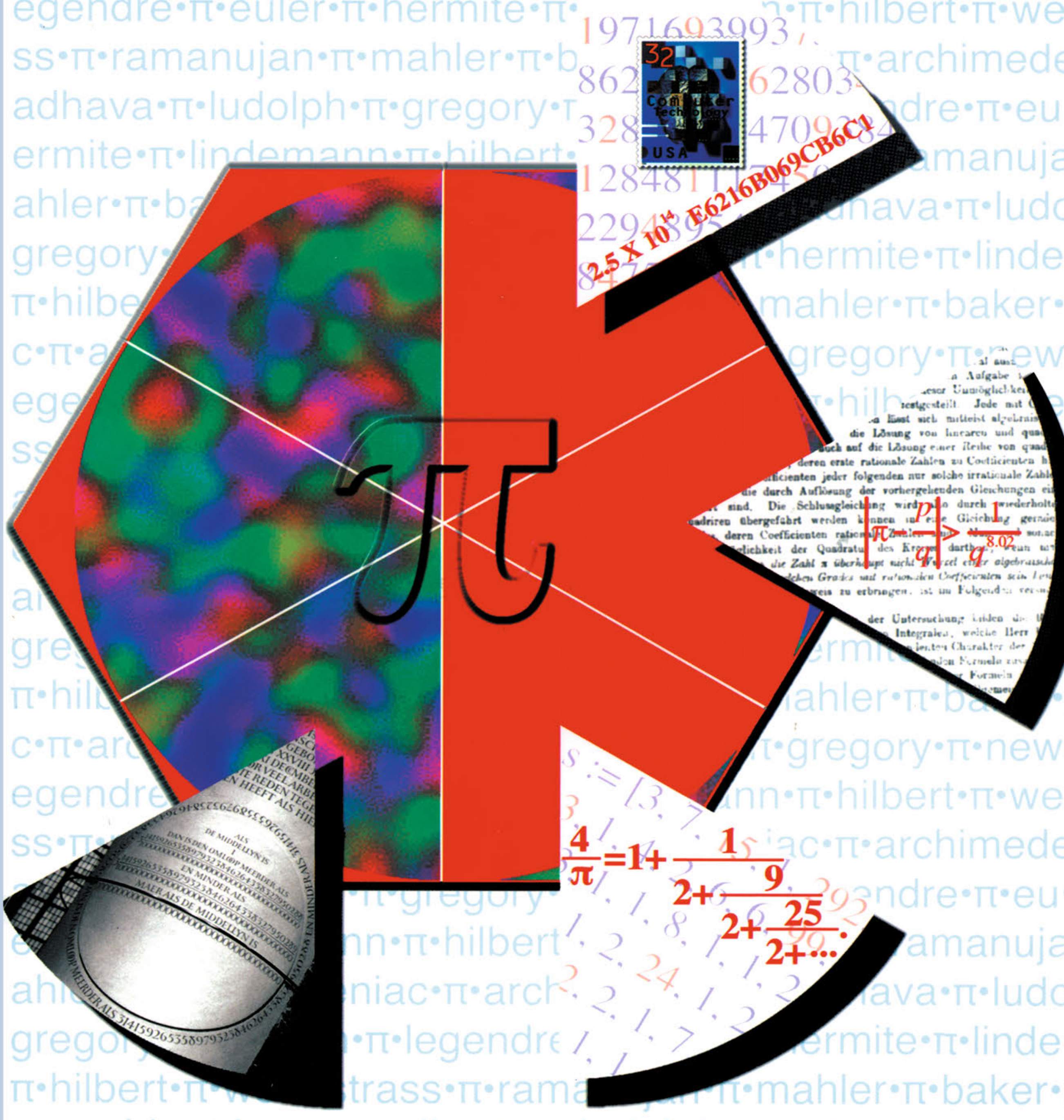
Third Edition

Lennart Berggren
Jonathan Borwein
Peter Borwein

Pi: A Source Book



Pi: A Source Book

Third Edition

Springer Science+Business Media, LLC

Lennart Berggren
Jonathan Borwein
Peter Borwein

Pi: A Source Book

Third Edition

Springer

Lennart Berggren
Department of Mathematics
Simon Fraser University
Burnaby, BC V5A 1S6
Canada

Jonathan Borwein
Faculty of Computer Science
Dalhousie University
6050 University Avenue
Halifax, Nova Scotia
Canada, B3H 1W5

Peter Borwein
Department of Mathematics
Simon Fraser University
Burnaby, BC V5A 1S6
Canada

Mathematics Subject Classification (2000): 01-00, 01A05, 01A75, 11-00, 11-03, 26-03, 65-03, 68-03

Library of Congress Cataloging-in-Publication Data
Pi, a source book / [edited by] Lennart Berggren, Jonathan Borwein, Peter Borwein.—3rd ed.
p. cm.
Includes bibliographical references and index.

1. Pi. I. Title: Pi. II. Berggren, Lennart. III. Borwein, Jonathan M. IV. Borwein, Peter B.
QA484.P5 2003
516.22–dc22 2003066023

Printed on acid-free paper.

ISBN 978-1-4419-1915-1 ISBN 978-1-4757-4217-6 (eBook)
DOI 10.1007/978-1-4757-4217-6

9 8 7 6 5 4 3 2 1 SPIN 10953327

springeronline.com

Preface to the Third Edition

Our aim in preparing this edition is to bring the material in the collection of papers in the second edition of this source book up to date. Moreover, several delightful pieces became available and are added.

This substantial supplement to the third edition serves as a stand-alone exposition of the recent history of the computation of digits of pi. It also includes a discussion of the thorny old question of normality of the distribution of the digits. Additional material of historical and cultural interest is included, the most notable being new translations of the two Latin pieces of Viète (translation of article 9 (Excerpt 1): Various Responses on Mathematical Matters: Book VII (1593) and (Excerpt 2): Defense for the New Cyclometry or "Anti-Axe"), and a thorough revision of the translation of Huygens's piece (article 12) published in the second edition.

We should like to thank Professor Marinus Taisbak of Copenhagen for grappling with Viète's idiosyncratic style to produce the new translation of his work. We should like to thank Karen Aardal for permission to use her photograph of Ludolph's new tombstone in the Pieterskerk in Leiden, the Smithsonian Institution for permission to reproduce a fine photo of ENIAC, and David and Gregory Chudnovsky for providing a "Walk on the digits of pi." We should also like to thank Irving Kaplansky for his gracious permission to include his "A Song about Pi." Finally, our thanks go to our colleagues whose continued interest in pi has encouraged our publisher to produce this third edition, as well as for the comments and corrections to earlier editions that some of them have sent us.

L. Berggren
J. Borwein
P. Borwein
Simon Fraser University
December 2003

Preface to the Second Edition

We are gratified that the first edition was sufficiently well received so as to merit a second. In addition to correcting a few minor infelicities, we have taken the opportunity to add an Appendix in which articles 9 and 12 by Viète and Huygens respectively are translated into English. While modern European languages are accessible to our full community—at least through colleagues—this is no longer true of Latin. Thus, following the suggestions of a reviewer of the first edition we have opted to provide a serviceable if fairly literal translation of three extended Latin excerpts. And in particular to make Viète's opinions and style known to a broader community.

We also record that in the last two years distributed computations have been made of the binary digits of π using an enhancement due to Fabrice Bellard of the identity made in article 70. In particular the binary digits of π starting at the 40 trillionth place are **0** 0000 1111 1001 1111. Details of such ongoing computations, led by Colin Percival, are to be found at www.cecm.sfu.ca/projects/pihex.

Corresponding details of a billion (2^{30}) digit computation on a single Pentium II PC, by Dominique Delande using Carey Bloodworth's desktop π program and taking under nine days, are lodged at www.cecm.sfu.ca/personal/jborwein/pi_cover.html. Here also are details of the computation of 2^{36} digits by Kanada et al. in April 1999.

We are grateful for the opportunity to thank Jen Chang for all her assistance with the cover design of the book. We also wish to thank Annie Marquis and Judith Borwein for their substantial help with the translated material.

Lennart Berggren
Jonathan Borwein
Peter Borwein
Simon Fraser University
July 5, 1999

Preface

Our intention in this collection is to provide, largely through original writings, an extended account of pi from the dawn of mathematical time to the present. The story of pi reflects the most seminal, the most serious, and sometimes the most whimsical aspects of mathematics. A surprising amount of the most important mathematics and a significant number of the most important mathematicians have contributed to its unfolding—directly or otherwise.

Pi is one of the few mathematical concepts whose mention evokes a response of recognition and interest in those not concerned professionally with the subject. It has been a part of human culture and the educated imagination for more than twenty-five hundred years. The computation of pi is virtually the only topic from the most ancient stratum of mathematics that is still of serious interest to modern mathematical research. To pursue this topic as it developed throughout the millennia is to follow a thread through the history of mathematics that winds through geometry, analysis and special functions, numerical analysis, algebra, and number theory. It offers a subject that provides mathematicians with examples of many current mathematical techniques as well as a palpable sense of their historical development.

Why a Source Book?

Few books serve wider potential audiences than does a source book. To our knowledge, there is at present no easy access to the bulk of the material we have collected.

Both professional and amateur mathematicians, whether budding, blooming, or beginning to wilt, can find in it a source of instruction, study, and inspiration. Pi yields wonderful examples of how the best of our mathematical progenitors have struggled with a problem worthy of their mettle. One of the great attractions of the literature on pi is that it allows for the inclusion of very modern, yet still highly accessible, mathematics. Indeed, we have included several prize winning twentieth century expository papers, and at least half of the collected material dates from the last half of the twentieth century.

While this book is definitely a collection of literature on, and not a history of, pi, we anticipate that historians of mathematics will find the collection useful. As authors we believe that one legitimate way of exhibiting the history of a concept is in gathering a coherent collection of original and secondary sources, and then to let the documents largely tell their own stories when placed in an appropriate historical and intellectual context.

Equally, teachers at every level will find herein ample supplementary resources: for many purposes from material for special topic courses to preparatory information for seminars and colloquia and guidance for student projects.

What Is Included?

We have chosen to include roughly 70 representatives of the accumulated literature on pi. In the Contents each piece is accorded a very brief but hopefully illuminating description. This is followed by an Introduction in which we highlight some further issues raised by the collection. Finally, since the pre-Newtonian study of pi presents many more problems for the reader than does the material after the time of Huygens, we have included an Appendix *On the Early History of Pi*. We have also provided two other Appendices. *Computational Chronology of Pi* offers a concise tabular accounting of computational records, and *Selected Formulae for Pi* presents a brief compendium of some of the most historically or computationally significant formulas for pi.

The pieces in the collection fall into three broad classes.

The core of the material is the accumulated mathematical research literature of four millennia. Although most of this comes from the last 150 years, there is much of interest from ancient Egypt, Greece, India, China, and medieval Islam. We trust that readers will appreciate the ingenuity of our earliest mathematicians in their valiant attempts to understand this number. The reader may well find this material as engrossing as the later work of Newton, Euler, or Ramanujan. Seminal papers by Lambert, Hermite, Lindemann, Hilbert and Mahler, to name but a few, are included in this category. Some of the more important papers on the number *e*, on zeta functions, and on Euler's constant have also been included as they are inextricably interwoven with the story of pi.

The second stratum of the literature comprises historical studies of pi, based on the above core sources, and of writings on the cultural meaning and significance of the number. Some of these are present here only in the bibliography such as Petr Beckmann's somewhat idiosyncratic monograph, *A History of Pi*. Other works on the subject are provided in extenso. These include Schepler's chronology of pi, some of Eves's anecdotes about the history of the number, and Engels' conjecture about how the ancient Egyptians may have computed pi.

Finally, the third level comprises the treatments of pi that are fanciful, satirical or whimsical, or just wrongheaded. Although these abound, we have exercised considerable restraint in this category and have included only a few representative pieces such as Keith's elaborate mnemonic for the digits of pi based on the poem "The Raven," a recent offering by Umberto Eco, and the notorious 1897 attempt by the state of Indiana[1] to legislate the value of pi.

Lennart Berggren
Jonathan Borwein
Peter Borwein
Simon Fraser University
September 6, 1996

[1] Oddly enough, the third page of this bill is apparently missing from the Indiana State Library and thus may now exist only in facsimile!

Some Points of Entry

For the reader looking for accessible points of introduction to the collection we make the following suggestions:

- As general introduction:

- As an introduction to irrationality and transcendence:

- As an introduction to elliptic integrals and related subjects:

- As an introduction to the computational issues:

- For a concise synopsis, the final "Pamphlet" makes an excellent self-contained entry. 721

Acknowledgments

We would like to thank, first of all, the publishers and authors who graciously granted permission to reproduce the works contained in this volume. Our principal debt, however, is to our technical editor, Chiara Veronesi, whose hard work and intelligent grasp of what needed to be done made the timely appearance of this book possible. We also wish to thank the publisher, Springer-Verlag, for its enthusiastic response to this project, as well as Ina Lindemann, our editor at Springer-Verlag, who saw the project through the press. Thanks, also, are due to David Fowler, who supplied copies of the Latin material contained herein from the work of John Wallis, as well as to David Bailey, Greg Fee and Yasumasa Kanada for helpful conversations about the project. Finally, we wish to thank the *Social Sciences Research Council of Canada* Small Grants Committee at Simon Fraser University for funding (Grant No. 410-86-0805) part of the cost of preparing this volume.

Lennart Berggren
Jonathan Borwein
Peter Borwein
Simon Fraser University
September 6, 1996

Contents

Introduction

As indicated in the Preface, the literature on pi naturally separates into three components (primarily research, history, and exegesis). It is equally profitable to consider three periods (before Newton, Newton to Hilbert and the Twentieth Century) and two major stories (pi's transcendence and pi's computation). With respect to computation, it is also instructive to consider the three significant methods which have been used: pre-calculus (Archimedes' method of exhaustion), calculus (Machin-like arctangent formulae), and elliptic and modular function methods (the Gaussian arithmetic–geometric mean and the series of Ramanujan type).

In the following introduction to the papers from the three periods, we have resisted the temptation to turn our *Source Book* into a "History of Pi and the Methods for Computing it." Accordingly, we have made no attempt to give detailed accounts of any of the papers selected, even when the language or style might seem to render such accounts desirable. Instead, we urge the reader seeking an account of 'what's going on' to either consult a reliable general history of mathematics, such as that of C. Boyer (in its most recent up-date by U. Merzbach) or V. Katz, or P. Beckmann's more specialized and personalized history of pi.

The Pre-Newtonian Period (Papers [1] to [15])

The primary sources for this period are, not surprisingly, more problematic than those of later periods, and for this reason we have included an additional appendix on this material. Our selections visit Egyptian, Greek, Chinese, and Medieval Arabo–European traditions. We commence with an excerpt from the Rhind Mathematical Papyrus from the Middle Kingdom of Egypt, circa 1650 B.C., representing some of what the ancient Egyptians knew about mathematics around 1800 B.C. By far the most significant ancient work—that of Archimedes of Syracuse (277–212 B.C.), which survives under the title *On the Measurement of the Circle* follows. It is hard to overemphasize how this work dominated the subject prior to the advent of the calculus.

We continue with a study of Liu Hui's third century A.D. commentary on the Chinese classic *Nine Chapters in the Mathematical Art* and of the lost work of the fifth century astronomer Zu Chongzhi. Marshall Clagett's translation of *Verba Filiorum*, the Latin version of the ninth century Arabic *Book of Knowledge of the Measurement of Plane and Spherical Figures* completes our first millennium extracts.

The next selection jumps forward 500 years and discusses the tombstone of Ludolph van Ceulen which recorded the culminating computation of pi by purely Archimedian techniques to 35 places as performed by Ludolph, using 2^{62}-gons, before 1615. We complete this period with excerpts from three great transitional thinkers: François Viète (1540–1603) whose work greatly influenced that of Fermat; John Wallis (1616–1703), to whom Newton indicated great indebtedness; and the Dutch polymath Christian Huygens (1629–1695), who correctly formalized Willebrord Snell's acceleration of Archimedes' method and was thus able to recapture Van Ceulen's computation with only 2^{30}-gons. In a part of this work, not reproduced here, Huygens vigorously attacks the validity of Gregory's argument for the transcendence of pi.

From Newton to Hilbert (Papers [16] to [24])

These comprise many of the most significant papers on pi. After visiting Newton's contribution we record a discussion of the arctangent series for pi variously credited to the Scot James Gregory, the German Leibniz, and to the earlier Indian Mādhava. In this period we move from the initial investigations of irrationality, by Euler and Lambert, to one of the landmarks of nineteenth century mathematics, the proof of the transcendence of pi.

The first paper is a selection from Euler and it demonstrates Euler's almost unparalleled—save for Ramanujan—ability to formally manipulate series, particularly series for pi. It is followed by an excerpt from Lambert and a discussion by Struik of Lambert's proof of the irrationality of pi, which is generally credited as the first proof of its irrationality. Euler had previously proved the irrationality of e. Lambert's proof of the irrationality of pi is based on a complicated continued fraction expansion. Much simpler proofs are to be found in [33], [48].

There is a selection from Shanks's self-financed publication that records his hand calculation of 607 digits of pi. (It is in fact correct only to 527 places, but this went unnoticed for almost a century.) The selection is included to illustrate the excesses that this side of the story has evoked. With a modern understanding of accelerating calculations this computation, even done by hand, could be considerably simplified. Neither Shanks's obsession with the computation of digits nor his error are in any way unique. Some of this is further discussed in [64].

The next paper is Hermite's 1873 proof of the transcendence of e. It is followed by Lindemann's 1882 proof of the transcendence of pi. These are, arguably, the most important papers in the collection. The proof of the transcendence of pi laid to rest the possibility of "squaring the circle," a problem that had been explicit since the late fifth c. B.C. Hermite's seminal paper on e in many ways anticipates Lindemann, and it is perhaps surprising that Hermite did not himself prove the transcendence of pi. The themes of Hermite's paper are explored and expanded in a number of later papers in this volume. See in particular Mahler [42]. The last two papers offer simplified proofs of the transcendence. One is due to Weierstrass in 1885 and the other to Hilbert in 1893. Hilbert's elegant proof is still probably the simplest proof we have.

The Twentieth Century (Papers [26] to [70])

The remaining forty-five papers are equally split between analytic and computational selections, with an interweaving of more diversionary selections.

On the analytic side we commence with the work of Ramanujan. His 1914 paper, [29], presents an extraordinary set of approximations to pi via "singular values" of elliptic integrals. The first half of this paper was well studied by Watson and others in the 1920s and 1930s, while the second half, which presents marvelous series for pi, was decoded and applied only more than 50 years later. (See [61], [62], [63].) Other highlights include: Watson's engaging and readable account of the early development of elliptic functions, [30]; several very influential papers by Kurt Mahler; Fields Medalist Alan Baker's 1964 paper on "algebraic independence of logarithms," [40]; and two papers on the irrationality of $\zeta(3)$ ([48], [49]) which was established only in 1976.

The computational selections include a report on the early computer calculation of pi—to 2037 places on ENIAC in 1949 by Reitwiesner, Metropolis and Von Neumann [34] and the 1961 computation of pi to 100,000 places by Shanks and Wrench [38], both by arctangent methods. Another highlight is the independent 1976 discovery of arithmetic–geometric mean methods for the computation of pi by Salamin and by Brent ([46], [47], see also [57]). Recent supercomputational applications of these and related methods by Kanada, by Bailey, and by the Chudnovsky brothers are included (see [60] to [64]). As of going to press, these scientists have now pushed the record for computation of pi beyond 17 billion digits. (See Appendix II.) One of the final papers in the volume, [70], describes a method of computing individual binary digits of pi and similar polylogarithmic constants and records the 1995 computation of the ten billionth hexadecimal digit of pi.

Extract from the Rhind Papyrus

Problem 50

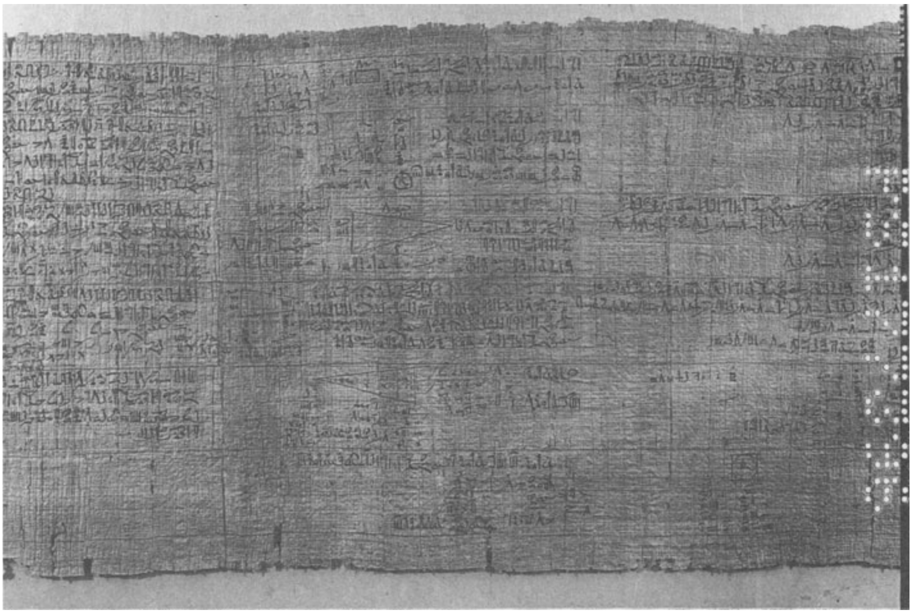
Example of a round field of diameter 9 khet. *What is its area?*

Take away $\frac{1}{9}$ of the diameter, namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. Therefore it contains 64 *setat* of land.

Do it thus:

	1	9
	$\frac{1}{9}$	1;
this taken away leaves 8		
	1	8
	2	16
	4	32
	\\8	64.

Its area is 64 *setat.*



<table>
<tr><td rowspan="6">Portions
of
Plates 78-82

Problems
56-60</td><td colspan="2">Plate 71 Problem 49</td><td rowspan="6">Portions
of
Plates 66-70

Problems
44-48</td></tr>
<tr><td colspan="2">Plate 72 Problem 50</td></tr>
<tr><td colspan="2">Plate 73 Problem 51</td></tr>
<tr><td colspan="2">Plate 74 Problem 52</td></tr>
<tr><td>Plate 75
Problem53</td><td>Plate 76
Problem 54</td></tr>
<tr><td colspan="2">Plate 77
Problem 55</td></tr>
</table>

HISTORIA MATHEMATICA 4 (1977), 137-140

QUADRATURE OF THE CIRCLE IN ANCIENT EGYPT

BY PROF. DR. HERMANN ENGELS,
TECHNICAL UNIVERSITY OF AACHEN

Summaries

The mathematicians of ancient Egypt approximated the area of a circle by a square with astonishing accuracy. The way to find this approximation is not handed down. In this paper a conjecture is given which seems to be much more simple than earlier attempts.

Die Mathematiker des alten Ägypten approximierten mit erstaunlicher Genauigkeit die Kreisfläche durch ein Quadrat. Es ist nicht überliefert, wie diese Approximation entstanden ist. In der vorliegenden Arbeit wird darüber eine Vermutung mitgeteilt, die wesentlich einfacher ist als bisherige Erklärungsversuche.

The mathematicians in ancient Egypt approximated the area of a circle by a square according to the rule: Shorten the diameter of the circle by $(1/9)$th to get the side of the square. This means a quadrature of the circle by $\pi r^2 = \pi(d/2)^2 \approx (8d/9)^2$, wherefrom the excellent approximation $\pi \approx (16/9)^2 = 3.1605$. The error is only 0.0189.

While M. Cantor [1907] still says that there is no way to understand this construction, there is an interesting conjecture of K. Vogel and O. Neugebauer [Becker and Hofmann 1951, 21] which uses a half-regular octogon that approximates the circle and nearly leads to the wanted solution [Vogel 1958]. But this conjecture seems to be too sophisticated. We here give a simpler one.

Cantor [1907] says that the Egyptian stone masons covered their designs and the walls in order to form a relief with orthogonal nets. Then the cutting points of the lattice lines and the contours of the design were carried over in fixed ratios. This technique seems to be the key to the comprehension of the Egyptian construction.

(1) If one attempts to draw a circle and a square intersecting this circle and having equal area, then nearly everybody intuitively gives a solution something like Figure 1.

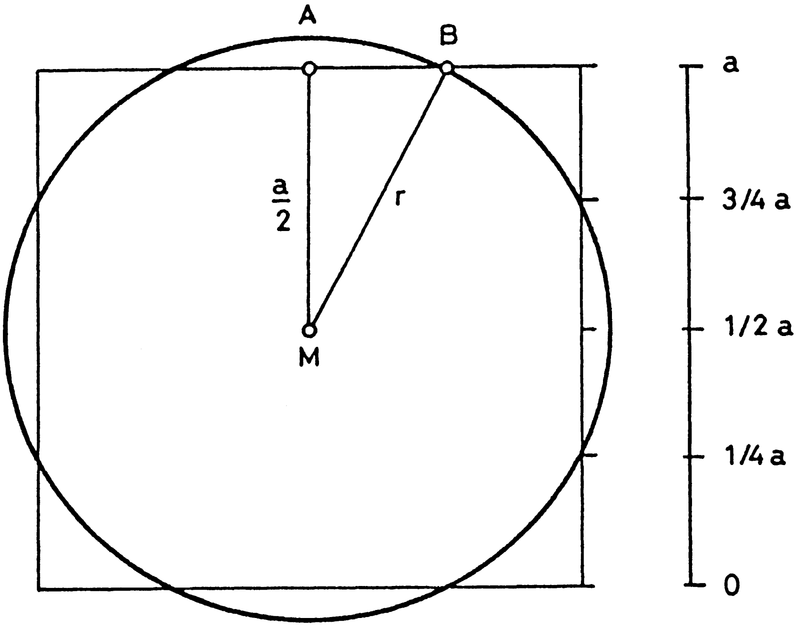


Figure 1

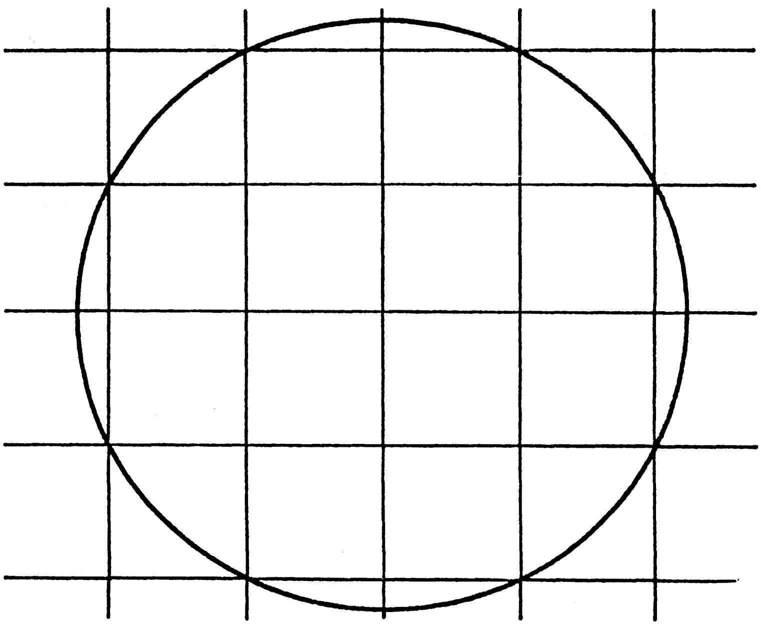


Figure 2

The points of intersection are found by taking a quarter and three quarters respectively of the length of the side. While this construction may arise from the feeling that this could be the exact solution (more sophisticated treatment of this problem only reinforces this feeling) and thus needs no geometrical knowledge, another way to Figure 1 is to use the above mentioned nets, as shown in Figure 2. We can assume that this technique was used for many centuries and hence that the probability of finding a picture like Figure 2 (and hence Figure 1) is nearly one.

Thus we have two very plausible and simple ways to realise this construction. The final Egyptian solution is to be found in a second step.

(2) The net-technique presents the possibility of getting a connection between $a/2$ and r in Figure 1. Assume, that a square is divided into 256 subsquares (Figure 3). Then it follows that $a = (8/9)d$, while $a = (2/\sqrt{5})d$ is correct. But the relative error ε is less than

$$.62 \text{ percent}$$

and is hardly noticeable even for a large diameter d.

This explanation of the Egyptian construction assumes two errors: an inaccurate determination of the square, and an

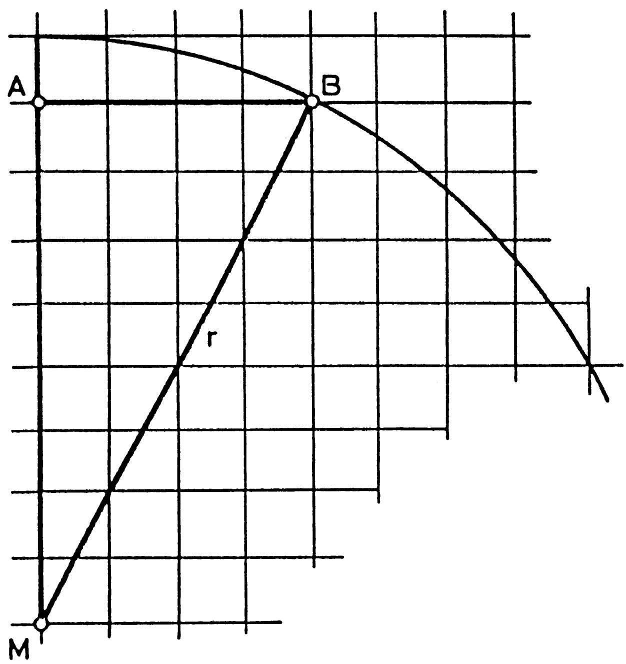


Figure 3

inaccurate calculation of *MB*. But both errors are not only very small but also diminish each other: The area of the circle passing through *B* (Figure 1) is $\pi(\sqrt{5}a/4)^2 = (0,99083a)^2 < a^2$, which is slightly too small. But with $8/9$ instead of $2/\sqrt{5}$, the result is the more accurate $\pi(9a/16)^2 = (0.99701a)^2$.

The fact that we have no record of the slightest hint of how to explain the Egyptian approximation $(16/9)^2$ may have many reasons, but if the construction was regarded as very simple, we would not expect to find any written explanation. The construction proposed is very simple and is based on the peculiarly Egyptian use of square nets.

REFERENCES

Becker, O and J E Hofmann 1951 *Geschichte der Mathematik* Bonn (Athenäum-Verlag) p. 21

Cantor, M 1907 *Vorlesungen über Geschichte der Mathematik,* vol. 1 Reprint 1965 New York (Johnson Reprint)

Vogel, K 1958 *Vorgriechische Mathematik, Part I: Vorgeschichte und Ägypten* Hannover

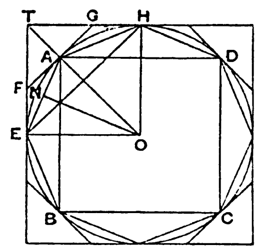
* * * * * * * * * * * * *

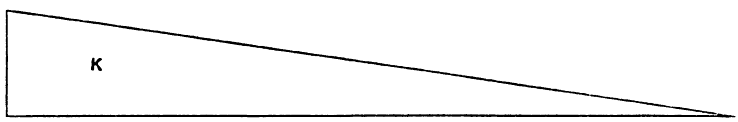
MEASUREMENT OF A CIRCLE.

Proposition 1.

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

Let $ABCD$ be the given circle, K the triangle described.





Then, if the circle is not equal to K, it must be either greater or less.

I. If possible, let the circle be greater than K.

Inscribe a square $ABCD$, bisect the arcs AB, BC, CD, DA, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over K.

Thus the area of the polygon is greater than K.

Let AE be any side of it, and ON the perpendicular on AE from the centre O.

Then ON is less than the radius of the circle and therefore less than one of the sides about the right angle in K. Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in K.

Therefore the area of the polygon is less than K; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than K.

II. If possible, let the circle be less than K.

Circumscribe a square, and let two adjacent sides, touching the circle in E, H, meet in T. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let A be the middle point of the arc EH, and FAG the tangent at A.

Then the angle TAG is a right angle.

Therefore
$$\begin{aligned} TG &> GA \\ &> GH. \end{aligned}$$

It follows that the triangle FTG is greater than half the area $TEAH$.

Similarly, if the arc AH be bisected and the tangent at the point of bisection be drawn, it will cut off from the area GAH more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of K over the area of the circle.

Thus the area of the polygon will be less than K.

Now, since the perpendicular from O on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle K; which is impossible.

Therefore the area of the circle is not less than K.

Since then the area of the circle is neither greater nor less than K, it is equal to it.

Proposition 2.

The area of a circle is to the square on its diameter as 11 *to* 14.

[The text of this proposition is not satisfactory, and Archimedes cannot have placed it before Proposition 3, as the approximation depends upon the result of that proposition.]

Proposition 3.

The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$.

[In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to $\sqrt{3}$ (one being less and the other greater than the real value) without any explanation as to how he arrived at them; and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated. These various approximations and the machinery of Greek arithmetic in general will be found discussed in the Introduction, Chapter IV.]

I. Let AB be the diameter of any circle, O its centre, AC the tangent at A; and let the angle AOC be one-third of a right angle.

Then $OA : AC\,[=\sqrt{3} : 1] > 265 : 153$ .... (1),

and $OC : CA\,[=2 : 1] = 306 : 153$ .... (2).

First, draw OD bisecting the angle AOC and meeting AC in D.

Now $CO : OA = CD : DA$, [Eucl. VI. 3]

so that $[CO + OA : OA = CA : DA$, or$]$

$CO + OA : CA = OA : AD$.

Therefore [by (1) and (2)]

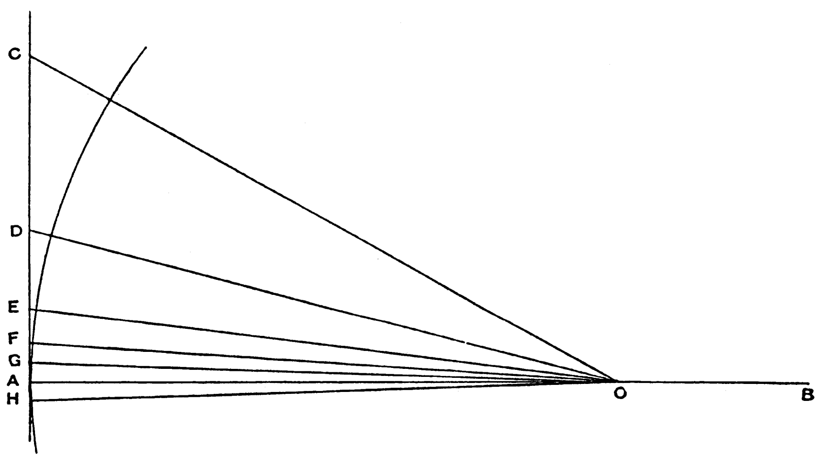
$OA : AD > 571 : 153$ .... (3).

Hence $OD^2 : AD^2\,[= (OA^2 + AD^2) : AD^2$

$> (571^2 + 153^2) : 153^2]$

$> 349450 : 23409$,

so that $OD : DA > 591\frac{1}{8} : 153$ .... (4).



Secondly, let OE bisect the angle AOD, meeting AD in E.

[Then $DO : OA = DE : EA$,

so that $DO + OA : DA = OA : AE$.]

Therefore $OA : AE\,[> (591\frac{1}{8} + 571) : 153$, by (3) and (4)]

$> 1162\frac{1}{8} : 153$.................... (5).

[It follows that

$$OE^2 : EA^2 > \{(1162\tfrac{1}{8})^2 + 153^2\} : 153^2$$
$$> (1350534\tfrac{33}{64} + 23409) : 23409$$
$$> 1373943\tfrac{33}{64} : 23409.]$$

Thus $$OE : EA > 1172\tfrac{1}{8} : 153 \qquad \text{.....................(6).}$$

Thirdly, let OF bisect the angle AOE and meet AE in F.

We thus obtain the result [corresponding to (3) and (5) above] that

$$OA : AF [> (1162\tfrac{1}{8} + 1172\tfrac{1}{8}) : 153]$$
$$> 2334\tfrac{1}{4} : 153 \qquad \text{.....................(7).}$$

[Therefore $$OF^2 : FA^2 > \{(2334\tfrac{1}{4})^2 + 153^2\} : 153^2$$
$$> 5472132\tfrac{1}{16} : 23409.]$$

Thus $$OF : FA > \dot{2}339\tfrac{1}{4} : 153 \qquad \text{.....................(8).}$$

Fourthly, let OG bisect the angle AOF, meeting AF in G.

We have then

$$OA : AG [> (2334\tfrac{1}{4} + 2339\tfrac{1}{4}) : 153, \text{ by means of (7) and (8)}]$$
$$> 4673\tfrac{1}{2} : 153.$$

Now the angle AOC, which is one-third of a right angle, has been bisected four times, and it follows that

$$\angle AOG = \tfrac{1}{48} \text{ (a right angle).}$$

Make the angle AOH on the other side of OA equal to the angle AOG, and let GA produced meet OH in H.

Then $$\angle GOH = \tfrac{1}{24} \text{ (a right angle).}$$

Thus GH is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since $$OA : AG > 4673\tfrac{1}{2} : 153,$$
while $$AB = 2OA, \quad GH = 2AG,$$
it follows that

$$AB : (\text{perimeter of polygon of 96 sides}) [> 4673\tfrac{1}{2} : 153 \times 96]$$
$$> 4673\tfrac{1}{2} : 14688.$$

But
$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$$
$$\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}}\right]$$
$$< 3\tfrac{1}{7}.$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{7}$ times the diameter AB.

II. Next let AB be the diameter of a circle, and let AC, meeting the circle in C, make the angle CAB equal to one-third of a right angle. Join BC.

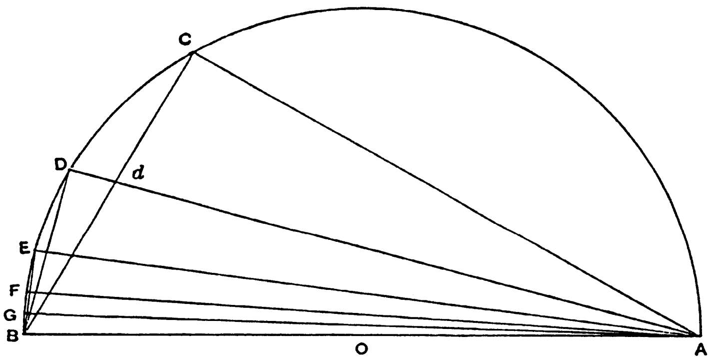
Then $\quad AC : CB \,[= \sqrt{3} : 1] < 1351 : 780.$

First, let AD bisect the angle BAC and meet BC in d and the circle in D. Join BD.

Then
$$\angle BAD = \angle dAC$$
$$= \angle dBD,$$

and the angles at D, C are both right angles.

It follows that the triangles ADB, $[ACd]$, BDd are similar.



Therefore
$$AD : DB = BD : Dd$$
$$[= AC : Cd]$$
$$= AB : Bd \qquad \text{[Eucl. VI. 3]}$$
$$= AB + AC : Bd + Cd$$
$$= AB + AC : BC$$

or
$$BA + AC : BC = AD : DB.$$

[But $AC : CB < 1351 : 780$, from above,

while $BA : BC = 2 : 1$

$= 1560 : 780.$]

Therefore $AD : DB < 2911 : 780$ ....(1).

[Hence $AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$

$< 9082321 : 608400.$]

Thus $AB : BD < 3013\frac{3}{4} : 780$ .... (2).

Secondly, let AE bisect the angle BAD, meeting the circle in E; and let BE be joined.

Then we prove, in the same way as before, that

$$\begin{aligned} AE : EB\,[&= BA + AD : BD \\ &< (3013\tfrac{3}{4} + 2911) : 780, \text{ by (1) and (2)}] \\ &< 5924\tfrac{3}{4} : 780 \\ &< 5924\tfrac{3}{4} \times \tfrac{4}{13} : 780 \times \tfrac{4}{13} \\ &< 1823 : 240 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(3). \end{aligned}$$

[Hence $AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$

$< 3380929 : 57600.$]

Therefore $AB : BE < 1838\frac{9}{11} : 240$ ....(4).

Thirdly, let AF bisect the angle BAE, meeting the circle in F.

Thus $$\begin{aligned} AF : FB\,[&= BA + AE : BE \\ &< 3661\tfrac{9}{11} : 240, \text{ by (3) and (4)}] \\ &< 3661\tfrac{9}{11} \times \tfrac{11}{40} : 240 \times \tfrac{11}{40} \\ &< 1007 : 66 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(5). \end{aligned}$$

[It follows that

$$\begin{aligned} AB^2 : BF^2 &< (1007^2 + 66^2) : 66^2 \\ &< 1018405 : 4356.] \end{aligned}$$

Therefore $AB : BF < 1009\frac{1}{6} : 66$ ....(6).

Fourthly, let the angle BAF be bisected by AG meeting the circle in G.

Then $$\begin{aligned} AG : GB\,[&= BA + AF : BF] \\ &< 2016\tfrac{1}{6} : 66, \text{ by (5) and (6)}. \end{aligned}$$

[And $\quad AB^2 : BG^2 < \{(2016\tfrac{1}{6})^2 + 66^2\} : 66^2$

$$< 4069284\tfrac{1}{36} : 4356.]$$

Therefore $\quad AB : BG < 2017\tfrac{1}{4} : 66,$

whence $\quad BG : AB > 66 : 2017\tfrac{1}{4}$ ....(7).

[Now the angle BAG which is the result of the fourth bisection of the angle BAC, or of one-third of a right angle, is equal to one-fortyeighth of a right angle.

Thus the angle subtended by BG at the centre is

$$\tfrac{1}{24} \text{ (a right angle).]}$$

Therefore BG is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

$$\text{(perimeter of polygon)} : AB\ [> 96 \times 66 : 2017\tfrac{1}{4}]$$

$$> 6336 : 2017\tfrac{1}{4}.$$

And $$\frac{6336}{2017\tfrac{1}{4}} > 3\tfrac{10}{71}.$$

Much more then is the circumference of the circle greater than $3\tfrac{10}{71}$ times the diameter.

Thus the ratio of the circumference to the diameter

$$< 3\tfrac{1}{7} \text{ but } > 3\tfrac{10}{71}.$$

Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. 88, No. 3, March 1981

ARCHIMEDES THE NUMERICAL ANALYST

G. M. PHILLIPS
The Mathematical Institute, University of St. Andrews, St. Andrews, Scotland

1. Introduction. Let p_N and P_N denote half the lengths of the perimeters of the inscribed and circumscribed regular N-gons of the unit circle. Thus $p_3 = 3\sqrt{3}/2$, $P_3 = 3\sqrt{3}$, $p_4 = 2\sqrt{2}$, and $P_4 = 4$. It is geometrically obvious that the sequences $\{p_N\}$ and $\{P_N\}$ are respectively monotonic increasing and monotonic decreasing, with common limit π. This is the basis of Archimedes' method for approximating to π. (See, for example, Heath [2].) Using elementary geometrical reasoning, Archimedes obtained the following recurrence relation, in which the two sequences remain entwined:

$$1/P_{2N} = \tfrac{1}{2}(1/P_N + 1/p_N) \tag{1a}$$

$$p_{2N} = \sqrt{(P_{2N}p_N)}. \tag{1b}$$

We note that these involve the use of the harmonic and geometric means. Beginning with $N = 3$ and applying the recurrence formula five times, Archimedes established the inequalities

$$3\tfrac{10}{71} < p_{96} < \pi < P_{96} < 3\tfrac{1}{7}. \tag{2}$$

His skill in obtaining rational numbers $3\frac{10}{71}$ and (the very familiar) $3\frac{1}{7}$ so close to the irrational numbers p_{96} and P_{96} can be more readily appreciated if we display all four numbers to four decimal places:

$$p_{96} = 3.1410, \qquad 3\tfrac{10}{71} = 3.1408$$

$$P_{96} = 3.1427, \qquad 3\tfrac{1}{7} = 3.1429.$$

2. Stability of the Recurrence Relation. In any thorough study of a recurrence relation we need to consider the question of *numerical stability*, that is, whether rounding errors are magnified by the recurrence relation. As an example, consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{2}{\pi}\int_0^\pi e^{\cos\theta}\cos n\theta\, d\theta. \tag{3}$$

(The a_n are the Chebyshev coefficients for e^x; see Clenshaw [1].) It is easily verified, on integrating (3) by parts, that this sequence satisfies the recurrence relation

$$a_{n+1} = a_{n-1} - 2na_n. \tag{4}$$

In principle, given a_0 and a_1, we may then use (4) to compute the value of any a_n. In practice, the recurrence relation (4) does not provide a satisfactory method of computing this sequence, because it is numerically unstable. To illustrate this, suppose we begin with $a_0 = 2.5321$ and $a_1 = 1.1303$, which are correct to 4 decimal places. Using (4) and rounding each a_n to 4 decimal places gives $a_2 = 0.2715$, $a_3 = 0.0443$, $a_4 = 0.0057$, $a_5 = -0.0013$, and $a_6 = 0.0187$. The true values, to 4 decimal places, are a_2 and a_3 as above and $a_4 = 0.0055$, $a_5 = 0.0005$, and $a_6 = 0.0000$. We can now see, on re-examining (4), that the error in a_{n+1} is approximately $(-2n)$ times the error in a_n, which shows why (4) is numerically unstable.

To examine the stability of (1) let us assume that, due to the effect of rounding errors, we actually compute numbers $\tilde{P}_{2N}$ and $\tilde{p}_N$ instead of P_{2N} and p_N, where

---

George M. Phillips did research in the theory of numbers under E. M. Wright at Aberdeen. He taught at Southampton (1963–1967) before moving to St. Andrews, where he is now Reader in Numerical Analysis. He has made research visits to the University of Texas at Austin and, on several occasions, to the University of Calgary. His main research interests are in numerical analysis and approximation theory. He is coauthor (with P. J. Taylor) of *Computers* (1969) and *Theory and Applications of Numerical Analysis* (1973).—*Editors*

$$\tilde{P}_{2N} = P_{2N}(1 + \delta), \tag{5a}$$

$$\tilde{p}_N = p_N(1 + \epsilon). \tag{5b}$$

We call δ and ϵ the relative errors in P_{2N} and p_N, respectively. To find the relative error in p_{2N}, we have

$$\tilde{p}_{2N} = \sqrt{(\tilde{P}_{2N}\tilde{p}_N)}. \tag{6}$$

Thus $\tilde{p}_{2N}$ (neglecting the rounding error incurred in evaluating the right side of (6)) is the number we would actually obtain, instead of p_{2N}. Substituting (5) into (6), we have

$$\frac{\tilde{p}_{2N} - p_{2N}}{p_{2N}} = (1 + \delta)^{1/2}(1 + \epsilon)^{1/2} - 1 \tag{7}$$

as the relative error in p_{2N}. Using binomial expansions in (7) we see that, for small values of δ and ϵ,

$$\frac{\tilde{p}_{2N} - p_{2N}}{p_{2N}} \simeq \tfrac{1}{2}(\delta + \epsilon). \tag{8}$$

An analysis of (1a) produces a result similar to (8), showing that rounding errors are not magnified by the recurrence relation, which is thus stable.

3. Rate of Convergence. We have a great advantage over Archimedes in being able to express P_N and p_N in terms of circular functions. It is easily verified that

$$p_N = N\sin(\pi/N) \tag{9}$$

and

$$P_N = N\tan(\pi/N). \tag{10}$$

From (9) and (10) we can justify that (1a) and (1b) are indeed correct and, further, from our familiarity with the Maclaurin series for $\sin\theta$ and $\tan\theta$, we can establish the rate of convergence of the sequences $\{p_N\}$ and $\{P_N\}$. Considering p_N first, we have from (9)

$$p_N = N\left[\left(\frac{\pi}{N}\right) - \frac{1}{3!}\left(\frac{\pi}{N}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{N}\right)^5 - \cdots\right] \tag{11}$$

so that, for large N,

$$\pi - p_N \simeq \frac{1}{6}\pi^3 \cdot \frac{1}{N^2}. \tag{12}$$

We could give a more precise form of (12) by writing down the first two terms of the series (11) plus a remainder term. We can now see from (8) that the error in p_{2N} is approximately one-quarter of the error in p_N. More precisely, we have

$$\lim_{N\to\infty} \frac{\pi - p_{2N}}{\pi - p_N} = \frac{1}{4}. \tag{13}$$

By considering the series for $\tan(\pi/N)$, we see that the errors in the sequence $\{P_N\}$ decrease at the same rate. An inspection of the values of p_N and P_N in Table 1 shows that one might guess this result. (An explanation of the last column of this table follows later.) Given the superb numerical skills of Archimedes, one is sorely tempted to conjecture that he must have been aware of the rate of convergence of his sequences.

4. "Faster" Convergence. We have just seen that the convergence of the sequences $\{P_N\}$ and $\{p_N\}$ is very slow, and it is interesting to consider how to improve on this. First we expand (10) in a Maclaurin series to give

$$P_N = N\left[\left(\frac{\pi}{N}\right) + \frac{1}{3}\left(\frac{\pi}{N}\right)^3 + \frac{2}{15}\left(\frac{\pi}{N}\right)^5 + \cdots\right]. \tag{14}$$

TABLE 1. The first few values of p_N, P_N, and u_N.

N	p_N	P_N	u_N
3	2.598076	5.196152	3.464102
6	3.000000	3.464102	3.154701
12	3.105829	3.215390	3.142349
24	3.132629	3.159660	3.141639
48	3.139350	3.146086	3.141596
96	3.141032	3.142715	3.141593
192	3.141452	3.141873	3.141593

We may now eliminate the terms in $1/N^2$ between (11) and (14) by writing

$$u_N = \frac{1}{3}(2p_N + P_N) = \pi + \frac{1}{20}\frac{\pi^5}{N^4} + \cdots, \tag{15}$$

so that

$$u_N - \pi \simeq \frac{1}{20}\frac{\pi^5}{N^4} \tag{16}$$

and u_N converges to π faster than p_N or P_N. The first few values of u_N are given in Table 1. If we re-calculate the numbers in Table 1 to greater accuracy, we find that u_{96} gives an approximation to π which is more accurate, by a factor greater than 1000, than either of Archimedes' approximations p_{96} and P_{96}.

The technique of eliminating the term in $1/N^2$ could also have been done between p_N and p_{2N} (or, equally, between P_N and P_{2N}). Thus, similarly to (16), we can show that, say,

$$v_N - \pi = \tfrac{1}{3}(4p_{2N} - p_N) - \pi$$

also behaves like a multiple of $1/N^4$ for large N. This process is called *extrapolation to the limit.* (See, for example, Phillips and Taylor [3].) This process can be repeated; that is, we can eliminate the term in $1/N^4$ between v_N and v_{2N}. In Table 2 we show the dramatic effect of repeated extrapolation to the limit. Note that the last two numbers in the final column of Table 2 give π correct to 9 decimal places, although it is only the effect of rounding error which has prevented us from achieving agreement to twice as many places of decimals. If we re-calculate the numbers p_N in Table 2 to 20 decimal places and carry out five extrapolations (rather than three given in the table), we obtain an approximation which differs from π by less than 10^{-18}. It is remarkable that such accuracy can be extracted from Archimedes' raw material.

TABLE 2. The effect of repeated extrapolation to the limit.

N	Extrapolated Values p_N	v_N	Repeated Extrapolation	
3	2.598 076 211			
6	3.000 000 000	3.133 974 596		
12	3.105 828 541	3.141 104 721	3.141 580 063	
24	3.132 628 613	3.141 561 970	3.141 592 454	3.141 592 650
48	3.139 350 203	3.141 590 733	3.141 592 651	3.141 592 654
96	3.141 031 951	3.141 592 534	3.141 592 654	3.141 592 654

5. Analysis of Convergence. In this final section we analyze the behavior of the recurrence relation (1) with arbitrary positive starting values. In divorcing (1) from its geometrical context, we shall change the notation and rewrite (1) in the form

$$1/Q_{N+1} = \tfrac{1}{2}(1/Q_N + 1/q_N) \tag{17a}$$

$$q_{N+1} = \sqrt{(Q_{N+1}q_N)}, \tag{17b}$$

beginning with arbitrary q_0, $Q_0 > 0$. We examine separately the two cases $0 < q_0 < Q_0$ and $0 < Q_0 < q_0$.

Case 1. For $0 < q_0 < Q_0$ we shall write

$$\frac{q_0}{Q_0} = \cos\theta, \qquad \alpha = \frac{q_0 Q_0}{(Q_0^2 - q_0^2)^{1/2}}, \tag{18}$$

so that

$$Q_0 = \alpha \tan\theta, \qquad q_0 = \alpha \sin\theta. \tag{19}$$

Substituting (15) into (13), we easily obtain

$$Q_1 = 2\alpha \tan\tfrac{1}{2}\theta, \qquad q_1 = 2\alpha \sin\tfrac{1}{2}\theta. \tag{20}$$

It follows that

$$Q_N = 2^N \alpha \tan(\theta/2^N), \qquad q_N = 2^N \alpha \sin(\theta/2^N), \tag{21}$$

and hence the sequences $\{Q_N\}$ and $\{q_N\}$ converge to the common limit

$$\alpha\theta = \frac{q_0 Q_0}{(Q_0^2 - q_0^2)^{1/2}} \cos^{-1}(q_0/Q_0). \tag{22}$$

The "Archimedes case" corresponds to $q_0 = 3\sqrt{3}/2$, $Q_0 = 3\sqrt{3}$.

Case 2. For $0 < Q_0 < q_0$ we write

$$\frac{q_0}{Q_0} = \cosh\theta, \qquad \alpha = \frac{q_0 Q_0}{(q_0^2 - Q_0^2)^{1/2}}, \tag{23}$$

so that

$$Q_0 = \alpha \tanh\theta, \qquad q_0 = \alpha \sinh\theta. \tag{24}$$

Substituting (24) into (17), we obtain

$$Q_1 = 2\alpha \tanh\tfrac{1}{2}\theta, \qquad q_1 = 2\alpha \sinh\tfrac{1}{2}\theta.$$

It follows that

$$Q_N = 2^N \alpha \tanh(\theta/2^N), \qquad q_N = 2^N \alpha \sinh(\theta/2^N),$$

and hence the sequences $\{Q_N\}$ and $\{q_N\}$ again converge to a common limit which, in this case, is

$$\alpha\theta = \frac{q_0 Q_0}{(q_0^2 - Q_0^2)^{1/2}} \cosh^{-1}(q_0/Q_0). \tag{25}$$

As an amusing application of this last result, let us choose

$$Q_0 = 2t, \qquad q_0 = t^2 + 1$$

for any positive $t \neq 1$. Then from (25) the sequences $\{Q_N\}$ and $\{q_N\}$ have common limit

$$\frac{2t(t^2 + 1)}{(t^2 - 1)} \log t.$$

This gives a simple method for evaluating $\log t$ and repeated extrapolation may be used to accelerate convergence. However, this is not proposed as a practical algorithm for computing $\log t$.

Acknowledgments. I am indebted to my colleagues J. M. Howie and J. J. O'Connor for interesting discussions on this topic. I also wish to thank John W. Wrench, Jr., for his valuable comments and for the trouble he took to re-work independently the calculations embodied in Tables 1 and 2.

References

1. C. W. Clenshaw, Chebyshev Series for Mathematical Functions, Mathematical Tables, vol. 5, National Physical Laboratory, H.M.S.O., London, 1962.

2. T. L. Heath, The Works of Archimedes, Cambridge University Press, 1897.

3. G. M. Phillips and P. J. Taylor, Theory and Applications of Numerical Analysis, Academic Press, 1973.

HISTORIA MATHEMATICA 13 (1986), 325–340

Circle Measurements in Ancient China

LAM LAY-YONG

Department of Mathematics, National University of Singapore, Singapore 0511, Republic of Singapore

AND

ANG TIAN-SE

Department of Chinese Studies, University of Malaya, Kuala Lumpur, Malaysia

This paper discusses the method of Liu Hui (3rd century) for evaluating the ratio of the circumference of a circle to its diameter, now known as π. A translation of Liu's method is given in the Appendix. Also examined are the values for π given by Zu Chongzhi (429–500) and unsurpassed for a millenium. Although the method used by Zu is not extant, it is almost certain that he applied Liu's method. With the help of an electronic computer, a table of computations adhering to Liu's method is given to show the derivation of Zu's results. The paper concludes with a survey of circle measurements in China. © 1986 Academic Press, Inc.

本文探讨公元263年刘徽如何求得现称为π的圆周率值及结论•刘徽的方法原文，以英文译出，作为附录•祖冲之（429－500）继后所得的圆周率值，其精密性，要等到一千年后才超越它•虽然演算的原文已经遗失，但他采用刘徽的方法，则几乎可以肯定•按照刘徽的方法，我们可以通过电脑得出一个计算表，说明祖率的来源•我们也对中国历代有关圆量法，加以讨论，作为本文的结束• © 1986 Academic Press, Inc.

Dieser Aufsatz erörtert Liu Huis (3. Jahrhundert n. Chr.) Methode, das Verhältnis des Umfanges eines Kreises zu seinem Durchmesser zu berechnen, das heute als π bekannt ist. Der Anhang enthält eine Übersetzung von Lius Methode. Ebenso werden die Werte von Zu Chongzhi (425–500) für π geprüft, die tausend Jahre lang nicht überboten wurden. Obwohl die von Zu verwandte Methode nicht mehr existiert, ist es fast sicher, daß er sich Lius Methode bediente. Mit Hilfe eines elektronischen Rechners wird eine Berechnungstabelle beigefügt, die sich an Lius Methode anlehnt, um die Ableitung von Zus Ergebnissen zu zeigen. Der Aufsatz schließt mit einem Uberblick über die Kreismessungen in China. © 1986 Academic Press, Inc.

AMS 1980 subject classifications: 01A25, 28-03.
KEY WORDS: Liu Hui, Zu Chongzhi, computations of π, decimal fractions, polygons, ratio of the circle circumference to the diameter.

China, like any other early civilization, had its fair share of men who tried to find as accurately as possible the area or the circumference of a circle. Two men stand out prominently among these: Liu Hui [a][1] of the 3rd century and Zu

[1] Lowercase letters in brackets indicate a Glossary listing (at the end of the paper).

0315-0860/86 $3.00

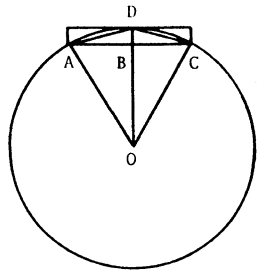


FIGURE 1

Chongzhi [b] of the 5th century. This paper first discusses the contributions of Liu and Zu to calculation of the ratio of the circumference to the diameter, now known as π, and their significance, and, second, offers a general survey of the evaluation of this ratio in China.

LIU'S METHOD

Problems 31 and 32 in chapter 1, entitled *fang tian* [c] (mensuration of fields), of the *Jiu zhang suanshu* [d] (*Nine chapters of the mathematical art*) [1] assume the area of a circle is half the circumference times half the diameter. In his commentary on problem 32, Liu Hui explained the derivation of this formula, discussed why the ratio of the circumference to the diameter was generally taken as 3, and then derived a more precise value for the ratio. Liu's commentary—divided into three sections for ease of reference in this paper—is summarized below and offered in translation in the Appendix [2].

Section 1. Start with a regular hexagon inscribed in a circle of radius 1 *chi* [e]. The product of one side of the hexagon, the radius, and 3 ($= \frac{1}{2} \times 6$) gives the area of an inscribed dodecagon. Repeat this process by taking the product of one side of the dodecagon, the radius, and 6 ($= \frac{1}{2} \times 12$) to obtain the area of an inscribed polygon of 24 sides. According to Liu's principle of exhaustion, if the process is repeated long enough, eventually a polygon will be reached whose sides are so short that it will coincide with the circle. This explains why the area of the circle is the product of half the circumference and the radius. The value of 3 to 1 for the ratio of the circumference to the diameter is imprecise as this is in fact the ratio of the perimeter of the hexagon to the diameter. However, this inaccuracy was passed down from generation to generation, Liu explains, because of an unwillingness to strive for accuracy.

Section 2. This section summarizes the stages of Liu's computation of a more precise value of the ratio. Let the side of an n-sided polygon $= a_n$ and its area $= A_n$. In Fig. 1, $n = 6$,

$$OC = \text{radius} = r = 10, \qquad BC = \tfrac{1}{2}a_6 = 5,$$

$$OB = b_6 = \sqrt{r^2 - \tfrac{1}{4}a_6^2} = 8.660254, \qquad DB = c_6 = r - b_6 = 1.339746,$$

$$DC^2 = a_{12}^2 = c_6^2 + \tfrac{1}{4}a_6^2 = 26.7949193445.$$

The same process is repeated for a dodecagon of side a_{12} as follows.

$$b_{12} = \sqrt{r^2 - \tfrac{1}{4}a_{12}^2} = 9.659258, \qquad c_{12} = r - b_{12} = 0.34742,$$
$$a_{24}^2 = c_{12}^2 + \tfrac{1}{4}a_{12}^2 = 6.8148349466.$$

The process continues:

$$b_{24} = 9.914448, \qquad c_{24} = 0.085552, \qquad a_{48}^2 = 1.7110278813,$$
$$a_{48} = 1.30806, \qquad A_{96} = 24ra_{48} = 313\tfrac{584}{625}.$$
$$b_{48} = 9.978589, \qquad c_{48} = 0.021411, \qquad a_{96}^2 = 0.4282154012,$$
$$a_{96} = 0.65438, \qquad A_{192} = 48ra_{96} = 314\tfrac{64}{625}, \qquad A_{192} - A_{96} = \tfrac{105}{625},$$
$$314\tfrac{64}{625} = A_{192} < A < A_{96} + 2(A_{192} - A_{96}) = 314\tfrac{169}{625},$$

where A = the area of the circle.

Hence $A \approx 314$ (to the nearest integer) and $S : A : s = 200 : 157 : 100$, where S = the area of the square circumscribing the circle and s = the area of the square inscribed in the circle. Therefore circumference/diameter = $\frac{157}{50}$.

Section 3. $A = A_{192} + \frac{36}{625} = 314\frac{4}{25}$. There is no clear explanation of how $\frac{36}{625}$ is derived apart from the fact that it bears some relation to the residual area $A_{192} - A_{96} = \frac{105}{625}$. It follows that $S : A : s = 5000 : 3927 : 2500$ and circumference/diameter $= \frac{3927}{1250}$. It is stated that this ratio can be verified by computing a_{1536} and hence deriving A_{3072}.

CONCLUSIONS FROM LIU'S METHOD

1. A problem in the Ahmes Papyrus (from ancient Egypt) gave the numerical area of a circle, and the excavations at Susa in 1936 revealed the old Babylonians' calculation of the circumference of a circle in relation to its inscribed hexagon [Neugebauer 1952, 47]. In instances such as these, historians calculate the value of π from the area or other recorded figures. Archimedes [Heath 1897, 93–98] and Liu Hui, however, are the only men from ancient history whose methods for obtaining the ratio of the circumference of a circle to its diameter are known to modern historians.

2. Both Archimedes' and Liu's methods employed regular polygons inscribed in the circle. Archimedes also had polygons circumscribing the circle. Both men assumed the principle of exhaustion, holding that eventually a polygon will be reached whose sides are so short that it will coincide with the circle. This principle was first stated by the Greek philosopher, Antiphon, in the 5th century B.C. when he began with a square inscribed in a circle [Heath 1921 I, 222].

3. The genius of Archimedes is displayed in his method. Without knowledge of subjects such as decimal notation and trigonometry, he was able to devise a method with inscribed and circumscribed polygons of 96 sides which gave the

values of π as greater than $3\frac{10}{71}$ but less than $3\frac{1}{7}$. His method of calculating the approximate values of irrational square roots is still in the realm of speculation. Compared with Archimedes' method, then, Liu's method is simple and elegant. He used only inscribed polygons, and each stage of his derivation is clear.

4. Archimedes' method is solely concerned with the evaluation of the perimeters of the inscribed and circumscribed polygons from which the circumference of the circle is deduced. It does not draw any conclusions about the area of the circle in relation to the ratio π [3]. By showing that the area of a circle is the product of half the circumference and the radius, Liu's method proved that the ratio of the area of a circle to the square of its radius is identical to the ratio of the circumference to the diameter, or, in other words, π.

5. One of the reasons for the simplicity of Liu's method is that he inherited a tradition using a decimal number system. The existence of a word–numeral decimal system in China can be traced to the oracle bone characters of the Shang dynasty [Needham 1959, 12–13]. Since the Warring States period (480 to 221 B.C.) counting rods, manifesting the place value of a decimal number system, were used for computation [Ang 1977, 97–98]. Liu's work serves as a fine example of the depth of the ancient Chinese understanding and handling of large numbers and decimal fractions as early as the 3rd century. In his text each numeral has a place name and on the counting board [4] each rod numeral has a place position relative to the other rod numerals. In the text, integral places of order $10, 10^2, 10^3, \ldots, 10^{11}$ are called *shi* [f], *bai* [g], *qian* [h], *wan* [i], *shi wan* [j], *bai wan* [k], *qian wan* [l], *yi* [m], *shi yi* [n], *bai yi* [o], and *qian yi* [p], respectively. For a decimal fraction, the names of the first five decimal places are given as *fen* [q], *li* [r], *hao* [s], *miao* [t], and *hu* [u], respectively. In calculating a value to more than five decimal places, Liu remarked that the "minute numbers" had no place names and so the numeral in the sixth decimal place had to be converted to a fraction. For example, $b_6 = 8$ *cun* [v] [5] 6 *fen* 6 *li* [0 *hao*] 2 *miao* $5\frac{2}{5}$ *hu* (see the Appendix, Section 2). In this fashion, the lengths of all b_n's are truncated at the sixth decimal place. This ensures a certain degree of accuracy for the values of subsequent a_n^2's which are truncated at the tenth decimal place. Without the "modern" notational decimal point, each set of a_n^2 numerals is considered by Liu in square *hu* [u] units. For computing the area A_{2n}, the value of a_n is obtained from a_n^2 by the square root method and is truncated at the fifth decimal place. For example, $a_{96}^2 = 4282154012$ square *hu* and $a_{96} = 6$ *fen* 5 *li* 4 *hao* 3 *miao* 8 *hu* (see the Appendix, Section 2). The ancient Chinese had devised a method for computing the square root of any number to any degree of accuracy. The earliest record of this method is found in the *Jiu zhang suanshu* [Qian 1963, 150; Wang & Needham 1955, 350–365]. Liu was therefore well aware that a number of $2n$ digits would give a square root with a number of n digits or *vice versa*.

6. Liu's method, written for computation by the counting rod system, illustrates the practicality and immense potential of this computational device. With this system, abstract ideas had to be transformed into concrete ones for the handling of counting rods on the counting board. Like the modern computer, the counting rod system encouraged algorithms such as the one devised by Liu. The first

decimal place was called *fen* [q] and, on the counting board, digits of the first and subsequent decimal places were designated to specified positions. The concept of decimal fractions existed, as it was merely an extension of the integral number system. In the same way the concept of zero in a notational form existed. If there were zero or no digit in a particular place value, the designated position on the counting board for that place value was left blank. P. Beckmann remarks that the Chinese discovery of the equivalence of the digit zero made them "far better equipped for numerical calculations than their western contemporaries" [Beckmann 1970, 27].

7. Both Archimedes and Liu had discovered methods that would enable men of later generations to calculate π to any desired degree of accuracy. With these methods, the number of decimal places to which π could be calculated was merely a matter of computational ability and perseverance. Thus, in 1593 François Viète and Adriaen van Roomen used Archimedes' method to calculate π to 9 and 15 decimal places, respectively. A few years later Ludolph van Ceulen computed π to 35 decimal places [Beckmann 1970, 98–99]. There is strong evidence that Zu Chongzhi used Liu's method to obtain his estimates of π. A mathematician, familiar with the counting rod system and as talented as Zu, would have no difficulty in applying Liu's method to an enlarged number or extending the number of decimal places in the computation and thereby calculating π to a higher degree of accuracy than Liu.

ZU CHONGZHI'S VALUES FOR π

Zu Chongzhi's values for π were not surpassed until a millenium later when al-Kashi evaluated π correctly to 16 decimal places [Youschkevitch & Rosenfeld 1973, 258]. It is interesting to note that Zu's fractional value of π in the form $\frac{355}{113}$ was also given by the Indians in the 15th century [Gupta, 1975, 3] and by Adriaan Anthoniszoon in the 16th century [Beckmann 1970, 98].

As an addendum to Liu Hui's commentary, Li Chunfeng [w] stated that Zu considered Liu's ratios inaccurate and therefore proposed to compute further [Qian 1963, 106]. Zu's concern for a better approximation for π was essentially due to his desire for the compilation of an astronomical system for the empire. There is no doubt that Zu incorporated the method for approximating π into his mathematical text, *Zhui shu* [x] (*Method of mathematical composition*), which is not extant. There remain now but quotations in the official histories. For example, the *Sui shu* [y] (*Standard history of the Sui dynasty*) [387–388] relates:

> In ancient mathematics, the ratio of the circumference to the diameter was taken to be 3 to 1 but this was only a rough estimate. Though various efforts had been made by Liu Xin [z], Zhang Heng [aa], Liu Hui, Wang Fan [ab] and Pi Yanzong [ac], the results obtained so far still lacked precision. Towards the end of the [Liu] Song period (420–479), Zu Chongzhi, a historian of Nanxu [ad] district, found further approximations. He took 100000000 [units] as 1 *zhang* [ae] along the diameter of a circle [of length 2 *zhang*] and found an upper value of 3 *zhang* 1 *chi* 4 *cun* [5] 1 *fen* 5 *li* 9 *hao* 2 *miao* 7 *hu* and a lower value of 3 *zhang* 1 *chi* 4 *cun* 1 *fen* 5 *li* 9 *hao* 2 *miao* 6 *hu* for the circumference [saying that] the true value must lie between the upper and lower limits. His "very close" ratio (*mi lu* [af]) was 355 to 113 and the "approximate" ratio (*yue lu* [ag]) was 22 to 7.

It is to be noted from the above account that numbers were enlarged or, in other words, extended to the left rather than the right in order to attain greater accuracy in calculation. Thus Zu used a radius of 1 *zhang*, taken as 100000000 units, while in Liu's figure the radius 1 *chi* equaled 1000000 *hu*. Obviously, it is easier to deal with larger whole units than with decimal fractions extended to more decimal places. From knowledge of Liu's method and Li Chunfeng's [w] statement, it seems logical to infer that Zu's method of finding the value of π between 3.1415926 and 3.1415927 was based on Liu's theory. Based on this assumption, the present authors proceeded to do the calculations on an electronic computer. Care was taken to devise a program for the computer which adhered to the method used by Liu Hui [6]. The following findings resulted.

1. According to Liu's method the values of b_n, a_n^2, and a_n are truncated at the sixth, tenth, and fifth decimal places, respectively. If Liu's procedure is extended and strictly followed, it is impossible to obtain Zu's estimates irrespective of the value of n. The simple reason is that Zu's estimates for π are up to the seventh decimal place.

2. If an adjustment is made to enlarge the radius from Liu's 10 units to 1000 units as specified in the *Sui Shu,* then the values of b_n and a_n^2 can be truncated, as in Liu's method, at the sixth and tenth decimal places, respectively. The procedure terminates when $n = 6144$, and, to obtain Zu's values for π, the values of a_{6144} and a_{12288} have to be truncated at the eighth and ninth decimal places, respectively. This is a deviation from Liu's method, where values of a_n are truncated at the fifth decimal place. However, an extension of decimal places should not have involved any difficulty for Zu. The figures obtained are shown in the table below.

$b_6 = 866.025403$
$c_6 = 133.974597$
$a_{12}^2 = 267949.1926413124$

$b_{12} = 965.925826$
$c_{12} = 34.074174$
$a_{24}^2 = 68148.3474941103$

$b_{24} = 991.444861$
$c_{24} = 8.555139$
$a_{48}^2 = 17110.2772768368$

$b_{48} = 997.858923$
$c_{48} = 2.141077$
$a_{96}^2 = 4282.1535299291$

$b_{96} = 999.464587$
$c_{96} = 0.535413$
$a_{192}^2 = 1070.8250495627$

$b_{192} = 999.866137$
$c_{192} = 0.133863$
$a_{384}^2 = 267.7241816933$

$b_{384} = 999.966533$
$c_{384} = 0.033467$
$a_{768}^2 = 66.9321654633$

$b_{768} = 999.991633$
$c_{768} = 0.008367$
$a_{1536}^2 = 16.7331113724$

$b_{1536} = 999.997908$
$c_{1536} = 0.002092$
$a_{3072}^2 = 4.1832822195$

$b_{3072} = 999.999477$
$c_{3072} = 0.000523$
$a_{6144}^2 = 1.0458208283$
$a_{6144} = 1.02265381$

$b_{6144} = 999.999869$
$c_{6144} = 0.000131$
$a_{12288}^2 = 0.2614552241$
$a_{12288} = 0.511326924$

From the above table and by Liu's method, where r = the radius, we have

$$\begin{aligned} A_{2288} &= r \times a_{6144} \times \tfrac{1}{2} \times 6144 = r \times 3141.592504 \\ &= r \times 3141.5925 \qquad \text{(truncated at the fourth decimal place)} \\ A_{24576} &= r \times a_{12288} \times \tfrac{1}{2} \times 12288 = r \times 3141.592621 \\ &= r \times 3141.5926 \qquad \text{(truncated at the fourth decimal place).} \end{aligned}$$

Again by Liu's method,

$$A_{24576} < A < A_{12288} + 2(A_{24576} - A_{12288})$$

$$3.1415926 < \frac{A}{r^2} < 3.1415927,$$

where A = the area of the circle.

The degree of accuracy for the approximation of π depends on the number of places to which b_n is calculated, and thereafter appropriate extensions of places are performed on c_n, a_n^2, and a_n. All these operations were known to the Chinese and for Zu these would have posed no problem except perhaps perseverance. Moreover, Liu's mathematical works were known to Zu. For instance, in one of the extant fragmentary records, it is noted that Zu and his son Zu Geng [ah] completed the well-known proof of the derivation of the volume of a sphere which was left unfinished by Liu [Lam & Shen 1985].

How Zu obtained what he called his "very close" ratio $\frac{355}{113}$ for π is not known. This value is correct to the sixth decimal place, and as for the value of the seventh decimal place a better indication is obtained from his other figures of 3.1415926 and 3.1415927.

A SURVEY OF CIRCLE MEASUREMENTS IN CHINA

Like all other early civilizations, the ancient Chinese took the value of the ratio of the diameter to the circumference as 3 in their mathematical calculations [7]. As time went by, they realized that this value was a rough approximation, and so tried to improve it. Liu Xin [z], an astronomer and calendar expert of the first century B.C., was said to have been one of the earliest to attempt the improvement. This was first mentioned in the *Sui shu* [y] [387–388], which did not provide any mathematical procedure and hence has led historians of mathematics to look for evidence elsewhere.

When Wang Mang [ai] ascended the throne toward the end of the Western Han (206 B.C.–A.D. 24), he commanded Liu Xin to construct a standard measure for the kingdom. Liu Xin produced a vessel cut from a solid bronze cylinder and called it the *Jia liang hu* [aj]. (For the words of the inscription on it, see the Appendix, Section 3.) It was estimated that about a hundred *Jia liang hu* were made for distribution throughout the entire empire [Sun 1955, 11]. One such vessel is still being kept in the Palace Museum in Beijing. Chinese historians of mathematics who had the opportunity to examine the vessel, including Li Yan [ak], Qian

Baocong [al], Li Naiji [am], Sun Zhifu [an], and Xu Chunfang [ao], thought that Liu Xin had contributed a new value of π. Their conclusion was drawn from the following procedure:

The diameter of the measure = $\sqrt{200} + 2 \times 0.095 = 14.332$ cun. Since the area is given as 162 sq. cun, $\pi(14.332/2)^2 = 162$ or $\pi = 3.1547$.

This argument of attributing the new value of π to Liu Xin by an inverse operation does not seem convincing. In a recent paper on the same subject Bai Shangshu [ap] [1982, 75–79] pointed out that in his study of six different kinds of standardized vessels for smaller capacities made by Wang Mang [ai], he found four different values of π, namely 3.1547, 3.1590, 3.1497, and 3.1679. From this inconsistency, Bai thought that it was unreasonable to suggest that Liu Xin assumed $\pi = 3.1547$. Yet Liu Xin, being an astronomer and calendar expert, would certainly not have used the ancient ratio of 3 in his mathematical calculations. His task of constructing a standard measure called for precision, particularly in dealing with solid objects. Furthermore, *Sui shu* [y] [387–388] says that Liu Xin was prompted to find a new value of π to replace the old one. Thus, although the theoretical record on his approach to the approximation of π is still wanting, it is not too farfetched to suggest that Liu Xin did have some kind of improved value of π before constructing the standard vessels.

About a century later, Zhang Heng [aa] (A.D. 78–139) made the first explicit effort to obtain a more accurate figure for π. The information comes from Liu Hui's commentary on a problem in the *Jiu zhang suanshu* [d] regarding the derivation of a diameter of a sphere from its volume [Qian 1963, 156]. Liu Hui pointed out that during the time of Zhang Heng the ratio of the area of a square to the area of its inscribed circle was taken as 4 : 3. Following the empirical ratio, it was thought that the volume of the cube to the volume of the inscribed sphere must also be in the ratio $4^2 : 3^2$, that is, $D^3 : V = 16:9$ or $V = \frac{9}{16}D^3$, where D is the diameter of the sphere and V, its volume. Hence, the formula for finding the diameter of a sphere from its volume is given in the *Jiu zhang suanshu* as $D = \sqrt[3]{16V/9}$. Zhang Heng realized that the value of the diameter obtained in this way fell short of the real value, and he attributed the discrepancy to the value taken for the ratio. He thought that this error could be corrected by adding an arbitrary value of $\frac{1}{16}D^3$ to the original formula, thus rectifying it to $V = \frac{9}{16}D^3 + \frac{1}{16}D^3 = \frac{5}{8}D^3$. This means that the ratio of the volume of the cube to that of the inscribed sphere is 8 : 5 and implies that the ratio of the area of the square to that of the circle is $\sqrt{8} : \sqrt{5}$. From this, π was calculated as $\sqrt{10}$.

According to a reference by Zu Geng [ah] cited in the *Kaiyuan zhan jing* [aq] (*Kaiyuan treatise on astrology*) [25b, 26a] of the 8th century, Zhang Heng compared the celestial circle to the width (i.e., diameter) of the earth in the proportion of 736 to 232, which gives π as 3.1724.

As both of Zhang Heng's values were on the high side, Wang Fan [ab] (217–257) investigated further the value of π. The *Song shu* [ar] (*Standard history of Liu Song dynasty*) [675] says that, "having tested that 1 was a little long as diameter for the circumference 3, he [Wang Fan] corrected the circumference to 142 and diameter to 45." This gives the value of π as $\frac{142}{45}$ or 3.155. While Wang Fan's method of arriving at such a figure was not given in the historical sources, Yan Dunjie [as] [1936a, 39–40] suggests that it could have been obtained soon after Liu Hui proposed the value of 3.14 for π. Since Wang Fan thought π greater than 3, a small fraction x should be added to 3, with $\pi = 3 + x$. As Liu Hui considered 3.14 a slightly low value for π, a small fraction y should be added to this, giving $\pi = 3.14 + y$. Taking $y = \frac{1}{10}x$, we have $3 + x = 3.14 + \frac{1}{10}x$, resulting in $x = \frac{7}{45}$ or $\pi = \frac{142}{45}$.

Yan Dunjie further suggested that perhaps the value of $\sqrt{2}/0.45$ for π given by Zhu Zaiyu [at] toward the end of the Ming dynasty could have been derived from Wang Fan's value. This is only a hypothesis. What appears certain, however, is that after the Han period there was considerable interest in a plausible method for approximating π based on theoretical foundation. The mathematician who in A.D. 263 succeeded in giving one was Liu Hui.

Liu strove for precision and refused "to follow the ancients" (*zhong gu* [au]). He aimed at "cutting the circle" continuously until "a limit is reached when the shape of the polygon coincides with that of the circle" so that the exact value might be attained. As a pragmatic mathematician he advocated the value of $\pi = \frac{157}{50}$, but, as a theoretician, he believed that the true value of π might be approached as closely as possible by successive approximations. Whether Liu succeeded in "cutting the circle" to the extent of attaining a 1536-sided polygon or not is open to speculation. In *Jiu zhang suanshu xichao tu shuo* [aw] (*Detailed diagrammatic explanations of the "Jiu zhang suanshu"*), Li Huang [av] (d. 1811) was the first to suggest that the ratio 3927 : 1250 was not Liu's contribution but Zu Chongzhi's. This sparked a great controversy involving several eminent historians of mathematics. The more cautious ones, including Li Yan [ak], Yan Dunjie [as], Du Shiran [ax], and He Luo [ay], recognized Liu's derivation of π only up to the 96-sided polygon and completely avoided the mention of $\frac{3927}{1250}$. Those who fervently believed that Liu had established $\frac{3927}{1250}$ for π were Qian Baocong [al], Xu Chunfang [ao], Wang Shouyi [az], Bai Shangshu [ap], Hua Luogeng [ba], He Shaogeng [bb], Mei Rongzhao [bc], Shen Kangshen [bd], and Li Naiji [am]. Opposing this view and crediting Zu Chongzhi with the invention were the mathematicians Yu Ningsheng [be], Yu Jieshi [bf], Sun Zhifu [an], Li Di [bg] [8], and Donald Wagner [1978, 206–208].

What appears ambivalent in Liu's commentary is the mention of *Jia liang hu* [aj] in the Jin armory. Liu was said to have written his commentary two years before the Wei [bh] kingdom was usurped by Sima Yan [bi], who established the Jin [bj] kingdom (A.D. 265–420). But whether this was a complete commentary on the whole text of *Jiu zhang suanshu* [d] is open to question. In fact elsewhere in *Sui shu* [y] [429], it says that Liu was making a study and a comparison of the *Jia liang hu* and the *hu* measure of his time when he wrote the commentary on chapter 5 (entitled *shang gong* [bk]) of the *Jiu zhang suanshu* in 263. Nevertheless, Liu continued to use $\pi = \frac{157}{50}$ for all the eight problems involving spherical and circular measurements in that chapter. Based on the fact that the same value for π was also used in the previous chapters (ten problems in chapter 1 and two problems in chapter 4), it may be assumed that Liu might not have had the opportunity of seeing a *Jia liang hu* before he set out to write his commentary on the *fang tian* chapter. This note about the *Jia liang hu* in the *fang tian* chapter could have been an addendum when Liu made a revision and updating of the whole commentary soon after the collapse of the Wei kingdom in 265. Therefore, as a mathematician whose life straddled two kingdoms, it was proper for him to mention the armory where the *Jia liang hu* was kept as the Jin armory. The revision would have spurred him to make a thorough investigation and calculation of the dimension of the *Jia liang hu*. He would have discovered that the vessel was not a perfect measure and that the value for π obtained from the vessel was almost the same as that he had derived earlier on. This realization led him to improve the value for π by continuing the process of calculation of areas of polygons up to 192 sides. He finally arrived at $\pi = \frac{3927}{1250}$ and was satisfied with it. It is no surprise that the great mathematician of the Tang dynasty Wang Xiaotong [bl] praised Liu as one who could stretch his "thought to the minutest detail" (*si ji hao mang* [bm]) [Qian 1963, 493].

Zu Chongzhi (429–500) came from a bureaucratic family of calendar experts [9]. The calendrical system of his time was the Yuanjia li [bn], compiled by Ho Chengtian [bo] (370–447). Zu found the system inaccurate and therefore unsuitable for civil use. He made two very bold attempts to rectify the system, by taking the precession of equinoxes (*sui cha* [bp]) into consideration, and by deviating from the traditional 19-year cycle (i.e., the Metonic Cycle) by putting 144 intercalary months in every 391 years. To gather the necessary data he needed precise mathematical techniques. It was for this reason that he had to reexamine the mathematical constants used in his computations.

Zu's expertise in astronomy and mathematics was unquestionable. Apart from successfully composing an astronomical system known as the *Daming li* [bq], he also wrote a mathematical text called *Zhui shu* [x], which was prescribed as a textbook for advanced students of mathematics in the official academies of the Tang dynasty. While the treatise on the *Daming li* is still available in the *Song shu* [ar] [192 ff.], the *Zhui shu* was purported to have been lost toward the end of the Northern Song dynasty (960–1127). The fact that students in the Tang academies had to devote four out of their seven years to the study of this text suggests that it must be an important and difficult treatise on mathematics. Ironically, as *Sui shu* [y] [388] says, "the official students were unable to understand the profoundity of the text so it was subsequently abandoned and neglected altogether."

Qian Baocong [1923, 56–57] and Sun [1955, 9] assigned Liu credit for $\pi = \frac{3927}{1250}$ and believed that Zu continued where Liu left off. The "very close" value of $\pi = \frac{355}{113}$ by Zu was a great advance in the history of mathematics (see [Yan 1936b, 518–519]). Mikami [1913, 50] even suggested that it should be named after Zu. As regards the familiar Archimedean ratio of $\frac{22}{7}$, the general consensus is that it was not Zu's discovery [10]. Prior to Zu it was thought that either Ho Chengtian [bo] or Pi Yancong [ac] (fl. 445) had already found a value of $\pi = 3.1428$ through an interest in astronomy. What is intriguing is the derivation of the "very close" value of $\pi = \frac{355}{113}$. Ho had earlier invented a "method of averaging days" (*tiao ri fa* [br]) in his astronomical system. He took two fractions, one known as the "strong ratio" and the other as the "weak ratio," to determine the fractional day part for the synodic period. It has been suggested by Qian [1923, 57–58] that Zu might have used a similar method, taking Liu's $\frac{157}{50}$ as the "weak ratio" and Ho's $\frac{22}{7}$ as the "strong ratio" such that $\frac{157}{50} < \pi < \frac{22}{7}$. By using Ho's method of averaging, he would have obtained the following ratio after the ninth iteration:

$$\frac{157 + (9 \times 22)}{50 + (9 \times 7)} = \frac{355}{113}.$$

Whether Zu obtained his "very close" ratio by the method suggested above is questionable. Apart from the extreme precision that was required for specific purposes, such as in calendrical calculations, mathematicians in general were quite happy with the approximate ratio of $\frac{22}{7}$. Li Chunfeng [w], for example, employed $\pi = \frac{22}{7}$ in almost all his mathematical commentaries. One can, in fact, say that after Zu Chongzhi the development of π was in limbo. It was not until the turn of the 14th century that new scholars probed into the value of π. One such person was the Yuan mathematician Zhao Youqin [bs], who returned to the question of Zu's approximation for π. Following Liu's use of inscribed polygons, Zhao Youqin continued the process persistently and reached a regular polygon of 16,384 ($= 4 \times 2^{12}$) sides to derive $\pi = 3.1415926$, thus confirming Zu's accuracy [Ruan 1799, 333–345].

Some interest in the evaluation of π was evidenced by the Ming mathematicians. Toward the end of the 16th and the beginning of the 17th century, Zhu Zaiyu [at] found $\pi = \sqrt{2}/0.45$ and $\pi = 3.1426968$, while Xing Yunlu [bt] adopted $\pi = 3.1126$ and $\pi = 3.12132034$. Then Chen Jinmo [bu] and Fang Yizhi [bv] used $\pi = 3.1525$ and $\pi = \frac{52}{17}$, respectively. Apart from these, there were other values such as $\frac{63}{20}$ and $\frac{25}{8}$. None of these was, however, as accurate as Zu's value [11].

There was another attempt to exhaust the value of π by the method of "cutting the circle." This is found in chapter 15 of *Shu li jing yun* [bw] (*Collected basic principles of mathematics*), which was commissioned by Emperor Kang Xi [bx] and edited by Mei Gucheng [by] and He Guozong [bz]. Starting from an inscribed hexagon of radius 10^{12}, the mathematicians here found that the length of one side of a regular 6×2^{23}-sided polygon was 121, with the sum of sides being 6283185307179. Again starting with a circumscribed regular hexagon and going up to the same number of sides, they found that the side and sum of sides remained the same. Hence, it was then fixed that $\pi = 3.14159265$ (a value correct to eight decimal places).

Following the arrival of the Jesuits, the traditional approach to the evaluation of π came to a halt. Mathematicians, like Ming Antu [ca] (d. 1765), Xiang Mingda [cb] (1789–1850), Li Shanlan [cc] (1811–1882), and Zeng Jihong [cd] (1848–1877), began to evaluate π by analytical methods.

APPENDIX: Translation of Liu's Method on Circle Measurement [Qian 1963, 103–106]

[*Section 1*]

If half the circumference [of a circle] is the length and half the diameter is the width, then the product of the length and width gives the area [of a circle]. Let the diameter of the circle be 2 *chi* [e]. The length of one side of a hexagon inscribed in the circle is equal to the radius. The ratio of the diameter to the perimeter [of the hexagon] is 1 to 3. Next, referring to the diagram [12], if the radius is multiplied by one side of the hexagon and then by 3, the product obtained is the area of an inscribed dodecagon. If the circle is cut (*ge* [ce]) again so that the radius is multiplied by one side of the dodecagon and then multiplied by 6, the product obtained is the area of an inscribed polygon of 24 sides. The more finely [the circle] is cut, the less loss there is [in area]. Cut it again and again until one is unable to cut further, that is, when the shape [of the polygon] coincides with that of the circle and there is no loss [in area]. Beyond one side of a polygon [from its mid-point to the circle], there is a remaining portion of the diameter. Multiply this remaining portion by a side of the polygon to give a rectangular area which extends beyond the arc [of the circle]. When a side of the inscribed polygon becomes so fine that it finally merges with the [circumference of the] circle, then there is no remaining portion of the diameter. When no remaining portion of the diameter is shown, this means that the area does not protrude out. When a side [of an inscribed polygon] is multiplied by the radius, [this implies that] in the process of extending the polygon, the number of sides is doubled. This is why half the circumference multiplied by the radius is the area of the circle. The relation of circumference and diameter considered in this manner gives the most satisfactory result.

The ratio of circumference to diameter is in fact not 3 to 1. The figure 3 attached to the circumference actually means that the perimeter of an inscribed hexagon is three times the diameter. Hence, if the ratio [i.e., 3] is used to compute the length of an arc, the result obtained is not the arc but the chord. The value of this ratio was, however, passed on from generation to generation indicating the reluctance of the people to strive for accuracy. The learners, too, just followed the ancients and simply learned the incorrect method. Nevertheless, without concrete examples, it is difficult to argue on [this inaccuracy]. Generally speaking, the forms of things around us are either square or round. If we understand the ratio of their forms at close range, we can also determine this ratio when viewed from afar. Considering the ratio in this manner, we can say that its use is indeed far and wide. For this reason, I have been using drawings for verifications in order to construct a more precise ratio. However, I fear that if only the method is given, the various numerical values involved may appear obscure and difficult to explain. Therefore, I do not consider it laborious here to show the derivation of the ratio in various stages with the necessary commentaries.

[*Section 2*]

Method of cutting an inscribed hexagon into an inscribed dodecagon. Put down [on the counting board] the diameter of the circle, 2 *chi* [e], and halve it to obtain 1 *chi*, which is the length of one side of the hexagon. Let the radius of length 1 *chi* be the hypotenuse (*xian* [cf]); half of a side of the hexagon, 5 *cun* [v] [5], be the *gou* [cg] (the shorter orthogonal side of a right-angled triangle); and it is required to find the *gu* [ch] (the longer orthogonal side). When 25 *cun*, which is the square of the *gou*, is subtracted from the square of the hypotenuse, there is a remainder of 75 *cun*. Find the square root up to the place values of *miao* [t] and *hu* [u]. Again move the number in the *fa* [ci] row back by one jump [of two places] to find the other "minute numbers" (*wei shu* [cj]) [13]. These "minute numbers" have no place names, so convert them into a fraction with 10 as the denominator so that they are approximated to $\frac{2}{5}$ *hu* [u]. Hence the *gu* is 8 *cun* 6 *fen* [q] 6 *li* [r] 2 *miao* [t] $5\frac{2}{5}$ *hu* [u]. Subtract this from the radius to leave a remainder of 1 *cun* 3 *fen* 3 *li* 9 *hao* [s] 7 *miao* $4\frac{3}{5}$ *hu* which we shall call the smaller *gou* [that is, the shorter orthogonal side of a smaller right-angled triangle]. Then half the side of the hexagon will be called the smaller *gu* and it is required to find the hypotenuse. The square [of the hypotenuse] is 267949193445 [square] *hu*, after the remaining terms are discarded. The square root of this gives the length of one side of a dodecagon.

Method of cutting an inscribed dodecagon into an inscribed polygon of 24 sides. Again, let the radius be the hypotenuse and half the side of the dodecagon be the *gou*. From this, find the *gu*. Put down the square of the smaller hypotenuse of the preceding [triangle] and divide it by 4 to obtain 66987298361 [square] *hu* after the remaining terms are discarded, and this is the square of the *gou*. Subtract this from the square of the hypotenuse and find the square root of the difference to obtain the *gu*, which is 9 *cun* 6 *fen* 5 *li* 9 *hao* 2 *miao* $5\frac{4}{5}$ *hu*. Subtract this from the radius to give a remainder of 3 *fen* 4 *li* 7 *miao* $4\frac{1}{5}$ *hu*, which is called the smaller *gou*; half the side of the dodecagon is called the smaller *gu* and from these, the smaller hypotenuse is derived. Its square is 68148349466 [square] *hu* after the remaining terms are discarded. The square root of this is the length of a side of a polygon of 24 sides.

Method of cutting an inscribed polygon of 24 sides into one of 48 sides. Again, let the radius be the hypotenuse and half the side of the polygon of 24 sides be the *gou*. From this, find the *gu*. Put down the square of the smaller hypotenuse of the preceding [triangle] and divide it by 4 to obtain 17037087366 [square] *hu* after the remaining terms are discarded, and this is the square of the *gou*. Subtract this from the square of the hypotenuse and find the square root of the difference to obtain the *gu*, which is 9 *cun* 9 *fen* 1 *li* 4 *hao* 4 *miao* $4\frac{4}{5}$ *hu*. Subtract this from the radius to give a remainder of 8 *li* 5 *hao* 5 *miao* $5\frac{1}{5}$ *hu*, which is called the smaller *gou*; half the side of the 24-sided polygon is called the smaller *gu* and from these, the smaller hypotenuse is derived. Its square is 17110278813 [square] *hu* after the remaining terms are discarded. The square root of this is 1 *cun* 3 *fen* 8 *hao* 6 *hu* neglecting the lower terms and this is the length of the smaller hypotenuse, which is also the length of a side of an inscribed polygon of 48 sides. Multiply this by the radius 1 *chi* and also by 24 to obtain 3139344000000 [square] *hu*. Divide by 10000000000 to obtain $313\frac{584}{625}$ [square] *cun* which is the area of an inscribed polygon of 96 sides.

Method of cutting an inscribed polygon of 48 sides into one of 96 sides. Again, let the radius be the hypotenuse and half the side of the polygon of 48 sides be the *gou*. From this find the *gu*. Put down the square of the hypotenuse of the preceding [triangle] and divide it by 4 to obtain 4277569703 [square] *hu* after the remaining terms are discarded, and this is the square of the *gou*. Subtract this from the square of the hypotenuse and find the square root of the remainder to obtain the *gu* which is 9 *cun* 9 *fen* 7 *li* 8 *hao* 5 *miao* $8\frac{9}{10}$ *hu*. Subtract this from the radius to give a remainder of 2 *li* 1 *hao* 4 *miao* $1\frac{1}{10}$ *hu* which is called the smaller *gou*; half the side of the 48-sided polygon is called the smaller *gu* and from these, the smaller hypotenuse is derived. Its square is 4282154012 [square] *hu* after the remaining terms are discarded. The square root of this is 6 *fen* 5 *li* 4 *hao* 3 *miao* 8 *hu*, neglecting the lower terms, and this is the length of the smaller hypotenuse which is also the length of a side of an inscribed polygon of 96 sides. Multiply this by the radius 1 *chi* and also by 48 to obtain 3141024000000 [square] *hu*. Divide the area by 10000000000 yielding $314\frac{64}{625}$ [square] *cun*, which is the area of an inscribed polygon of 192 sides. Subtract the area of the 96-sided polygon from this to obtain a remainder of $\frac{105}{625}$ [square] *cun* and this is called a residual area (*cha mi* [ck]). Double this amount [so that the numerator of the fraction] becomes 210 and this amount represents the area outside the 96-sided polygon extending beyond the arcs [of the circle]. This can be said to be the sum of the products of the chord (*xian* [cf]) and the sagitta (*shi* [cl]). Add this amount to the area of the 96-sided polygon to obtain $314\frac{169}{625}$ square *cun* which gives an area covering the circle and protruding beyond it. This is the reason why 314 [square] *cun* which is the area of the 192-sided polygon to the nearest integer is taken as the standard area of the circle after the smaller terms are discarded. Divide the area of the circle by its radius, 1 *chi*, and then double this to obtain 6 *chi* 2 *cun* 8 *fen* which gives the circumference. Squaring the diameter gives a square area of 400 [square] *cun*. Compare the area of the circle with this, and obtain the ratio of the areas of the circle to the [circumscribed] square as 157 to 200. If a square has 200 [square units] then its inscribed circle has 157 [square units]. The rate for the circle actually contains some smaller units as well. Referring to the diagram of the arc of a circle, we see a circle inscribed in the square and a square inscribed in the circle. The area of the inner square is half the area of the outer square. If the area of the circle is 157 [square units] then the area of the inscribed square is 100 [square units]. Next, if the diameter, 2 *chi*, is compared with the circumference, 6 *chi* 2 *cun* 8 *fen*, the ratio of the circumference to the diameter is obtained as 157 to 50. Actually, the circumference still has other smaller units.

[Section 3]

There was in the armory of the Jin dynasty a copper *hu* [cm] measure constructed by Wang Mang [ai] during the Han dynasty. The inscription on it reads: "The standardized *chia liang hu* [aj] measure has a square with each side 1 *chi* long, and outside it a circle. The distance from each corner of the square to the circle is 9 *li* 5 *hao*. The area of the circle is 162 [square] *cun*, the depth is 1 *chi* and the volume is 1620 [cubic] *cun*. The measure has a capacity for 10 *dou* [cn] (bushels)." If we use the present ratio [i.e., $\frac{157}{50}$] to compute, we obtain the area [of the circle] as 161 *cun* and a fraction. The two values are very close indeed. But as the present ratio is not precise, we have to consider the area of the dodecagon as a basis to augment or diminish the area of $\frac{105}{625}$ [square] *cun*. This yields $\frac{36}{625}$ [square] *cun*. Add this to the area of the 192-sided polygon to obtain $314\frac{4}{25}$ [square] *cun* as the area of the circle. Put down the square of the diameter 400 [square] *cun* and compare this with the area of the circle. If the area of the circle is 3927 then that of the square is 5000. Taking this as the ratio [i.e., $\frac{3927}{5000}$] we can say that the area of the inscribed circle in the square of 5000 [square units] is 3927 [square units] and the area of the square inscribed in the circle of area 3927 [square units] is 2500 [square units]. Divide the area of the circle $314\frac{4}{25}$ [square] *cun* by its radius 1 *chi* and double the result to obtain 6 *chi* 2 *cun* $8\frac{8}{25}$ *fen* which gives the circumference. Take the diameter 2 *chi* and compare it with the circumference, getting 1250 for the diameter and 3927 for the circumference. The ratio [of the circumference to the diameter] thus obtained has been exhausted to very minute numbers. However, for practical purposes, the former ratio [i.e., $\frac{157}{50}$] is still a simpler one. The area of a 3072-sided polygon is computed from a side of a 1536-sided polygon. When the lower terms are discarded, the ratio obtained is the same as the previous one. In this way, the ratio is again verified.

NOTES

1. The *Jiu zhang suanshu* is printed in [Qian 1963, 81–258]. The data assigned to this book is between 200 B.C. and A.D. 200; see [Needham 1959, 24–25].

2. There is a fairly large literature on Liu's method written in Chinese, of which one of the more up-to-date is [Bai Shangshu 1983, 35–53]. In English, Liu's method is found in [Mikami 1913, 47–49], which contains some misprints, and, more briefly, in [He Shaogeng 1983, 90–98]. In our translation, we have tried to give as close an interpretation of the original text [Qian 1963, 103–106] as possible.

3. However, the ancient Greeks were aware that the area of a circle to the square of its radius is a constant ratio [Euclid, Bk. XII, Prop. 2].

4. The phrase "counting board" means any flat surface suitable for the placement of rod numerals and should not be confused with the specially constructed counting boards of medieval Europe.

5. Note that 10 *cun* [v] = 1 *chi* [e] and 10 *chi* [e] = 1 *zhang* [ae].

6. We would like to thank Lam Chih Chao and Lam Chih Ming for the software program.

7. This is evident in the earliest mathematical texts of *Zhou bi suanjing* [co] (*The arithmetic classic of the gnomon and the circular paths of heaven*) and *Jiu zhang suanshu*, as also in the *Kao gong ji* [cp] (*Artificer's record*) of the *Zhou li* [cq] (*Record of the rites of the Zhou dynasty*).

8. For an objective review of the polemic of this problem, see [Li Di 1982, 35–44].

9. For a brief biography of Zu Chongzhi, see [Li Di 1962]. See also [Ruan Yuan 1799 353, 91–105].

10. See [Qian 1923, 59; Yan 1936b, 43; Sun 1955, 9].

11. For a brief survey of this period, see [Xu 1957, 48].

12. The diagram is missing, but it should be similar to Fig. 1.

13. For the square-root extraction procedure, see [Wang & Needham 1955, 350–356; Lam 1969, 93–97].

REFERENCES

Ang Tian Se. 1977. Chinese computation with the counting-rods. *Papers on Chinese Studies*, University of Malaya **1**, 97–109.

Bai Shangshu [ap]. 1982. Cong Wang Mang liang qi dao Liu Xin yuan lu [cr] [From Wang Mang's measuring vessel to Liu Xin's ratio of circumference to diameter]. *Beijing daxue xuebao*, No. 2, 75–79.

——— 1983. *Jiu zhang suanshu zhushi* [cs] [Annotations on *Nine chapters of the mathematical art*]. Beijing: Kexue chubanshe.

Beckmann, P. 1970. *A history of π (pi)*. All page references are to the second edition, Boulder, Colo.: Golem, 1971.

Gupta, R. C. 1975. Some ancient values of pi and their use in India. *Mathematics Education* **9,** 1–5.

He Shaogeng [bb]. 1983. Method for determining segment areas and evaluation of π. In *Ancient China's technology and science*. Compiled by the Institute of the History of Natural Sciences, Chinese Academy of Sciences. Beijing: Foreign Language Press.

Heath, T. L. (ed.). 1897. *The works of Archimedes*. All page references are to the reissued edition, New York: Dover, 1953.

——— 1921. *A history of Greek mathematics*. 2 vols. London/New York: Oxford Univ. Press (Clarendon).

Lam Lay-Yong. 1969. The geometrical basis of the ancient Chinese square-root method. *Isis* **61,** 92–101.

Lam Lay-Yong and Shen Kangsheng. 1985. The Chinese concept of Cavalieri's principle and its applications. *Historia Mathematica* **12,** 219–228.

Li Di [bg]. 1962. Da kexuejia Zu Chongzhi [ct] [The great scientist Zu Chongzhi]. Shanghai: Renmin chubanshe.

——— [bg]. 1982. Jiu zhang suanshu zhengming wentide gai shu [cu] [A summary of the various views on the Jiu zhang suanshu]. In *Jiu zhang suanshu yu Liu Hui* [cw] [The *Jiu zhang suanshu* and Liu Hui], Wu Wenjun [cv], ed., pp. 28–50. Beijing: Shifan daxue chubanshe.

Mikami, Y. 1913. *The development of mathematics in China and Japan*. All page references are to the second edition, New York: Chelsea, 1974.

Needham, J. 1959. *Science and civilisation in China*. Vol. 3. Cambridge: Cambridge Univ. Press.

Neugebauer, O. 1952. *The exact sciences in antiquity*. All page references are to the second edition, Providence, R.I.: Brown Univ. Press, 1957.

Qian Baocong [al]. 1923. Zhongguo suan shu zhong zhi zhoulu yanjiu [cx] [A study of π in Chinese mathematical texts]. In *Qian Baocong kexueshi lunwen xuanji* [cy] [Selected essays of Qian Baocong on the history of Chinese science], pp. 50–74. All page references are to the 1983 book edition, Beijing: Kexue chubanshe 1983.

——— (ed.). 1963. *Suanjing shi shu* [cz] [Ten mathematical manuals]. Shanghai: Zhonghua shuju.

Qudan Xida [da]. 729. *Kaiyuan zhan jing* [aq] [Kaiyuan treatise on astrology].

Ruan Yuan [db]. 1799. *Chou ren zhuan* [dc] [Biographies of mathematicians and scientists]. In *Guoxue jiben congshu*. Taipei: Shangwu yinshuguan, 1968.

Song shu [ar]. [Standard history of the Liu Song dynasty]. 1973 edition. Beijing: Zhonghua shuju.

Sui shu [y] [Standard history of the Sui dynasty]. 1973 edition. Beijing: Zhonghua shuju.

Sun Zhifu [an]. 1955. Zhongguo gudai shuxue jia guanyu yuanzhoulu yanjiu de chengjiu [dd] [Achievements in the research of π by ancient Chinese mathematicians]. *Shuxue Tongbao,* No. 5, 5–12.

Wagner, D. 1978. Doubts concerning the attribution of Liu Hui's commentary on the *Chiu-chang suan-shu*. *Acta Orientalia* **39,** 199–212.

Wang, L., and Needham, J. 1955. Horner's method in Chinese mathematics: Its origins in the root-extraction procedures of the Han dynasty. *T'oung Pao* **43,** 345–401.

Xu Chunfang [ao]. 1957. *Zhong suan jiade jihexue yanjiu* [de] [Research on geometry by Chinese mathematicians]. Hong Kong.

Yan Dunjie [as]. 1936a. Sui shu luli zhi Zu Chongzhi yuan lu ji shi shi [df] [An analysis of records for π by Zu Chongzhi as found in the harmonics and calendar chapters of *Sui shu*]. *Xue yi zazhi* [dg] (Shanghai) **15,** No. 10, 27–57.

——— 1936b. Zhongguo suanxuejia Zu Chongzhi ji qi yuanzhou lu zhi yanjiu [dh] [Chinese mathematician Zu Chongzhi and his work on π]. *Xue yi zazhi* [dg] (Shanghai) **15,** No. 5, 37–50.

Youschkevitch, A. P., and Rosenfeld, B. A. 1973. Al-Kashi (or Al-Kashani), Ghiyath al-din Jamshid Mas'ud. In *Dictionary of scientific biography*, Vol. 7, pp. 255–262. New York: Scribner's.

GLOSSARY

a 刘徽	b 祖冲之	c 方田	d 九章算术
e 尺	f 十	g 百	h 千
i 万	j 十万	k 百万	l 千万
m 亿	n 十亿	o 百亿	p 千亿
q 分	r 厘	s 毫	t 秒
u 忽	v 寸	w 李淳风	x 缀术
y 隋书	z 刘歆	aa 张衡	ab 王蕃
ac 皮延宗	ad 南徐	ae 丈	af 密率
ag 约率	ah 祖暅	ai 王莽	aj 嘉量斛
ak 李俨	al 钱宝琮	am 励乃骥	an 孙炽甫
ao 许纯舫	ap 白尚恕	aq 开元占经	ar 宋书
as 严敦杰	at 朱载堉	au 踵古	av 李潢
aw 九章算术细草图说		ax 杜石然	ay 何洛
az 王守义	ba 华罗庚	bb 何绍庚	bc 梅荣照
bd 沈康身	be 余宁生	bf 余介石	bg 李迪
bh 魏	bi 司马炎	bj 晋	bk 商功
bl 王孝通	bm 思极毫芒	bn 元嘉历	bo 何承天
bp 岁差	bq 大明历	br 调日法	bs 赵友钦
bt 邢云路	bu 陈荩谟	bv 方以智	bw 数理精蕴
bx 康熙	by 梅谷成	bz 何国宗	ca 明安图
cb 项名达	cc 李善兰	cd 曾纪鸿	ce 割
cf 弦	cg 勾	ch 股	ci 法
cj 微数	ck 差幕	cl 矢	cm 斛
cn 斗	co 周髀算经	cp 考工记	cq 周礼
cr 从王莽量器到刘歆圆率		cs 九章算术注释	ct 大科学家祖冲之
cu 九章算术争鸣问题的概述		cv 吴文俊	cw 九章算术与刘徽
cx 中国算书中之周率研究		cy 钱宝琮科学史论文选集	
cz 算经十书	da 瞿昙悉达	db 阮元	dc 畴人传
dd 中国古代数学家关于圆周率研究的成就		de 中算家的几何学研究	
df 隋书律历志祖冲之圆率记事释		dg 学艺杂志	
dh 中国算学家祖冲之及其圆周率之研究			

[V.] THE RATIO OF THE DIAMETER OF ANY CIRCLE TO ITS CIRCUMFERENCE IS ONE [THAT IS, IS THE SAME FOR ALL CIRCLES].

For example, let there be two different circles, the circles ABG, DEZ [see Fig. 38]. Let their diameters be BG, the diameter of circle ABG, and EZ, the diameter of circle DEZ. I say, therefore, that

$$\text{diameter } BG/\text{circum } ABG = \text{diameter } EZ/\text{circum } DEZ,$$

which is demonstrated as follows:

If the two ratios were not equal, then let line BG/line $ABG = EZ/HU$, where line HU is longer or shorter than line DEZ. First I shall posit that it is shorter, if that is possible. And I shall bisect line HU at T and I shall erect perpendicularly upon line HU at point H a line equal to

35 et impossibile *om. Ar.*
36–37 medietatis...circuli: – ج ه – (*EG*)
37 linee...ipsum: – ا ب ج – محيط (*circumference ABG*)
38 *post* circuli *add. Ar.* – ا ب ج – (*ABG*)
39–45 Et....figure: وقد بان منه ان سطح نصف [الكرة فى نصف] القطر فى نصف اى قوس فرض يكون مساويا لمساحة القطلع الذى يحيط بـه تلك القوس و نصفا قطرين يمرّ ان بطر فيها ((The section in brackets is in the Arabic edition but, quite properly, is not in the Paris Arabic manuscript.)) (*And it is already known from this that the multiplication of the radius by the half of any assigned arc is equal to the area of the sector contained by this arc and two radii extending to the extremities of the arc.*)
5 EZ: – د ه – (*ED*)
5–8 Dico...demonstratur *om. Ar.*
9 Si... una: فان لم يكن كما ادعينا (*And so if it is not as we claimed*)
10, 13 EZ: – ه د – (*DE*)
12 si...possibile *om. Ar.*

stantem super lineam *HU* orthogonaliter, que sit linea *HK*. Et complebo quadratum *KT*. Et quoniam linea *HK* est equalis medietati linee *EZ*, et linea *HT* est brevior medietate linee *DZE*, erit quadratum *KT* minus superficie circuli *DEZ*. Verum, proportio linee *KH* ad lineam *HT* est sicut proportio medietatis linee *BG* ad medietatem linee *ABG*. Et multiplicatio linee *KH* in lineam *HT* est superficies *KT*. Et multiplicatio medietatis linee *BG* in medietatem linee *ABG* est superficies
c. 2 circuli *ABG*. Ergo proportio superficiei circuli *ABG* / ad quadratum *KT* est sicut proportio medietatis linee *BG* ad lineam *KH* duplicata. Sed proportio medietatis linee *BG* ad lineam *KH* multiplicata est sicut proportio linee *BG* ad duplum linee *KH* duplicata. Sed duplum linee *KH* est equale linee *EZ*. Ergo proportio superficiei circuli *ABG* ad quadratum *KT* est sicut proportio linee *BG* ad lineam *EZ* duplicata, ac proportio superficiei circuli *ABG* ad superficiem circuli *DEZ* est sicut proportio *BG* ad *EZ* duplicata, sicut declaravit Euclides. Ergo proportio superficiei circuli *ABG* ad superficiem circuli *DEZ*, et ad quadratum *KT*, est una. Ergo sunt equales. Sed quadratum *KT* iam fuit minus superficie circuli *DEZ*, quod quidem est contrarium et impossibile. Non est ergo linea *HU* brevior linea *DEZ*.

Et per huiusmodi dispositionem scitur quod linea *HU* non est longior linea *DEZ*. Et cum linea *HU* non sit longior neque brevior linea *DEZ*, tunc est equalis ei. Et proportio linee *BG* ad lineam *ABG*

14 linea *om. T*
15 quoniam: quia *T*
16 DZE: DEZ *H*
17 KH *ZmHRT* HK *P*
18 BG: AG *H*
19 KH: HK *H*
20 medietatis *om. T* / BG: KG *H*
21 *post* ABG[1] *add. H* Sit igitur proportio medietatis linee BG ad lineam KH multiplicata sicut proportio linee BG ad duplum linee KH duplicata (*istud totum om. PMaT hic; cf. lineas 23–24*) / Ergo: et ergo *T*
21–22 proportio... sicut: sicud est *T*
21 ABG[2]: ADG *H*
22 lineam *om. T* / duplicata: multiplicata *T*
23–24 Sed... duplicata *om. H hic sed cf. var. lineae 21*
23 medietatis *om. T*
24 linee[2] *om. T* / KH: HK *T*
24–25 Sed.... Ergo *om. T*
25 est: erit *H*
27 ac *PZm* aut *H* a *Ma*
28 sicut[2]... Euclides: per euclidem *T*
31 quidem *om. T*
32 Non... HU: Igitur linea HU non est *T* / ergo *om. H*
33–34 Et... DEZ *om. Ma*
33–35 Et.... ei: nec longior igitur est equalis *T*

15–16 Et...DZE *om. Ar.*
21–37 Et....voluimus: فنسبة سطح – ك ط – الى دائرة – ا ب ج – كنسبة – ط ح – اعنى نصف – د ه – الى نصف – ب ج – مثناة و هى نسبة – د ه – الى – ب ج – مثناة و قد بين اقليدس ان نسبة – د ه – الى – ب ج – مثناة كنسبة دائرة – د ز ه – الى دائرة – ا ب ج – فنسبة سطح – ك ط – الى – دائرة – ا ب ج – كنسبة دائرة – د ه ز – اليها فسطح – ك ط – مساو لدائرة –

1/2 line *EZ*. This is line *HK*. And I shall complete square *KT*. And since line *HK* = 1/2 line *EZ*, and line *HT* < 1/2 line *DZE*, square *KT* < area of circle *DEZ*. Now line *KH*/line *HT* = 1/2 line *BG*/1/2 line *ABG*, and (*KH* · *HT*) = area *KT*, and (1/2 line *BG* · 1/2 line *ABG*) = area circle *ABG*, and area circle *ABG*/square *KT* = (1/2 line *BG*/ line *KH*)^2. But (1/2 *BG*/ line *KH*)^2 = (line *BG*/2 line *KH*)^2, and 2 line *KH* = line *EZ*. Therefore, area circle *ABG*/square *KT* = (*BG*/*EZ*)^2, and area circle *ABG*/area circle *DEZ* = (*BG*/*EZ*)^2, as Euclid showed. Therefore, area circle *ABG*/area circle *DEZ* = area circle *ABG*/square *KT*. Therefore, area circle *DEZ* = square *KT*. But earlier [it was inferred from the assumption that] square *KT* < area circle *DEZ*. This indeed is a contradiction and is impossible. Therefore, line *HU* is not less than line *DEZ*.

By a similar procedure it is [also] known that line *HU* is not longer than line *DEZ*. And since line *HU* is not longer and is not shorter than line *DEZ*, then it is equal to it; and line *BG*/line *ABG* = *ZE*/*HU*. And

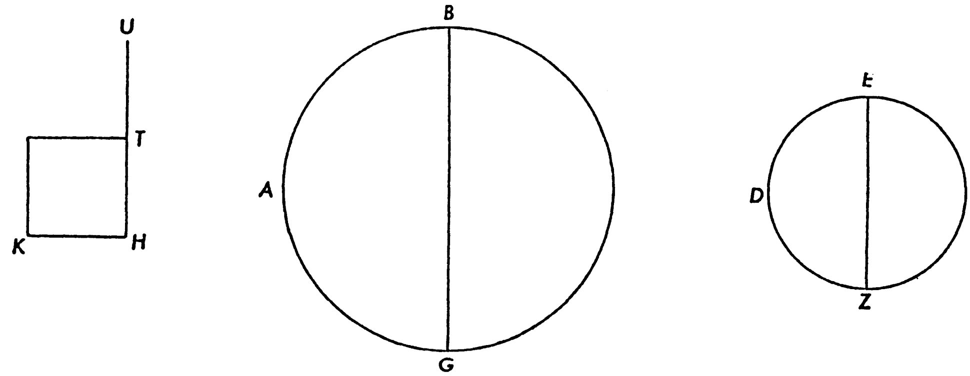


Fig. 38

Note: I have changed diameter *EZ* from a horizontal to a vertical orientation, in conformity with the drawing accompanying the Arabic text.

د ه ز – وكان اصغر منها هذا خلف فليس خطّ –
ح و – اقصر من محيط – د ه ز – و بمثل هذا
التدبير تبين انه ليس اطول منه فاذا نسبة –
د ه – الى محيط – د ه ز – كنسبة – ب ج – الى
محيط – ا ب ج – وكذلك فى كل دائرتين غيرهما
و ذلك ما اردناه

(*And so* KT^2/*circle ABC* = *TH* / ($\frac{1}{2}$ *BG*)^2 = ($\frac{1}{2}$ *DE* / $\frac{1}{2}$*BG*)^2 = (*DE*/*BG*)^2. *And Euclid has already demonstrated that* (*DE*/*BG*)^2 = *circle DZE*/*circle ABG*. *And so* KT^2 / *circle ABG* = *circle DEZ* / *circle ABG*. *And so* KT^2 = *circle DEZ*. *But it was smaller than it. This is a contradiction. And so line HU is not shorter than circumference DEZ; and by a similar disposition it is demonstrated that it is a lot longer than it. And so DE* / *circum DEZ* = *BZ* / *circum ABG and it is thus for any two circles other than these. And this is what we wished.*)

est sicut proportio *ZE* ad *HU*. Et linea *HU* est equalis linee *DEZ*. Iam ergo ostensum est quod proportio diametri omnis circuli ad lineam continentem ipsum est una. Et illud est quod demonstrare voluimus.

[VI.] CUM ERGO IAM MANIFESTUM SIT ILLUD QUOD NARRAVIMUS, TUNC OPORTET UT OSTENDAMUS PROPORTIONEM DIAMETRI CIRCULI AD LINEAM CONTINENTEM IPSUM.

Et operabimur in hoc per modum quo operatus est in eo Archimenides. Nam nullus illius scientie invenit aliquid usque ad hunc nostrum tempus preter ipsum in eo quod nobis apparuit. Et iste modus in inveniendo proportionem diametri ad lineam continentem, etsi non ostendat proportionem unius eorum ad alterum ita ut per eam raciocinetur secundum veritatem, tamen significat proportionem unius eorum ad alterum ad quemcunque finem voluerit inquisitor huius scientie de propinquitate, scilicet si voluerit inquisitor scire proportionem unius eorum ad alterum, verbi gratia, ut perveniat in propinquitate illius ad hoc, ut non sit inter ipsam et inter veritatem proportionis unius eorum ad alterum, cum posita fuerit diametrus unum,
57r c. 1 nisi minus minuto, quod est pars sexagessima diametri, possit / illud. Et si voluerit pervenire in propinquitate illius ad hoc, ut non sit ei finis inter ipsam et inter veritatem proportionis unius eorum ad alterum nisi minus secundo, quod est pars sexagesima minuti, possit illud. Et si voluerit ut perveniat in propinquitate illius ad quemcunque

36 ZE: linee ZE *H* / Et: sed *T* / linee DEZ *tr. T*
37 Iam...quod: ergo *T* / diametri omnis *tr. T*
38 Et...est *om. T*
38–39 demonstrare voluimus *HT tr. PZm* declarare voluimus *Ma*
1 [VI]: 7 *MaR*
1–24 Cum.... illud: Que igitur sit proportio dyametri circuli ad lineam continentem ipsum operabimur sicud Archimenides solus, ita quod non fallatur inquisitor in propinquitate veritatis proportionis ad alteram nisi minus minuto, quod est pars 60ⁿ dyametri. Et si voluerit quod non medium nisi secundo servando (*! del. ?*), quod est pars 60ⁿ minuti, et post illa ut perveniat ad quantum[cun]que finem voluerit computator raciocinari. *T*
1–173 Cum....quod² *om. S*
3 ad: et *H* /
4 ipsam *H* /
5 in hoc *om. H* / in eo *om. H*
6 ad hunc *om. H*
10 *de* significat *scr. P mg et Zm supra* i. ostendit / *post* significat *add. MaR* i. ostendit
11–12 huius...inquisitor *om. R* (*sed in PZmHMa*)
18 *de* proportionis *scr. P. mg. et Zm supra* vel mensure / *ante* proportionis *add. Ma* vel mensure
19 pars sexagesima *tr.* H

line HU = line DEZ. Therefore, it has now been demonstrated that the ratio of the diameter of every circle to its circumference is one. And this is what we wished to demonstrate.

[VI.] HENCE, SINCE WHAT WE HAVE RECOUNTED HAS NOW BECOME EVIDENT, THEN WE MUST SHOW [THAT IS, FIND] THE RATIO OF THE DIAMETER OF A CIRCLE TO ITS CIRCUMFERENCE.

And we shall proceed in this matter by the method which Archimedes used for it. For up to our time no one except him has discovered any knowledge of this, so far as we have seen. And this method of finding the ratio of the diameter to the circumference, although it does not reveal a true ratio that can be reckoned with, still does yield a ratio of the one to the other which is an approximation to any limit the investigator of this subject desires. That is, if the investigator wishes to know the ratio of the one to the other approximately so that, for example, between it and the true ratio there is less than a minute, i.e., a sexigesimal part of the diameter when the diameter is posited as one, that could be done. And if he wished to find an approximation of this to that so that less than a second, i.e., 1/60 of a minute, exists between it and the true ratio, that could be done. And if one wished to achieve an approximation of one to the other to any

1–4 Cum...ipsum: ثم لتبين نسبة القطر الى المحيط (*Then let us investigate the ratio of the diameter to the circumference*)

5 Et...in hoc *om. Ar.*

12–24 scilicet....illud *om. Ar.*

finem voluerit post illa duo, possit illud per illud quod narravit Archimenides. Et usi sunt hoc modo propinquitatis in omni computatione in qua cadunt radices surde, cum computator vult raciocinari per quantitatem eius. Et erit hoc ita. Incipiamus ergo declarare illud.

Lineemus ergo circulum *ATB*, cuius diameter sit *AB*, et ipsius centrum sit punctum *G* [Fig. 39]. Et protraham ex centro *G* lineam *GZ* continentem cum linea *GB* tertiam anguli recti. Et erigam super punctum *B* linee *GB* lineam *BZ* orthogonaliter. Manifestum est igitur quod arcus qui subtenditur angulo *BGZ* est medietas sexte circuli *ATB* et quod linea *BZ* est medietas lateris exagoni continentis circulum *ATB*. Et dividam angulum *BGZ* in duo media cum linea *GE*. Et dividam angulum *BGE* in duo media per lineam *GU*. Et dividam angulum *BGU* in duo media per lineam *GD*. Et dividam angulum

24 *de* Et...ita *scr. P mg. et Zm mg.* (*et add. MaR post* eius): in alio, cum ergo hoc sit ita / *ante* Incipiamus *add. MaR mg.* 8
25 circulus *T* / sit *om. T*
25–26 et...punctum: centrum *T*
26 centrum sit *ZmHMa mg. P*
28 igitur *om. T*
29 sexte *PZm* seste *Ma* sexti *H*
31 BGZ: GBZ *T*
32 Et...GU *om. T* / BGE: BEG *H*
33 BGU *PZmMa* BGN *HT* (-N *pro* -U *hic et ubique in HT*)

---

27 GZ: – ج د – (*GD*) ((Note: The Arabic printed text has د (*D*) everywhere Gerard has Z; I am not noting any other place. Incidentally, the Paris Arabic MS has ح (*H*) for ج (*G*) and an ambiguous mark))

33 BGU: – ب ج – (*BG*) ((Note: This is – ب ح ر – in the Paris MS.)) / GD: – ج ز – (*GZ*) ((Note: Arabic text has ز (*Z*) wherever Gerard has D.))

desired limit beyond these two, that could be done by the method which Archimedes has recounted. And this method of approximation is used in every computation involving surd roots when a computator wishes to calculate with such a quantity. And it will be thus. Therefore, let us begin to show this.

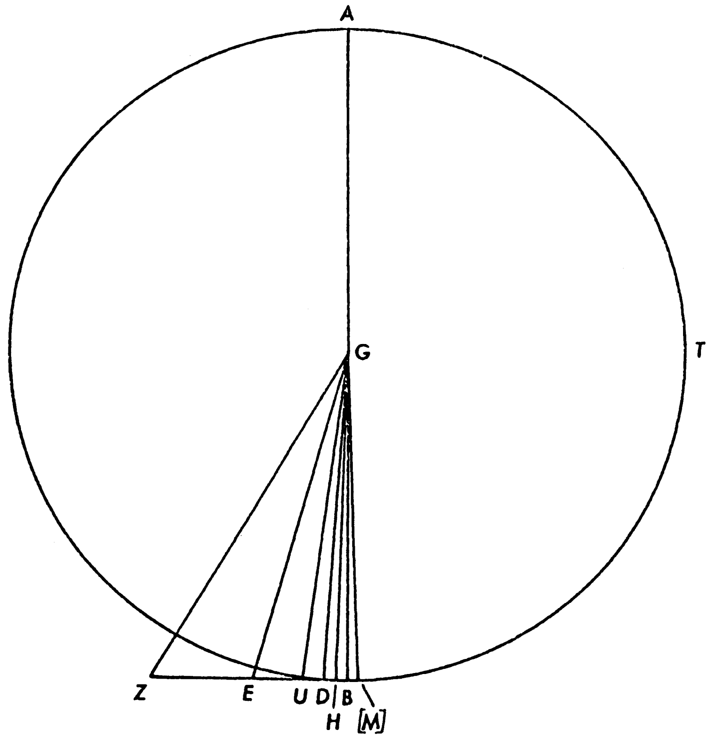


Fig. 39

Note: In MSS *P* and *H*, both halves of the proof are represented on one drawing, although that drawing is repeated. For Fig. 39 I have left off the inscribed figure covering the second half of the proof (see Fig. 40). Also, line *GM*, which is in the original Archimedean proof, is missing in this text and its drawing.

Let us draw circle *ATB*, whose diameter is *AB* and whose center is point *G* [see Fig. 39]. And I shall protract from center *G* line *GZ*, which contains with line *GB* a third of a right angle. And I shall erect line *BZ* perpendicularly on point *B* of line *GB*. It is evident, therefore, that the arc which is subtended by ∠ *BGZ* is 1/2 of 1/6 of circle *ATB* and that line *BZ* is 1/2 of a side of a hexagon containing circle *ATB*. Then I shall bisect ∠ *BGZ* by line *GE*, and ∠ *BGE* by line *GU*, and ∠ *BGU* by line

BGD in duo media per lineam *GH*. Manifestum est igitur quod arcus qui subtenditur angulo *BGH* est pars centessima et nonagessima secunda circuli *ATB*, et quod linea *BH* est medietas lateris figure habentis nonaginta sex latera continentis circulum *ATB*.

Cum ergo hoc sit ita, tunc nos ponemus lineam *GZ* trecentum et sex propter facilitatem usus huius numeri in eo quod computatur. Cum ergo fuerit linea *GZ* trecentum et sex, erit quadratum eius nonaginta tria millia et sexcentum et triginta sex. Et erit linea *BZ* centum et quinquaginta tria, quoniam angulus *BGZ* est tertia anguli recti et angulus *GBZ* est rectus. Et erit quadratum linee *BZ* viginti tria millia et quadringenta et novem. Et quadratum linee *GB* septuaginta millia et ducenta et viginti septem. Ergo linea *GB* est plus ducentis et sexaginta quinque. Sed proportio duarum linearum *BG*, *GZ* agregatarum ad *BZ* est sicut proportio *GB* ad *BE*, propterea quod linea *GE* dividit angulum *BGZ* in duo media. Et due linee *BG*, *GZ* agregate sunt plus quingentis et septuaginta uno. Et linea *BZ* est centum et quinquaginta tria. Ergo proportio *GB* ad *BE* est maior proportione quingentorum et septuaginta unius ad centum et quinquaginta tria. Ergo linea *GB* erit plus quingentis et septuaginta uno, cum fuerit *BE* centum et quinquaginta tria. Et quadratum *GB* est plus trecentis et viginti sex millibus et quadraginta uno. Et quadratum *BE* est viginti tria millia et quadringenta et novem. Ergo quadratum *GE* est plus trecentis et quadraginta novem millibus et quadringentis et quinquaginta. Ergo linea *GE* est plus quingentis et nonaginta uno et octava
2 / unius.

34 igitur *PZmHMa om. T* ergo *R*
36 est: erit *H*
38–39 Cum...'sex: tunc *T*
38 trecenta *H*
39 huius *om. T*
40 fuerit: fuit *H* / Cum...eius: ponamus lineam GZ trecentum et 6 cuius linee quadratum necessario erit *T*
41 milia *H hic et ubique*
42 quoniam: quia *T*
43 BZ *ZmMaR* GZ *PH*
43–44 viginti...novem *del. m. rec. P et add. mg.* 23409
44 quadringenta *HMaT* quadreginta *R* quadriginta *P*
45 et[1] *om. H* / Ergo: igitur *T* / GB : BG *Zm* / *cum* ducentis *desinit T*
46 GZ: EZ *H*
51 ad: ac *H*
52 fuerit: fuerit positum *H*
53–54 trecentis...millibus: 300 et 26000 *H*
55 quadratum : 4 *H*
56 trecentis...millibus: 300 et 49000 *H*

38 Cum...ita *om. Ar.*
39 computatur: تبين (*investigated*)
40 Cum...sex *om. Ar.*
42–43 et[2]...rectus: – ج ب د – (*GBD*)

GD, and $\angle BGD$ by line GH. It is evident, therefore, that the arc which is subtended by angle BGH is $1/192$ of circle ATB and that line BH is $1/2$ of a side of a figure having 96 sides which contains circle ATB.

Since, therefore, this is so, then let us assume line GZ to be 306 because of the facile utility of this number for computation. Therefore, since GZ is 306, $GZ^2 = 93{,}636$. And line $BZ = 153$, since $\angle BGZ = (1/3) \cdot 90°$ and $\angle GBZ = 90°$. And $BZ^2 = 23{,}409$, and $GB^2 = 70{,}227$. Hence, line $GB > 265$. But $(BG + GZ)/BZ = GB/BE$ because line GE bisects $\angle BGZ$. And $(BG + GZ) > 571$ and line $BZ = 153$. Hence, $(GB/BE) > (571/153)$. Hence, line $GB > 571$, when $BE = 153$. And $GB^2 > 326{,}041$. And $BE^2 = 23{,}409$. Hence, $GE^2 > 349{,}450$. Hence, line $GE > 591\frac{1}{8}$.

APPENDIX.

By K. Balagangadharan

I

The main theorems of the foregoing article are collected below; underneath each theorem, its enunciations in the original śloka form and in English are given. In Theorems 3–12, C denotes the circumference of a circle whose diameter is D. The abbreviations employed to denote the references are all those of the article.

I may mention at the outset that the translations appended to the ślokas are not literal, aiming as they do at clarity rather than mere verbal faithfulness to their originals. As for the ślokas themselves, my sources have been the Trivandrum edition of *Karaṇa-paddhati* and the MSS. of *Tantrasaṅgraha* in the Trippunittura Sanskrit College Library and the Adyar Library. All my quotations from *T*, barring those under Theorems 6, 11, 12, 13, follow the Trippunittura MS., the quotations under these theorems following Whish and the Adyar MS.

Theorem 1. $\text{arc}\tan t = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \ldots$ $\qquad (|\text{arc}\tan t| \leqslant \pi/4)$

Ślokā :

व्यासार्धेन हृतादभीष्टगुणतः कोटचाप्तमाद्यं फलं
ज्यावर्गेण विनिघ्नमादिमफलं तत्तत्फलं चाहरेत् ।
कृत्या कोटिगुणस्य तत्र तु फलेष्वेकत्रिपञ्चादिभि-
र्भक्तेष्वोजयुतैस्त्यजेत्समयुतिं जीवाधनुश्शिष्यते ॥

[*K* (T.S.S.), 19, chap. VI.]

Translation.—[Take any circular arc, as in the accompanying figure, whose "abscissa" is not less than its "ordinate".] Multiply the "ordinate" of the arc by the semidiameter and divide it by the "abscissa". This gives the first term. Multiply this term by the square of the "ordinate" and divide it by the square of the abscissa; a second term results. Repeat the process of multiplying by the square of the ordinate and dividing by the square of the abscissa. Thus obtain successive terms and divide them in order by the odd integers 1, 3, 5, . . . If now the terms whose order is odd are added to, and the terms whose order is even subtracted from the preceding, what remains is the circumference.

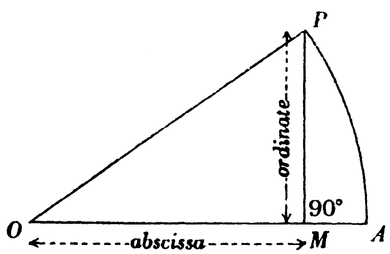


Fig. 3.

That is to say, in the figure (with $A\hat{O}P \leqslant 45°$).

$$\text{arc } AP = OP\left\{\frac{PM}{OM} - \frac{1}{3}\frac{PM^3}{OM^3} + \frac{1}{5}\frac{PM^5}{OM^5} - \ldots\right\}$$

Remarks.—(i) The words "ordinate" and "abscissa" and "arc" in the English translation do duty for the Sanskrit ज्या (literally bowstring) कोटि and धनुस् (bow) respectively. (ii) The restriction on the length of the arc is mentioned by the commentator.

THEOREM 2. *If, in Fig.* 3, $A\hat{O}P < 225'$, *then*

$$PM \doteqdot \text{arc } AP - \frac{1}{6}\frac{(\text{arc } AP)^3}{OP^2},$$

$$\text{arc } AP \doteqdot PM + \frac{1}{6}\frac{PM^3}{OM^2}.$$

Ślokā :

स्वल्पचापघनषष्ठभागतो विस्तरार्धकृतिभक्तवर्जितम् ।
शिष्टचापमिह शिञ्जिनी भवेत् तद्युतोऽल्पगुणोऽसकृद्धनुः ॥

[K * (T.S.S.), 19, chap. VI.]

Translation.—The arc, when small, diminished by the sixth part of its own cube divided by the square of the semidiameter, becomes the ordinate; and often, when small, the arc is equal to the ordinate increased by its cube divided by six times the square of the semidiameter.

Remarks.—(i) As in Theorem 1, the restriction on the arc is due to the commentator. (ii) The English "ordinate" now serves for the Sanskrit शिञ्जिनी.

THEOREM 3. $C = \sqrt{12}\, D\left\{1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \ldots\right\}$

Ślokā :

व्यासवर्गाद्रविहतात्पदं स्यात्प्रथमं फलम् ।
ततस्तत्तत्फलाच्चापि यावदिच्छं त्रिभिर्हरेत् ॥
रूपाद्ययुग्मसंख्याभिर्लब्धेष्वेषु यथाक्रमम् ।
विषमाच्च युते त्यक्ते युग्मयोगे वृतिर्भवेत् ॥

[T, chap. II.]

Translation.—Extract the square root of twelve times the diameter squared. This is the first term. Dividing the first term repeatedly by 3, obtain other terms: the second after one division by 3, the third after one more division and so on. Divide the terms in order by the odd integers 1, 3, 5, . . . ; add the odd-order terms to, and subtract the even-order terms from, the preceding. The result is the circumference.

THEOREM 4. $C = 4D - 4D\left(\frac{1}{3} - \frac{1}{5}\right) - 4D\left(\frac{1}{5} - \frac{1}{7}\right) - \ldots$

* The same verse occurs in T, chap. II, with the last pāda replaced by स्पष्टता भवति चाल्पतावशात् ।

Ślokā :

व्यासाच्चतुर्घ्नाद्बहुशः पृथक्स्थात् त्रिपञ्चसप्ताद्ययुगाहृतानि ।
व्यासे चतुर्घ्ने क्रमशस्त्वृणं स्वं कुर्यात्तदा स्यात्परिधिस्सुसूक्ष्मः ॥

[*K* (T.S.S.), 16, chap. VI.]

Translation.—Divide four times the diameter by each of the odd integers 3, 5, 7, . . . Take away every quotient whose order is even from the one preceding it. Subtract from four times the diameter the combined result of all such small operations. This gives the exact value of the circumference.

THEORRM 5. $$C = 3D+4D\left\{\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\cdots\right\}$$

Ślokā :

व्यासाद्वनसंगुणितात् पृथगाप्तं त्र्याद्ययुग्विमूलघनैः ।
त्रिगुणव्यासे स्वमृणं क्रमशः कृत्वापि परिधिरानेयः ॥

[*K* * (T.S.S.), 16, chap. VI.]

Translation.—Divide four times the diameter separately by the cubes of the odd integers, from 3 onwards, diminished by these integers themselves. These quotients alternately add to and subtract from thrice the diameter. The circumference is again obtained.

THEOREM 6. $$C = 16D\left\{\frac{1}{1^5+4\,.\,1}-\frac{1}{3^5+4\,.\,3}+\frac{1}{5^5+4\,.\,5}-\cdots\right\}$$

Ślokā :

समपञ्चाहतयो यरूपाद्ययुजाश्चतुर्घ्ना मूलयुताः ।
ताभिष्षोडशगुणिताद्व्यासात्पृथगाहृते तु विषमयते ॥
समफलयोगे त्यक्ते स्यादिष्टव्याससंभवः परिधिः ।

[*T*, chap. II.]

Translation.—Sixteen times the diameter is divided by the fifth powers of each of the odd integers 1, 3, 5,......, increased by these integers themselves. The quotients obtained in this order are added to, or subtracted from, the preceding, according as their order is odd or even. What remains after these operations is the circumference.

THEOREMS 7 & 8. $$C = 8D\left\{\frac{1}{2^2-1}+\frac{1}{6^2-1}+\frac{1}{10^2-1}+\cdots\right\}$$

$$C = 4D-8D\left\{\frac{1}{4^2-1}+\frac{1}{8^2-1}+\cdots\right\}$$

Ślokā :

द्व्यादेश्चतुरादेर्वा चतुरधिकानां निरेकवर्गास्स्युः ।
हाराः कुञ्जरगुणितो विष्कंभः स्वमितिकल्पितो भोज्यः ॥
फलयुतिराद्ये वृत्तं भाज्यदलं फलविहीनमन्यत्र ।

[*T*, chap. II.]

* This stanza again is found in *T*, chap. II, with two minor alterations: वनसंगुणित being replaced by वारिधिनिहत and त्रिगुण by त्रिघ्न ।

Translation.—One less than the squares of the successive integers starting with 2 or 4 and increasing steadily by 4, are the successive divisors. Eight times the diameter is divided separately by these and the results are added together to give the circumference in the first case. In the second case the similar sum is subtracted from four times the diameter to give the circumference.

THEOREM 9. $C = 3D+6D\left\{\frac{1}{(2\,.\,2^2-1)^2-2^2}+\frac{1}{(2\,.\,4^2-1)^2-4^2}+\frac{1}{(2\,.\,6^2-1)^2-6^2}+\cdots\right\}$

Ślokā :

वर्गैर्युजां वा द्विगुणैर्निरेकै-
वर्गीकृतैर्वर्जितयुग्मवर्गैः ।
व्यासं च षड्घ्नं विभजेत्फलं स्वं
व्यासे त्रिनिघ्ने परिधिस्तदा स्यात् ॥

[*K* (T.S.S.), 17, chap. VI.]

Translation.—Divide six times the diameter separately by the squares of, twice the squares of the even integers (2, 4, 6, ...) minus 1, diminished by the squares of the even integers themselves. The sum of the resulting quotients increased by thrice the diameter is the circumference.

THEOREM 10. $C \doteqdot 4D\left\{1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{n}\mp\frac{(n+1)/2}{(n+1)^2+1}\right\}$,

where n is odd and large.

Ślokā :

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते ।
त्रिशरादिविषमसंख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात् ॥
यत्संख्ययात्र हरणे कृते निवृत्ता हृतिस्तु जामितया ।
तस्या ऊर्ध्वगतायास्समसंख्याया तद्दलं गुणोऽन्ते स्यात् ॥
तद्वर्गः रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत् ।
तस्यामाप्तं स्वमृणे कृते धने शोधनञ्च करणीयम् ॥
सूक्ष्मः परिधिः सा स्यात् बहुकृत्वो हरणतोऽतिसूक्ष्मश्च ।

[*T*, chap. II.]

Translation.—Multiply the diameter by 4. Subtract from it and add to it alternately the quotients obtained by dividing four times the diameter by the odd integers 3, 5, 7, . . . Let the process stop at a certain stage giving rise to a "finite sum". Multiply four times the diameter by half the even integer subsequent to the last odd integer used as a divisor and then divide by the square of the even integer increased by unity. The result is the correction to be added to or subtracted from our finite sum, the choice of addition or subtraction depending on the sign of the last term in the sum. The final result is the circumference determined more exactly than by taking a large number of terms, i.e. terms going beyond the stage at which we stopped.

THEOREM 11. $C \doteqdot 4D\left\{1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{n}\mp\frac{\left(\frac{n+1}{2}\right)^2+1}{\left[(n+1)^2+4+1\right]\left(\frac{n+1}{2}\right)}\right\}$

n being odd and large.

Ślokā :

अस्मात्सूक्ष्मतरोऽन्यो विलिख्यते कश्चनापि संस्कारः ।
अन्ते समसंख्यादलवर्गस्सैको गुणस्स एव पुनः ॥
युगगुणितो रूपायुतस्समसंख्यादलहतो भवेद्धारः ।
त्रिशरादिविषमसंख्याहरणात्परमे तदेव वा कार्यम् ॥

[T, chap. II.]

Translation.—Next is given another correction more precise than the foregoing. The square of half the even integer next greater than the last odd-integer divisor, increased by unity, is a multiplier. This multiplier multiplied by 4, then increased by unity and then multiplied by the even integer already defined, gives a divisor. Multiply and divide four times the diameter by our multiplier and divisor respectively. The result is an improvement on our previous correction.

THEOREM 12. $C \doteqdot 2D+4D\left\{\frac{1}{2^2-1}-\frac{1}{4^2-1}+\cdot\cdot\cdot\pm\frac{1}{n^2-1}\mp\frac{1}{2[(n+1)^2+2]}\right\}$

n being even and large.

Ślokā :

द्व्यादियुजां विकृतयो व्येका हारा द्विनिघ्ना विष्कंभे ।
धनमृणमन्तेऽन्त्योर्ध्वगतौजकृतिर्द्विसहितहारो द्विघ्नः ॥

[T, chap. II.]

Translation.—Divide four times the diameter separately by the squares of the even integers (2, 4, 6, . . .) diminished by unity. The quotients alternately add to and subtract from twice the diameter, the process terminating at a certain stage and defining a "finite sum". Take the odd integer subsequent to the last even integer squared, square it, add 2 to the square, double the sum, and with the result thus obtained, divide four times the diameter. This quotient added to or subtracted from the finite sum defined, leads to a corrected value of the circumference.

II

The passage from the work *Āryabhaṭīya of Āryabhaṭācārya with the Bhāṣya of Nīlakaṇṭhasomsutvān*, referred to in footnote 18 of the article is part of the following.

कुतः पुनर्वास्तवीं संख्यामुत्सृज्यासन्नैवेहोक्ता । उच्यते । तस्या वक्तुमशक्यत्वात् । कुतः । येन मानेन मीयमानो व्यासो निरवयवः स्यात्, तेनैव मीयमानः परिधिः पुनः सावयव एव स्यात् । येन च मीयमानः परिधिर्निरवयवस्तेनैव मीयमानो व्यासोऽपि सावयव एव, इत्येकेनैव मानेन मीयमानयोरुभयोः क्वापि न निरवयवत्वं स्यात् । महान्तमध्वानं गत्वाप्यल्पावयवत्वमेव लभ्यम् । निरवयवत्वं तु क्वापि न लभ्यमिति भावः ॥ [*A* I (T.S.S.), 41*f*.]

Translation.—Why then is it that discarding the exact value, only the approximate one has been mentioned here? This is the answer: because it (the exact value) cannot be mentioned. If the diameter, measured with respect to (by comparison with) a particular unit of measurement, is commensurable, with respect to that same unit of measurement, the circumference is incommensurable (the circumference cannot be *exactly* measured by the same unit); and if with respect to any unit the circumference is commensurable, then, with respect to that same unit, the diameter is incommensurable. Thus there will

6

never be commensurability for both with respect to the same unit of measurement. Even by going a long way, only the "degree of commensurability" can be made very small, absolute commensurability can never be attained.

III

Prince Rāma Varma (12th prince of Cochin) has brought to light a fact which goes some way towards substantiating the conjecture that the proofs in *Yukti-Bhāṣa* are almost as old as *Tantrasaṅgraha*. He has in his possession a Sanskrit manuscript commentary on Bhaskara's *Līlāvati*, by an unknown Keralīya. This commentary, which bears the title *Kriyākramakarī*, contains the proofs of many of our theorems in forms which lead one to suppose that they are the originals of the proofs in *Yukti-Bhāṣa*. There is a verse in the commentary:

नारायणं जगदनुग्रहजागरूकं
श्रीनीलकण्ठमपि सर्वविदं प्रणम्य ।
व्याख्यां क्रियाक्रमकरीं रचयामि लीला-
वत्याः कथञ्चिदहमल्पधियां हिताय ॥

the first two pādas of which are identical with those of the third benedictory stanza in the commentary attached to *Tantrasaṅgraha*. This identity corroborates the statement in the verse that the author of *Kriyākramakarī* is a student of Nīlakaṇṭha and its Kali day of compilation (suggested by the underlined chronogram in the verse) is the 1681915th. Thus *Kriyākramakarī* would seem to be a work belonging to the same period as *Tantrasaṅgraha* (4602 Kali era); and it is not unlikely that a close study of it will lead to valuable conclusions regarding the origin of the mathematical arguments in *Yukti-Bhāṣa*.

CORRESPONDENCE.

LUDOLPH (OR LUDOLFF OR LUCIUS) VAN CEULEN.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—In the archives of St. Pieter's Kerk in Leiden, Holland, this epitaph is recorded :

HIC IACET SEPULTUS MR. LUDOLFF VAN CEULEN, PROFESSOR BELGICUS, DUM VIVERET MATHEMATICARUM SCIENTIARUM IN ATHENAEO HUIUS URBIS, NATUS HILDESHIMIA ANNO 1540, DIE XXVIII IANUARII, ET DENATUS XXXI DECEMBRIS, 1610, QUI IN VITA SUA MULTO LABORE CIRCUMFERENTIAE CIRCULI PROXIMAM RATIONEM AD DIAMETRUM INVENIT SEQUENTEM. QUANDO DIAMETER EST 1, TUM CIRCULI CIRCUMFERENTIA PLUS EST QUAM

$$\frac{314159265358979323846264338327950288}{100000000000000000000000000000000000}$$

ET MINUS QUAM

$$\frac{314159265358979323846264338327950289}{100000000000000000000000000000000000};$$

SED QUANDO DIAMETER EST

$$100000000000000000000000000000000000,$$

TUM EST CIRCULI CIRCUMFERENTIA PLUS QUAM

$$314159265358979323846264338327950288$$

& MINUS QUAM

$$314159265358979323846264338327950289.$$

On my first visit to Holland in 1935 I tried to locate van Ceulen's tombstone, in the hope of presenting the Mathematical Association with a rubbing of this interesting inscription ; but the grave had changed hands several times, and the coveted epitaph, if still in existence at all, was facing downwards on the underside of some stone recording on its upper surface the usual entirely fictional virtues of some lesser Dutchman, who employed his leisure in some more conventional way than the calculation of π by a method not far in advance of that which Archimedes had employed eighteen centuries before.

In preparation for a renewed attempt to recover the original epitaph, I have lately appealed to Dr. C. de Jong, President of the "Liwenagel" (Leeraren in Wiskunde en Natuurwetenschappen aan Gymnasia en Lycea), roughly the equivalent of our Mathematical Association. His answer, I think, will be of some interest to our members.

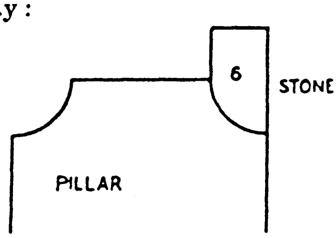
"LEIDEN, 21*st March*, 1938.

DEAR MR. HOPE-JONES,

It was a great pleasure to me to be able to help you in your attempts to discover the epitaph of Ludolph van Ceulen in St.

T

Pieter's Church at Leiden. With the help of Miss Le Poole I have succeeded in finding out some points, which will surely interest you. We have discovered that Ludolph's grave was exchanged for another grave by his widow, Dec. 31st, 1610. In the year 1626, Aug. 10th, the grave was sold by the Church-masters to Jonkheer Christoffel van Sac, and afterwards to Mr. Adriaen van Hogeveen (1718). According to the archives, Ludolph's first grave was nr 6 in the 'High Choir'. Now, after a long search, I have found a piece of a tombstone there, *carrying the number* 6, *but nothing else.* Part of this stone has been cut off so as to fit to one of the great pillars of the Church, in this way :



For this reason I doubt if it will be worth while to turn the stone upside-down ; for, in the most favourable case, you will find only part of van Ceulen's epitaph, and certainly not the whole of it.

I shall be very glad to help you further if you want so. In this case I would like you to give me further directions. I regret that I shall not be in town during the coming School Holidays.

Yours sincerely,

C. de Jong."

It is presumably through some error in the archives that van Ceulen's widow is recorded as having exchanged his grave on the same day on which, according to his epitaph, he was " denatus ", or " disborn ".

I hope that I may speak for all members of the Mathematical Association, not only in passing a vote of censure on the Vandals who destroyed such a treasure, but even more in thanking Dr. de Jong most heartily for his co-operation in solving the mystery of its disappearance.

Yours truly,

W. Hope-Jones.

23rd March, 1938.

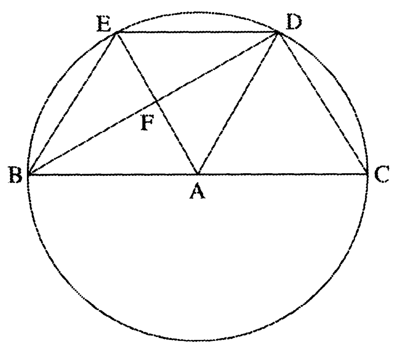
CAPVT XVIII.

Polygonorum circulo ordinate inscriptorum ad circulum ratio.

QUadravit parabolen Archimedes inscriptione continua triangulorum existentium ἐν λόγῳ ῥητῷ. Quoniam enim triangulo maximo parabolæ inscripto, superinscripsit triangula in continua ratione ad maximum illud constanter subquadrupla in infinitum: Ideo conclusit parabolen esse maximi illius trianguli sesquitertiam. At ita circulum quadrare nescivit Antiphon, quoniam circulo inscripta continue triangula existunt ἐν λόγῳ ἀῤῥήτῳ, & vago. An igitur circulus non poterit quadrari? Si enim figura composita ex triangulis in ratione subquadrupla ad datum maximum triangulum constitutis in infinitum, sit ad idem sesquitertia, infinitorum aliqua scientia est. Et figura quoque plana poterit componi ex triangulis circulo in infinitum continue inscriptis ἐν λόγῳ, licet ἀῤῥήτῳ, & vago. Et composita illa ad maximum triangulum inscriptum aliquam habebit rationem. Valebunt autem Euclidæi adserentes angulum majorem acuto & minorem obtuso non esse rectum. Circa hæc, ut liceat liberius Philosophari de incerta illa & inconstanti polygoni cujusvis ordinate inscripti ad polygonum infinitorum laterum, seu, si placet, circulum, ita propono.

PROPOSITIO I.

Si eidem circulo inscribantur duo ordinata polygona, numerus autem laterum vel angulorum primi, sit subduplus ad numerum laterum vel angulorum secundi: erit polygonum primum ad secundum, sicut apotome lateris primi ad diametrum.



Apotomen lateris voco subtensam peripheriæ, quam relinquit è semicirculo ea cui latus subtenditur.

In circulo igitur cujus A centrum, diameter BC, inscribatur polygonum quodcunque ordinatum, cujus latus sit BD. Secta vero circumferentia BD bifariam in E, subtendatur BE. Itaque inscribatur aliud polygonum ordinatum cujus latus sit BE. Numerus igitur laterum vel angulorum polygoni primi, erit subduplus ad numerum laterum vel angulorum secundi. Connectatur autem DC. Dico polygonum primum cujus latus BD ad polygonum secun-

ſecundum cujus latus BE vel ED eſſe, ut DC ad BC. Iungantur enim DA, ED. Conſtat igitur polygonum primum tot triangulis B A D, quot exiſtunt latera vel anguli polygoni primi. Polygonum autem ſecundum conſtat totidem trapeziis BEDA. Polygonum igitur primum ad polygonum ſecundum ſe habet, ut triangulum B A D ad trapezium BEDA. Quod quidem trapezium B E D A dividitur in duo triangula BAD, BED, quorum baſis communis eſt B D. Triangula autem quorum eadem eſt baſis ſunt ut altitudines. Agatur itaque ſemidiameter A E, ſecans B D in F. Quoniam igitur circumferentia B D ſecta eſt bifariam in E, acta A E ſecat B D ad rectos angulos. Itaque A F eſt altitudo trianguli B D A, & F E altitudo trianguli BED. Quare triangulum B A D ad triangulum BED eſt, ut AF ad E F, & componendo triangulum BAD ad triangula B A D, B E D ſimul juncta, id eſt trapezium B E D A, ſicut AF ad AE. Qua adeo in ratione erit etiam polygonum primum ad polygonum ſecundum. Sed AF ad AE ſeu AB eſt, ut DC ad BC. Eſt enim angulus BDC rectus ſicut BFA. & ideo ſunt parallelæ A F, D C. Eſt igitur polygonum primum, cujus latus BD, ad polygonum ſecundum, cujus latus BE vel ED, ſicut D C ad B C. Quod erat oſtendendum.

PROPOSITIO II.

Si eidem circulo inſcribantur polygona ordinata in infinitum, & numerus laterum primi ſit ad numerum laterum ſecundi ſubduplus, ad numerum vero laterum tertii ſubquadruplus, quarti ſuboctuplus, quinti ſubſexdecuplus, & ea deinceps continua ratione ſubdupla.

Erit polygonum primum ad tertium, ſicut planum ſub apotomis laterum polygoni primi & ſecundi ad quadratum a diametro.

Ad quartum vero, ſicut ſolidum ſub apotomis laterum primi ſecundi & tertii polygoni ad cubum à diametro.

Ad quintum, ſicut plano-planum ſub apotomis laterum primi ſecundi tertii & quarti ad quadrato-quadratum à diametro

Ad ſextum, ſicut plano-ſolidum ſub apotomis laterum primi ſecundi tertii quarti & quinti polygoni ad quadrato-cubum à diametro.

Ad ſeptimum, ſicut ſolido-ſolidum ſub apotomis laterum primi ſecundi tertii quarti quinti & ſexti polygoni ad cubo-cubum à diametro. Et eo in infinitum continuo progreſſu.

Sit enim apotome lateris polygoni primi B, ſecundi C, tertii D, quarti F, quinti G, ſexti H. Et ſit diameter circuli Z. Ex antecedente igitur propoſitione polygonum primum ad polygonum ſecundum erit, ut B ad Z. Itaque quod fit ex B in polygonum ſecundum, erit æquale ei quod fit ex Z in polygonum primum; polygonum vero ſecundum ad tertium erit, ut C ad Z. Et per conſequens, quod fit ſub polygono ſecundo & B, id eſt quod fit ſub primo & Z ad id quod fit ſub polygono tertio & B, ſicut C ad Z. Quare quod fit ſub polygono primo & Z quadrato, æquale eſt ei quod fit ſub polygono tertio & plano B in C. Eſt igitur polygonum primum ad polygonum tertium, ſicut planum B in C ad Z quadratum. Et quod fit ſub tertio & plano B in C, æquale erit ei quod fit ſub primo & Z quadrato. Rurſus ex eadem antecedente propoſitione eſt, ut polygonum tertium ad polygonum quartum, ſicut D ad Z. Et per conſequens. Quod fit ſub tertio & plano B in C, id eſt quod fit ſub primo & Z quadrato ad id quod fit ſub quarto & plano B in C, eſt ſicut D ad Z. Quare quod fit ſub primo & Z cubo, æquale erit ei quod fit ſub quarto & ſolido B in C in D. Eſt igitur polygonum primum ad quartum, ſicut B in C in D ad Z cubum. Eademque demonſtrationis methodo erit ad quintum, ſicut B in C in D in F ad Z quadrato-quadratum. Ad ſextum, ſicut B in C in D in F in G ad Z quadrato-cubum. Ad ſeptimum, ſicut B in C in D in F in G in H ad Z cubo-cubum. Et eo conſtanti in infinitum progreſſu.

COROLLARIVM.

Itaque quadratum circulo inſcriptum erit ad circulum, ſicut latus illius quadrati ad poteſtatem diametri altiſſimam adplicatam ad id quod fit continue ſub apotomis laterum octogoni, hexdecagoni, polygoni triginta duorum laterum, ſexaginta quatuor, centum viginti octo, ducentorum quinquaginta ſex, & reliquorum omnium in ea ratione angulorum laterumve ſubdupla.

Sit enim quadratum circulo inſcriptum polygonum primum, octogonum erit ſecundum, hexdecagonum tertium, polygonum triginta duorum laterum quartum, & eo continuo ordine. Itaque erit, ut quadratum circulo inſcriptum ad polygonum extremum ſeu infinitorum laterum, ſicut quod fit ſub apotomis laterum tetragoni, octogoni, hexdecagoni, & reliquorum omnium in ea ratione ſubdupla in infinitum, ad poteſtatem diametri altiſſimam. Et per adplicationem communem, ſicut apotomes lateris quadrati ad poteſtatem diametri altiſſimam adplicatam ad id quod fit ſub apotomis laterum octogoni, hexdecagoni, & reliquorum omnium in ea ratione ſubdupla in infinitum. Eſt autem apotome lateris quadrati circulo inſcripti ipſi lateri æqualis, & polygonum infinitorum laterum circulus ipſe.

Sit circuli diameter 2. Latus quadrati ei circulo inſcripti ſit $\sqrt{2}$, *quadratum ipſum 2. Apotome lateris octogoni* $\sqrt{2+\sqrt{2}}$. *Apotome lateris hexdecagoni* $\sqrt{2+\sqrt{2+\sqrt{2}}}$. *Apotome lateris polygoni triginta duorum laterum* $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$. *Apotome lateris polygoni ſexaginta quatuor laterum* $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$. *Et eo continuo progreſſu.*

Sit autem diameter 1. Circulus 1 N. Erit $\frac{1}{2}$ *ad 1 N, ſicut* $\sqrt{\frac{1}{2}}$ *ad unitatem adplicatam ad id quod ſit ex* $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}$, *in* $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}}$, *in* $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}}}$, *in* $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}}}}$.

Sit diameter X. Circulus A planum. Erit X quadratum $\frac{1}{2}$ ad A planum, ſicut L. X quadrati $\frac{1}{2}$ ad X poteſtatum maximam adplicatam ei quod fit ex radice binomia X quadrati $\frac{1}{2}$, + radice X quadrato-quadrati $\frac{1}{2}$, in radicem binomiam X quadrati $\frac{1}{2}$, plus radice binomiæ X quadrato-quadrati $\frac{1}{2}$, + radice X quadrato-quadrato-quadrato-quadrati $\frac{1}{2}$, in radicem binomiæ X quadrati $\frac{1}{2}$, plus radice binomiæ X quadrato-quadrati $\frac{1}{2}$, + radice binomiæ X quadrato-quadrato-quadrato-quadrati $\frac{1}{2}$, + radice X quadrato-quadrato-quadrato-quadrato-quadrato-quadrato-quadrato-quadrati $\frac{1}{2}$, in radicem &c. in infinitum obſervata uniformi methodo.

CAPVT XIX.

Πρόχειρον, *ſeu ad uſum Mathematici Canonis methodica.*

AT vos, ô nobiles ſiderum obſervatores, miſſa facta matæotechnia ad veram Cyclometriam revoco, hoc eſt, ad legitimum Mathematici Canonis uſum. Ut enim vulgo peccatur in ejus fabrica, ſic etiam in uſu. Itaque dum renovatur meus ad Canonem inſpectionum liber, Analyticæ meæ methodi, qua ſoleo expedire me à triangulis planis ac ſphæricis, ultro copiam facio ad excitandum veſtra ſtudia per aliquod laborum, quos ſuſtinetis in abacis Aſtronomicis, ſublevamen. Neque vero obruent vos multitudine præcepta. Tum enim negocium viginti & uno δεδομένοις fere abſolvo.

Δεδο-

Δεδομένον I.

TRIANGULI PLANI RECTANGULI

Datis angulis, dantur latera in partibus Canonis.

Enimvero,

Ex canonica serie prima:

Perpendiculum fiet simile sinui anguli acuti. Basis, sinui reliqui è recto. Hypotenusa, sinui toto.

Vel,

Ex serie secunda:

Perpendiculum fiet simile prosinui anguli acuti. Basis, sinui toto. *Hypotenusa, transsinuosæ anguli acuti.*

Vel denique,

Ex serie tertia:

Perpendiculum fiet simile sinui toto. *Basis, prosinui anguli reliqui. Hypotenusa, transsinuosæ ejusdem.*

Vel etiam,

Mixtim ex Canonis serie trina:

Perpendiculum fiet simile differentiæ inter transsinuosam anguli acuti & sinum reliqui è recto. Basis, sinui acuti. Hypotenusa, prosinui ejusdem.

Vel,

Perpendiculum fiet simile differentiæ inter sinum anguli acuti & transsinuosam reliqui è recto. Basis, sinui reliqui è recto. Hypotenusa, prosinui ejusdem.

Vel denique,

Perpendiculum fiet simile transsinuosæ anguli acuti. Basis, transsinuosæ reliqui è recto. Hypotenusa, adgregato prosinus acuti & prosinus reliqui è recto.

Δεδομένον II.

Trianguli plani rectanguli.

Data hypotenusa ac perpendiculo vel base, dantur anguli,

Enimvero erit,

Ut hypotenusa ad sinum totum, *ita perpendiculum ad sinum anguli acuti. Et ita basis ad sinum reliqui è recto.*

Aliter erit,

Ut basis ad sinum totum, *ita hypotenusa ad transsinuosam anguli acuti.*

Vel,

Ut perpendiculum ad sinum totum, *ita hypotenusa ad transsinuosam anguli reliqui è recto.*

III.

Trianguli plani rectanguli.

Datis perpendiculo & base, dantur anguli.

Enimvero erit,

Ut basis ad perpendiculum, ita sinus totus *ad prosinum anguli acuti.*

Vel,

Ut perpendiculum ad basin, ita sinus totus *ad prosinum anguli reliqui è recto.*

TRIAN-

FRANCISCI VIETÆ

MVNIMEN

ADVERSVS NOVA CYCLOMETRICA,

Seu,

ΑΝΤΙΠΕΛΕΚΥΣ

FRANCISCI VIETÆ

MUNIMEN

ADVERSUS NOVA CYCLOMETRICA,

Seu,

ΑΝΤΙΠΕΛΕΚΥΣ.

LUSERUNT illi operam infeliciter, qui suis, quas Seculas vocant, figuris conati sunt circulum triginta sex segmentis hexagoni adæquare. Quid enim certi ex magnitudinibus plane incertis poterant resolvendo consequi? Æqualia æqualibus addant vel subtrahant, per æqualia dividant aut multiplicent, invertant, permutent, ac denique per quoscunque proportionum gradus deprimant, vel attollant, hilum sua Zetesi non proficient. sed in vicium, quod Logici appellant αἴτημα τοῦ αἰτήματος, Diophantæi ἀνισότητα, incident, aut demum falso scipsos deludent calculo, ut præsensissent, si qua lux eis adfulsisset veræ analyticæ doctrinæ. Sunt autem imbelles, qui μονοστόμους istas bipennes reformidant, & jam ab iis sauciatum deflent Archimedem. Sed vivit Archimedes. Neque enim cum offendunt ψευδογραφήματα περὶ τὸ ἀληθὲς, ψευδοψηφοφορίαι, Anapodixes, verba magnifica. Quo tamen undique sint tutiores,

Nubigeros clypeos, intactaque cædibus arma,

sed δυσαπελέκητα, quibus primum sese muniant, profero, subministraturus πολεμικὰ, si forte hostium ferocior audacia est.

PROPOSITIO I.

AMBITUS dodecagoni circulo inscripti, minorem habet rationem ad diametrum, tripla sesquioctava.

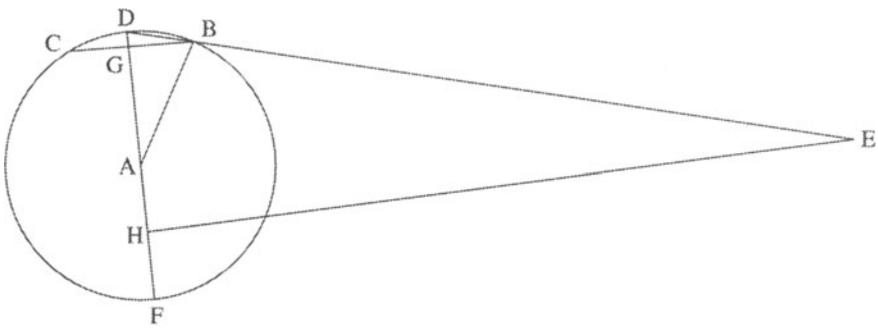
Centro A intervallo quocunque A B describatur circulus B C D, in quo sumatur B C circumferentia hexagoni, quæ secetur bifariam in D, & subtendatur D B. Est igitur D B latus dodecagoni, quo duodecuplato in E, erit D E æqualis ambitui dodecagoni circulo B D C inscripti. Agatur autem diameter D F. Dico D E ad D F, rationem habere minorem tripla sesquioctava.

Jungantur enim B C, B A, ipsamque B C diameter D F secet in G. Ergo bifariam & ad angulos rectos secabit. Triangulo autem D B G construatur simile triangulum D E H.

Quoniam recta B C subtenditur circumferentiæ hexagoni, ideo B A seu D A ipsi B C est æqualis. Quare constituta A C seu B C partium octo, sit B G earundem quatuor. Quadratum vero abs A G est 48, & ideo A G sit major $6\frac{12}{13}$. Ipsa autem D G minor $1\frac{1}{13}$. Et cum constituta sit D E duodecupla ipsius D B, erit quoque E H duodecupla

Hhh 2 ipsius

ipsius B G, & D H duodecupla ipsius D G. Quare erit E H earundem partium 48. D H vero minor 13. Immo minor $12\frac{12}{13}$. Quadratum autem à latere 48, est 2304; abs 13 ve-



ro, 169. Quæ duo quadrata conficiunt 2473, non etiam 2500, quadratum à latere 50. Quare recta D E, cujus quadratum æquale est quadratis E H, D H, minor est 50. At vero ratio 50 ad 16; est tripla sesquioctava accurate. Ratio igitur D E ad D F, minor est tripla sesquioctava. Quod erat ostendendum.

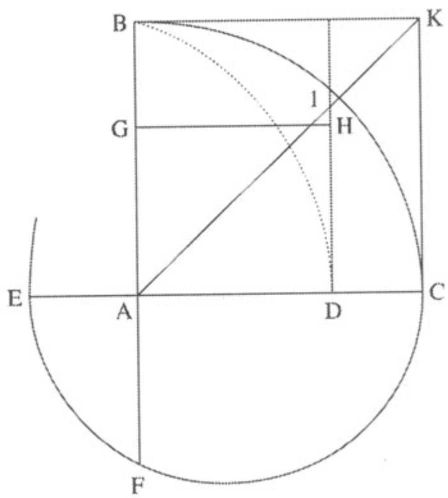
Omnino Arithmetica tam scientia est quam Geometria. Magnitudines rationales rationalibus numeris, irrationales irrationalibus commode designantur. Qui per numeros magnitudines metitur, si suo calculo alias is deprehendit, quam re ipsa sint, non arti sed artificis culpa est.

Immo vero, ait Proclus, est ἀριθμητικὴ ἀκριβεστέρα γεωμετρίας. *Accurate supputanti constituta diametro partis unius, ambitus dodecagoni inscripti sit latus binomiæ* $72 - \sqrt{3888}$. *Qui contra pronunciaverit, errat vel in mensuris Geometra, vel in numeris Epilogista.*

Ambitus autem circuli ad diametrum majorem esse tripla sesquioctava sicuti minorem tripla sesquiseptima non dubitavit hactenus Mathematicorum schola. Id enim vere demonstravit Archimedes. Non igitur è falso Epilogismo inducendum fuit ἀπόπημα ὀφθαλμοφανές. *Lineam rectam esse circulari iisdem terminis contenta majorem, contrarium sumente Archimede* ἐκ τῆς κοινῆς ἐννοίας, *& ipsum etiam demonstrante Eutocio, ac generaliter definiente* πασῶν τῶν τὰ πέρατα ἐχουσῶν γραμμῶν ἐλαχίστην εἶναι τὴν εὐθεῖαν.

PROPOSITIO II.

Semidiametri circuli à quadrataria divisæ pars à centro ad quadratariam, major est media proportionali inter semidiametrum & duas quintas semidiametri.



Sit quadrans circuli A B C, quadrataria B D; sumatur A E æqualis duabus quintis semidiametri A B vel A C; media vero proportionalis inter A E, A C, sit A F. Dico A D majorem esse quam A F.

Ex iis enim, quæ de quadrataria à Pappo demonstrata sunt, semidiameter A B seu A C, me-

A C, media est proportionalis inter circumferentiam B C & A D. Sit A B partium 7. Circumferentia B C, quæ quadrans est perimetri, erit minor 11. Nam diametro existente 14, perimeter minor est 44. Sit autem A B 35. Circumferentia B C minor erit 55. Quod vero fit sub A D, B C, æquale est quadrato ex A B. Quare erit A D major $22\frac{3}{11}$. Qualium autem A B, id est A C, valet 35, talium est A E 14; A F vero minor $22\frac{3}{22}$. Erit igitur A D major quam A F. Quod erat ostendendum.

Itaque si ex diametro A B *abscindatur recta* A G *ipsi* A F *æqualis, & compleatur parallelogrammum* G H D A, *ipsum erit* ἑτερόμηκες, *non quadratum. Et cum complebitur quadratum* B C, *acta diagonia* B K *non transibit per* H, *sed per aliquod* I *punctum remotius à* D *puncto. Quod ad vitandum Pseudographema præstabat adnotasse.*

PROPOSITIO III.

Quadratum ab ambitu circuli, minus est decuplo quadrati à diametro.

Sit enim diameter 7. Diametri quadratum erit 49. Ipsius vero decuplum 490. At ambitus circuli minor erit 22, & proinde quadratum ab ambitu minus 484.

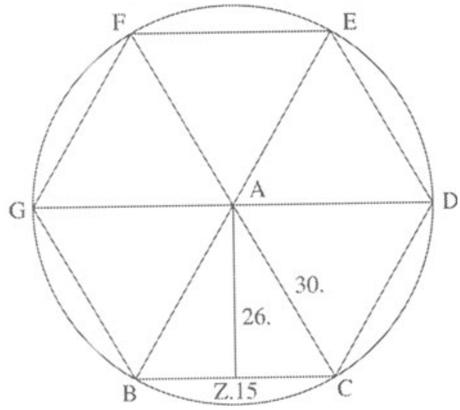
Fuit autem hæc Arabum in quadrando circulo jamdiu explosa sententia, Quadratum ab ambitu circuli *esse* decuplum quadrati à diametro. *Neque vero ferendus est, qui adversus demonstrantem Archimedem* ἀντιγραφικῶς *Anapodicta proposuerit.*

PROPOSITIO IV.

Circulus ad hexagonum ei inscriptum rationem habet majorem, quam sex ad quinque.

Circulo, cujus A centrum, inscribatur hexagonum B C D E F G. Dico circulum cujus A centrum ad hexagonum BCDEFG rationem habere majorem, quam sex ad quinque.

Iunctis enim A B, A C, B C, cadat in B C perpendicularis A Z.



Quoniam igitur in triangulo A B C crura A B, A C æqualia sunt, basis secta est bifariam in Z & sunt æquales B Z Z C. Triangulum autem æquilaterum est A B C. Crura enim ambo sunt semidiametri. Sed & basis, cum sit latus hexagoni, semidiametro est æqualis. Constituta igitur semidiametro B A seu A C 30, sit B Z seu Z C 15, A Z vero sit minor 26, cujus quadratum est 676. Differentia vero quadratorum A B, B Z est duntaxat 675. Quod fit porro sub B Z, A Z rectangulum, triangulo B A C est æquale. Ducatur itaque 15 in 26, fiunt 390. Qualium igitur quadratum A B erit 900, talium triangulum A B C erit minus 390, vel (omnibus divisis per 30) existente quadrato A B 30, triangulum A B C erit minus 13. Iungantur A D, A E, A F, A G. Constat igitur hexagonum B C D E F G triangulis sex æqualibus ipsi B A C. Quare qualium quadratum A B erit 30, talium hexagonum erit minus 78. Vel qualium quadratum A B erit quinque; talium hexagonum erit minus partibus tredecim.

Hhh 3 At

At vero, eſt ut perimeter circuli ad diametrum, ita quod fit ſub perimetro circuli & quadrante diametri ad id quod fit ſub diametro & quadrante diametri. Sed id quod fit ſub perimetro circuli & quadrante diametri, eſt æquale circulo. Quod autem fit ſub diametro & quadrante diametri, ipſum eſt quadratum à ſemidiametro. Ergo eſt ut perimeter ad diametrum, ita circulus ad quadratum è ſemidiametro. Qualium autem diameter eſt 1, talium perimeter major eſt $3\frac{10}{71}$, & tanto manifeſtius major $3\frac{10}{80}$ ſeu $3\frac{1}{8}$. Qualium igitur quadratum ſemidiametri A B erit quinque, ut ante, talium circulus erit major $15\frac{5}{8}$, Hexagonum autem in iiſdem partibus fuit minus 13. Quare circulus ad hexagonum ei inſcriptum majorem habebit rationem quam $15\frac{5}{8}$ ad 13, id eſt, quam 125 ad 104, ſeu 6 ad $4\frac{124}{125}$, & tanto evidentius majorem, quam ſex ad quinque. Quod erat oſtendendum.

Non igitur κατὰ τὸ πρᾶγμα circulum quadrant, qui cum hexagono & quintæ parti hexagoni ſtatuunt æqualem, cum ſit major ſecundum limites ab Archimede ἐκ τῶν ἰδίων ἀρχῶν præſtitutos. Scholæ autem noſtræ Platonicæ ſunt, ô profeſſores candidi. Quare ne principiis Geometricis obluctamini. Et vero ut circulum truncarunt πλεκτικῶς, ſic in damni accepti compenſationem caudæ ſuæ hirundinum acutiorem verſus partem jam decurtentor.

PROPOSITIO V.

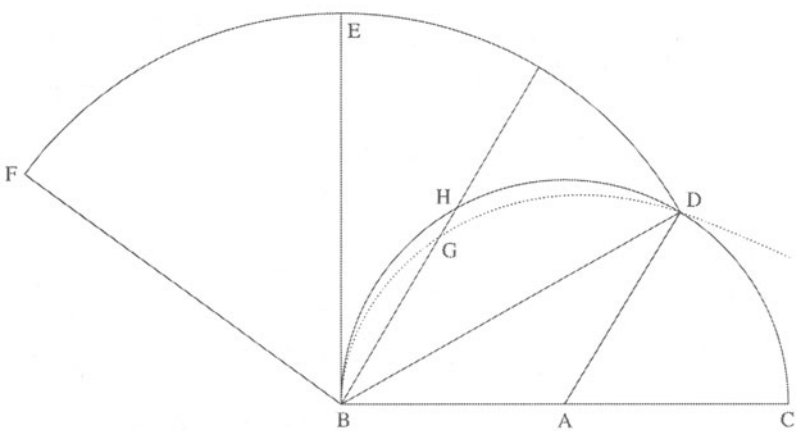
Triginta ſex hexagoni ſegmenta majora ſunt circulo.

Quoniam enim circulus ad hexagonum ei inſcriptum majorem habet rationem, quam ſex ad quinque, ſeu as ad dextantem, ideo differentia inter circulum & hexagonum erit major ſextante circuli. Sed differt circulus ab hexagono per ſex ſegmenta hexagoni. Sex igitur ſegmenta hexagoni ſuperant ſextantem circuli, atque adeo triginta ſex ſegmenta erunt aſſe circulo ve majora. Quod erat oſtendendum.

PROPOSITIO VI.

Omne ſegmentum circuli majus eſt ſextante ſectoris ſimilis, ſimiliterque deſcripti in eo circulo, cujus ſemidiameter baſi ſegmenti propoſiti eſt æqualis.

In deſcripto ſub A centro, circulo B D C, ſubtendatur quævis circumferentia BD; tangat autem circulum recta B E, & centro B intervallo B D deſcribatur circulus alter DEF.



Circumferentia igitur E D ſimilis erit ſemiſſi circumferentiæ B D. Itaque ſumatur D I circumferentia ipſius DE dupla, & jungantur BF, AD. Similes igitur erunt ſectores BAD FBD. Dico ſegmentum circuli B D C contentum recta B D & circumferentia, cui ea ſubtenditur, eſſe majus ſextante ſectoris F B D.

Deſcri

Deſcribatur enim linea ſpiralis, cujus principium B, tranſitus per D, exiſtente BD tanta parte principii converſionis BEZ, quanta pars eſt angulus E B D quatuor rectorum. Sectoris igitur E B D tertia pars eſt ſpatium contentum recta B D & ſpirali. Id enim poſt Archimedem Pappus demonſtravit propoſitione XXII. libri IV. Mathematicarum collectionum. Sectoris vero FBD ſpacium idem erit pars conſequenter ſextupla. duplus enim conſtructus eſt ſector F B D ad ſectorem EBD. Neque vero ſpiralis concurret cum circulari. Id enim eſſet abſurdum. Sed neque ſpiralis in progreſſu egredietur circulum priuſquam ad D punctum pervenerit. Secetur enim angulus E B D utcunque à recta B G H, intercipiente ſpiralem in G, circumferentiam in H. Recta igitur BD ad rectam BG erit, ut angulus EBD ad angulum EBG; id eſt, ut circumferentia BD ad circumferentiam BH, ex conditionibus helicῶn. At major eſt ratio circumferentiæ BD ad circumferentiam BH, quam ſubtenſæ BD ad ſubtenſam BH. Majores enim circumferentiæ ad minores majorem habent rationem, quam rectæ ad rectas, quæ iiſdem circumferentiis ſubtenduntur. Quare recta BH rectam BG excedet. Idemque in quibuſlibet rectis, angulum EBD ſecantibus, accidet. Itaque tranſibit ſpiralis ſub circumferentia B D, & aliquod ſpacium inter ſe & circumferentiam relinquet. Quo quidem ſpacio ſegmentum circuli contentum recta B D & circumferentia, excedit ſpacium, quod ab eadem recta & ſpirali comprehenditur, & ſextanti ſectoris F B D oſtenſum eſt æquale. Segmentum igitur illud erit majus ſextante ſectoris F B D. Quod erat demonſtrandum.

COROLLARIUM.

Atque hinc quoque manifeſtum eſt, triginta ſex ſegmenta hexagoni eſſe circulo majora.

Quando enim eveniet B D eſſe ſegmentum hexagoni, ſectores F B D, B A D erunt æquales, quoniam ſuorum circulorum ſemidiametri B D, A D erunt æquales. Sex igitur ſegmenta hexagoni erunt ſectore B A D majora, atque adeo triginta ſex ſegmenta majora ſex ſectoribus, id eſt, toto circulo.

Potuit non minus generale Theorema, per parabolas, aut potius ea, quibus parabola quadrantur, Geometrica media demonſtrandum ita proponi, Omne ſegmentum circuli majus eſt ſeſquitertio trianguli iſoſcelis ipſi ſegmento immota baſe inſcripti. *Secundum quod ſtatim adparebit majorem eſſe rationem triginta ſex ſegmentorum hexagoni ad circulum, quam* 48 *ad* 47. *Immo etiam accuratius ſupputanti ſola triginta quatuor ſegmenta, & ſpacium paulo majus beſſe ſegmenti, ſed minus dodrante, deprehendentur complere circulum. Licet autem hyperochen ſegmenti ſupra trientem unciæ circuli ita oculis exhibere.*

PROPOSITIO VII.

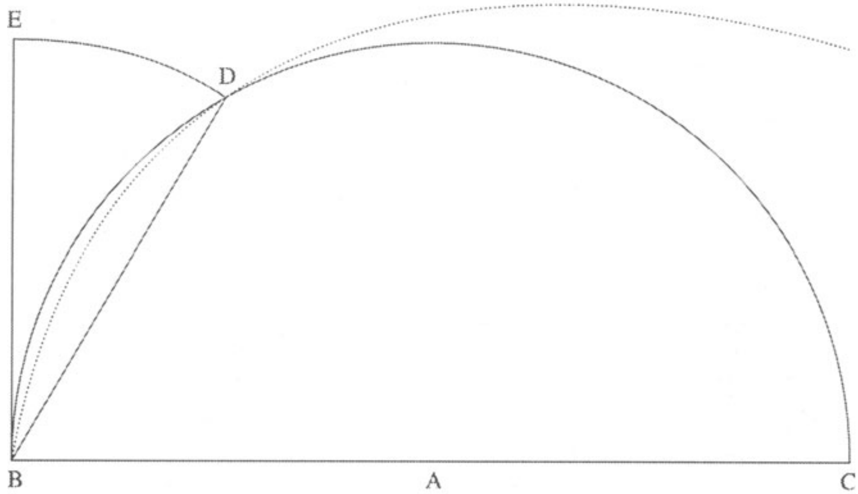
In dato circulo à ſegmento hexagoni trigeſimam ſextam partem ipſius circuli abſcindere.

Sit datus circulus, cujus A centrum, diameter B C, ſegmentum hexagoni B D. Oportet in dato circulo BDC à ſegmento hexagoni B D contento recta B D & circumferentia cui ſubtenditur, trigeſimam ſextam partem ipſius circuli B D C abſcindere.

Tangat circulum recta B E, & deſcribatur linea ſpiralis, cujus principium B, tranſitus per D, exiſtente BD tanta parte principii converſionis BEZ, quanta pars angulus EBD eſt quatuor rectorum, & centro B intervallo B D deſcribatur circulus D E. Sectoris igitur EBD tertia pars eſt ſpatium contentum recta BD & ſpirali. Eſt autem BD æqualis ſemidiametro B A. eſt enim BD latus hexagoni ex hypotheſi, & angulus EBD eſt triens recti, cum ſit B D circumferentia amplitudo beſſis recti. Sector igitur EBD eſt uncia circuli, & ſpatium conſequenter ſpirale BD triens unciæ, id eſt, trigeſima ſexta pars circuli. Tranſibit autem ſpiralis per ſegmentum, non etiam concurret cum circulari, vel circularem

rem

rem abſcindet in progreſſu abs B antequam ad punctum D pervenerit, ut eſt demonſtra-



tum. In dato igitur circulo B D C abſciſſa eſt à ſegmento hexagoni B D trigeſima ſexta pars ipſius circuli. Quod facere oportebat.

Atque hoc ſcuto tandem ſeptemplici mollis & hebetis ſecuriclæ acies ſatis obtuſa eſto.

Quod ſi qui ipſius πελεκεωμαχίας hypotyposin deſiderent, in his ne vacent, brevibus paginis eam conſpiciunto.

ANALYSIS CIRCULI

ſecundum πελεκητάς.

I.

Circulus conſtat ſex ſcalpris hexagoni.

II.

Scalprum hexagoni conſtat ſegmento hexagoni & triangulo hexagoni, ſeu majore.

III.

Triangulum hexagoni ſeu majus conſtat ſegmento hexagoni & ſecuricla.

IV.

Securicla conſtat duobus ſegmentis hexagoni & complemento ſecuriclæ.

V.

Complementum ſecuriclæ conſtat ſegmento hexagoni & reſiduo ſegmenti.

VI.

Rurſus complementum ſecuriclæ conſtat triangulo minore & reſiduo triangul minoris. Eſt autem triangulum minus quinta pars trianguli hexagoni, ſeu majoris.

LEM

LEMMATA DVO VERA,

Primum.

Decem minora triangula æqualia ſunt ſex ſegmentis hexagoni & duobus complementis ſecuriclæ.

Quorum enim triangulum hexagoni conſtat ſegmento & ſecuricla, ſecuricla vero duobus ſegmentis & complemento: ideo duo triangula hexagoni conſtant ſex ſegmentis & duobus complementis. Sed duo triangula hexagoni ſeu majora æqualia ſunt decem minoribus. Ergo decem minora triangula æqualia erunt ſex ſegmentis & duobus complementis. Quod erat oſtendendum.

Secundum.

Quadraginta minora triangula æqualia ſunt circulo & duobus complementis ſecuriclæ.

Quoniam enim circulus æquatur ſex ſcalpris hexagoni, ſex autem ſcalpra æqualia ſint ſex triangulis hexagoni & ſex ſegmentis, ſex porro triangula hexagoni valeant triginta triangula minora: ideo circulus æquatur triginta triangulis minoribus & ſex ſegmentis. Utrobique addantur duo complementa ſecuriclæ. Circulus igitur una cum duobus complementis ſecuriclæ æquabitur triginta triangulis minoribus & ſex ſegmentis & duobus complementis. Sed ſex ſegmenta & duo complementa valent decem minora triangula per antecedens Lemma: Ergo quadraginta minora triangula æquantur ſex ſegmentis hexagoni & duobus complementis ſecuriclæ. Quod erat oſtendendum.

ΨΕΥΔΑΡΙΟΝ.

Dico triangulum minus æquari ſuo reſiduo.

Ὡς Απόδειξις.

Quoniam enim circulus cum duobus complementis ſecuriclæ (quæ quidem valent duo triangula minora, & duo reſidua minoris trianguli) æquantur triginta ſex minoribus triangulis & inſuper quatuor. Utrinque auferantur duo triangula minora. Illic cum auferentur de duobus complementis, relinquent duo reſidua trianguli. Hic cum auferentur è quatuor triangulis, relinquent duo triangula. Ergo duo reſidua æquantur duobus triangulis.

Elenchus ἀσυλλογιστίας.

Ab æqualibus totis non ab æqualium parte auferenda æqualia ſunt, ut quæ relinquuntur maneant æqualia. Auferre ex æqualium parte eſt adſumere reliquum è toto reliquo eſſe æquale, ut hic circulum æquari triginta ſex minoribus triangulis. Illud vero pernegatur, & eſt falſiſſimum. Sibi demonſtranda concedere, eſt velle videri demonſtrative errare.

AD ΨΕΥΔΑΡΙΟΝ ALIVD, LEMMATA DVO VERA,

Primum.

Viginti quatuor quartæ trianguli hexagoni & ſex ſegmenta ſunt æqualia viginti quatuor ſegmentis & ſex complementis ſecuriclæ.

Quoniam enim triangulum hexagoni conſtat tribus ſegmentis & complemento ſecuriclæ, circulus autem componatur ex ſex triangulis & ſex ſegmentis: ideo viginti quatuor ſegmenta cum ſex complementis circulum adæquant. Et quia quatuor quartæ integrum componunt, æquabunt quoque circulum viginti quatuor quartæ trianguli hexagoni una cum ſex ſegmentis. Quæ autem uni æquantur æqualia ſunt inter ſe. Quare viginti quatuor quartæ trianguli hexagoni & ſex ſegmenta æqualia ſunt viginti quatuor ſegmentis & ſex complementis ſecuriclæ. Quod erat oſtendendum.

I i i Secun-

Secundum.

Si fuerint tres magnitudines inæquales, quarum media ſumpta viceſies & quater, & addita minimæ ſexies ſumptæ, eandem magnitudinem componat quam minima ſumpta viceſies & quater, & addita maximæ ſexies ſumptæ: differentia inter quadruplum mediæ & triplum minimæ erit maximæ æqualis.

Sit enim minima B, media D, maxima A. Ergo ex hypotheſi B 6, plus D 24. æquabitur A 6 plus B 24. Utrinque auferatur B 24. Igitur D 24 minus B 18 æquabitur A 6. Et omnibus per ſex diviſis, D 4 minus B 3 æquabitur A. Quod ipſum eſt quod enunciatur.

ΑΝΑΠΟΔΕΙΚΤΟΝ ΘΕΩΡΗΜΑ.

Sunt tres inæquales figuræ planæ & inter ſe commenſurabiles; minima, ſegmentum hexagoni; media, quadrans trianguli hexagoni; maxima, complementum ſecuriclæ hexagoni.

Potuit inæqualitas, & inæqualitatis gradus demonſtrari; at ſymmetriam aſymmetriamve nemo demonſtraverit, quin triangulum hexagoni aliudve rectilineum circulo primum comparaverit. Ea vero comparatio adhuc neſcitur, εἴγε ἐπιστητὸς ἐστι, καὶ ἐν θεῶν γούνασι κεῖται.

ΨΕΥΔΟΠΟΡΙΣΜΑ.

Itaque qualium quadrans trianguli hexagoni erit partium quinque, talium ſegmentum eſſe quatuor neceſſe eſt.

Ὡς ἀπόδειξις.

Sit enim ſegmentum hexagoni B, quadrans trianguli hexagoni D, complementum ſecuriclæ Z. Quoniam igitur tres ſunt inæquales magnitudines, atque harum B minima, D media, Z maxima, & ſe habent inter ſe ut numerus ad numerum. Eſto D partium quinque, talium B erit trium aut quatuor, & nihil præterea. Sit autem, ſed ſi fieri poſſit, partium trium; ex primo igitur & ſecundo Lemmate erit Z undecim. Itaque complementum conſtabit duobus ſegmentis & dodrante ſegmenti. Senſus autem repugnat. Quare eſt B quatuor.

Elenchus ἀσυλλογισίας.

Poſita D magnitudine partium quinque, poteſt oſtendi B major eſſe partibus tribus. An vero ideo B erit quatuor, conceſſo etiam eo, quod neſcitur, habere ſe B ad D, ut numerum ad numerum? Omnino ea concluſio aſyllogiſtica eſt. Quid enim ſi B ſtatuatur quatuor partium cum aliqua rationali fractiuncula. An quatuor cum ſemiſſe ſe non habere ad quinque, ut numerum ad numerum, hoc eſt, ut 9 ad 10, alius quam ἀλογιστικὸς ἢ ἀγεωμέτρητος negaverit? Sanè poſita D partium 11, fit B paulo major 9; Z vero paulo minor 17, ſecundum limites Archimedæos. Ex his autem duobus ψευδαρίοις dimanarunt reliqua πλεκητῶν κατὰ τὸ ἐμβαδὸν τοῦ κύκλου καὶ τὰς τῶν στερεῶν ἐπιφανείας σφάλματα.

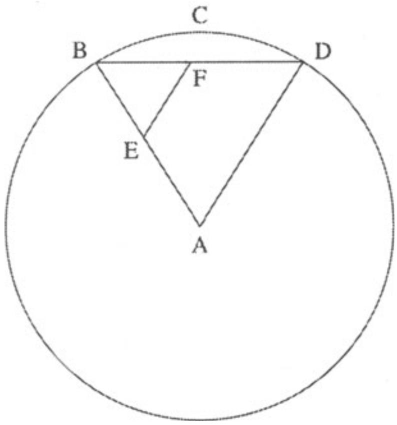
Πέρας τοῦ Ἀντιπελέκεως.

SECUN-

SECVNDÆ ΠΕΛΕΚΥΟΜΑΧΙΑΣ hypotyposis, ἐκ τοῦ προσθηκιδίου.

IN circulo cujus A centrum ſumatur circũferentia hexagoni BCD, & connectantur AB, AD, BD. Ex AB autem abſcindatur recta, cujus quadratum ad quadratum AB ſe habeat, ut unum ad quinque. Sit illa BE, & per E agatur ipſi AD parallela, ſecans BD in F.

Itaque triangulum BEF triangulo BAD fiat æquiangulum, & ejuſdem ſubquintuplum.



LEMMA I. VERVM.

Triginta ſeptem triangula BEF majora ſunt circulo BCD.

In adnotatis enim ad Mathematicum Canonem oſtenſus eſt circulus ad quadratum ſemidiametri ſe habere proxime, ut 31, 415, 926, 536 ad 10,000,000.000. Poſito autem latere A B, id eſt, ſemidiametro, particularum 100,000, trianguli A B D æquilateri altitudo eſt 86, 602 $\frac{54,038}{100,000}$.

Itaque.

Triangulum ABD fit	4, 330, 127, 019
Triangulum BEF.	866, 025, 404
Triginta ſeptem triangula BEF,	32, 042, 939, 948
Excedentia circulum	31, 415, 926, 536
Per particulas	627, 013, 412

LEMMA II. VERVM.

Circulus BCD non eſt major triginta ſex ſegmentis BCDF.

Quinimo circulus B C D longe minor eſt triginta ſex ſegmentis B C D F. Sector enim B A D ſexta pars eſt totius circuli

Itaque

Qualium circulus eſt	31, 415, 926, 536
Talium ſector B A D eſt	5, 235, 987, 756
Auferatur triangulum A B D earundem	4, 330, 127, 019
Relinquitur ſegmentum hexagoni ſpaciumve mixtilineum B C D F.	905, 860, 737
Ter duodena autem talia ſegmenta ſunt	32, 610, 986, 532
Excedentia circulum per particulas	1, 195, 059, 996

ΨΕΥΔΟΠΟΡΙΣΜΑ.

Ergo triginta ſeptem triangula B E F ſunt majora triginta ſex ſegmentis B C D F.

Elenchus ἀσυλλογισίας.

In Grammaticis, dare navibus Auſtros, & dare naves Auſtris, ſunt æque ſignificantia. Sed in Geometricis, aliud eſt adſumpſiſſe circulum B C D non eſſe majorem triginta ſex ſegmentis B C D F, aliud circulo B C D non eſſe majora triginta ſex ſegmenta BCDF. Illa adſumptiuncula vera eſt, hæc falſa.

Cum igitur ita arguo
Triginta ſeptem triangula majora ſunt circulo.
Sed triginta ſex ſegmenta non ſunt majora circulo.
Ergo triginta ſeptem triangula majora ſunt triginta ſex ſegmentis.
Syllogiſtice concludo, ſed falſo, quia falſum adſumo.

Pecco autem in leges Logicas cum in hanc formulam ſyllogiſmum inſtituo.
Circulus minor eſt triginta ſeptem triangulis.
Circulus non eſt major triginta ſex ſegmentis.
Ergo triginta ſeptem triangula ſunt majora triginta ſex ſegmentis.

Eſt autem ὀφθαλμικὸν σφάλμα, non διανοητικόν. Cum enim initio vere propoſuiſſent Cyclometræ circulum non eſſe majorem triginta ſex ſegmentis hexagoni, legerunt ex poſtfacto non eſſe minorem, atque inde ſuum elicuerunt falſum Corollarium.

FINIS

FRAN-

13 WALLIS. COMPUTATION OF π BY SUCCESSIVE INTERPOLATIONS

After 1650, analytic methods began to receive more attention and to replace geometric methods based on the writings of the ancients. This was due partly to the acceptance into geometry of those algebraic methods that Descartes and Fermat had introduced, and partly to the still very active interest in numerical work—interpolation, approximation, logarithms—a heritage of the sixteenth and early seventeenth centuries. This tradition was strong in England, where Napier and Briggs had labored.

This analytic method advanced rapidly through the efforts of John Wallis (1616–1703), of Emmanuel College, Cambridge, who in 1649 became the Savilian professor of geometry at Oxford. He was one of the founders of the Royal Society and, through his work, influenced Newton, Gregory, and other mathematicians. In his *Arithmetica infinitorum* (Oxford, 1655), he led explorations into the realms of the infinite with daring analytic methods, using interpolation and extrapolation to obtain new results. The title of the book shows the difference between Wallis' method—he called it "arithmetica"; we would say (with Newton) "analysis"—and the geometric method of Cavalieri. First Wallis derived Cavalieri's integral in an original way. Thereupon, he plunged into a maelstrom of numerical work and, with fine mathematical intuition to guide him in his interpolations, arrived at the infinite product for π that bears his name. See J. F. Scott, *The mathematical work of John Wallis* (Taylor and Francis, Oxford, 1938); also A. Prag, "John Wallis," *Quellen und Studien zur Geschichte der Mathematik* (*B*) I (1931), 381–412.

Proposition 39.[1] Given a series of quantities that are the cubes of a series of numbers continuously increasing in arithmetic proportion (like the series of cubic numbers), which begin from a point or zero (say 0, 1, 8, 27, 64, . . .); we ask for the ratio of this series to the series of just as many numbers equal to the highest number of the first series.

[1] In previous propositions Wallis has derived the limit

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^k}{n^{k+1}} = \frac{1}{k+1}$$

for $k = 1, 2$. This Proposition 39 prepares for the case $k = 3$; it shows Wallis's typical inductive and analytic method.

The investigation is carried out by the inductive method, as before. We have

$$\frac{0 + 1 = 1}{1 + 1 = 2} = \frac{2}{4} = \frac{1}{4} + \frac{1}{4};$$

$$\frac{0 + 1 + 8 = 9}{8 + 8 + 8 = 24} = \frac{3}{8} = \frac{1}{4} + \frac{1}{8};$$

$$\frac{0 + 1 + 8 + 27 = 36}{27 + 27 + 27 + 27 = 108} = \frac{4}{12} = \frac{1}{4} + \frac{1}{12};$$

$$\frac{0 + 1 + 8 + 27 + 64 = 100}{64 + 64 + 64 + 64 + 64 = 320} = \frac{5}{16} = \frac{1}{4} + \frac{1}{16};$$

$$\frac{0 + 1 + \cdots + 125 = 225}{125 + \cdots + 125 = 750} = \frac{6}{20} = \frac{1}{4} + \frac{1}{20};$$

$$\frac{0 + \cdots + 125 + 216 = 441}{216 + \cdots + 216 = 1512} = \frac{7}{24} = \frac{1}{4} + \frac{1}{24};$$

and so forth.

The ratio obtained is always greater than one-fourth, or $\frac{1}{4}$. But the excess decreases constantly as the number of terms increases; it is $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{20}$, $\frac{1}{24}$, . . . There is no doubt that the denominator of the fraction increases with every consecutive ratio by a multiple of 4, so that the excess of the resulting ratio over $\frac{1}{4}$ is the same as 1 : 4 times the number of terms after 0, etc.

Proposition 40. *Theorem.* Given a series of quantities that are the cubes of a series of numbers continuously increasing in arithmetic proportion beginning, for instance, with 0, then the ratio of this series to the series of just as many numbers equal to the highest number of the first series will be greater than $\frac{1}{4}$. The excess will be 1 divided by four times the number of terms after 0, or the cube root of the first term after 0 divided by four times the cube root of the highest term.

The sum of the series $0^3 + 1^3 + \cdots + l^3$ is $\frac{l+1}{4} l^3 + \frac{l+1}{4l} l^3$, or, if m is the number of terms, $\frac{m}{4} l^3 + \frac{m}{4l} l^3 = \frac{1}{4} ml^3 + \frac{1}{4} ml^2$. This is apparent from the previous reasoning.

If, with increasing number of terms, this excess over $\frac{1}{4}$ diminishes continuously, so that it becomes smaller than any given number (as it clearly does), when it goes to infinity, then it must finally vanish. Therefore:

Proposition 41. *Theorem.* If an infinite series of quantities which are the cubes of a series of continuously increasing numbers in arithmetic progression, beginning, say, with 0, is divided by the sum of numbers all equal to the highest and equal in number, then we obtain $\frac{1}{4}$. This follows from the preceding reasoning.

Proposition 42. *Corollary.* The complement AOT [Fig. 1] of half the area of the cubic parabola therefore is to the parallelogram TD over the same arbitrary base and altitude as 1 to 4.

Indeed, let AOD be the area of half the parabola AD (its diameter AD, and the corresponding ordinates DO, DO, etc.) and let AOT be its complement.

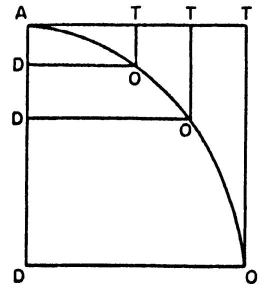


Fig. 1

Since the lines DO, DO, etc., or their equals AT, AT, etc. are the cube roots[2] of $AD, AD, \ldots$, or their equals $TO, TO, \ldots$, these TO, TO, etc. will be the cubes of the lines $AT, AT, \ldots$ The whole figure AOT therefore (consisting of the infinite number of lines TO, TO, etc., which are the cubes of the arithmetically progressing lines $AT, AT, \ldots$) will be to the parallelogram ATD (consisting of just as many lines, all equal to the greatest TO), as 1 to 4, according to our previous theorem. And the half-segment AOD of the parabola (the residuum of the parallelogram) is to the parallelogram itself as 3 is to 4.

In Proposition 44 the result of these considerations on the quotient of the two series $\sum_{i=1}^{n} i^k$ and $\sum n^k$ ($n + 1$ terms) is laid down in a table for $k = 0, 1, 2, \ldots, 10$. Wallis discriminates for i^k between the series of equals ($k = 0$), of the first order ($k = 1$), of the second order ($k = 2$), and so forth (*series aequalium*, *primanorum*, *secundanorum*, and so forth).

Proposition 54. *Theorem.*[3] If we consider an infinite series of quantities beginning with a point or 0 and increasing continuously as the square, cube, biquadratic, etc. roots of numbers in an arithmetic progression (which I call the series of order $k = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$), then the ratio of the whole series to the series of all numbers equal to the highest number is expressed in the following table:

k	*Result*		
$\frac{1}{2}$	$\frac{1}{3}$		$1\frac{1}{2}$
$\frac{1}{3}$	$\frac{1}{4}$,	or as 1 to	$1\frac{1}{3}$
$\vdots$	$\vdots$		$\vdots$
$\frac{1}{10}$	$\frac{1}{11}$		$1\frac{1}{10}$

---

[2] Wallis uses the terms *ratio subduplicata*, *subtriplicata* etc., to denote square, cubic, etc., roots; the *ratio subduplicata* of A^2/B^2 is A/B. These terms are not classical, and may be medieval. Wallis uses them here and in his *Mathesis universalis* (Oxford, 1657), chap. 30. The term *duplicate ratio* is classical; see Euclid, *Elements*, Book V, Definition 9: if $a/b = b/c$, then a/c has the duplicate ratio of a/b, hence $a/c = a^2/b^2$. Similarly, *triplicate ratio* in Definition 10 means the ratio of cubes. See G. Eneström, "Ueber den Ursprung des Termes 'ratio subduplicata'," *Bibliotheca mathematica* [3] *4* (1903), 210–211; *6* (1905), 410; *12* (1911–12), 180–181.

[3] Propositions 54 and 59 are supplementary, with the tabulation of $\int_0^1 x^k\,dx = 1/(k+1)$ for all positive rational k.

Wallis calls these series of order $\frac{1}{2}, \frac{1}{3}, \ldots$ *series subsecundanorum, subtertianorum,* etc. Proposition 59 gives the full table for $k = p/q$ (Table 1).

Table 1

	q \ p	0	1	2	3	⋮	10
	1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	⋮	$\frac{1}{11}$
Quadraticae	2	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	⋮	$\frac{2}{12}$
Cubicae	3	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	⋮	$\frac{3}{13}$
	...	...	...	...	...	⋮	...
Decimanae	10	$\frac{10}{10}$	$\frac{10}{11}$	$\frac{10}{12}$	$\frac{10}{13}$	⋮	$\frac{10}{20}$
		Aequalium	Primarorum	Secundarum	Tertiarum		

Wallis's table uses for p the terms *aequalium, primanorum,* etc., and for q the terms *quadraticae, cubicae,* etc. if $q = 2$, $q = 3$, etc.

Proposition 64. If we take an infinite series of quantities, beginning with a point or 0, continuously increasing in the ratio of any power, an integer or a rational fraction, then the ratio of the whole to the series of as many numbers equal to the highest number is 1 divided by the index of this power $+1$.[4]

At the end of the explanation Wallis adds: "If we suppose the index irrational, say $\sqrt{3}$, then the ratio is as 1 to $1 + \sqrt{3}$, etc."

In Prop. 87 we have the analogous result for negative powers (the term "negative" is used).

Proposition 108. If two series be given, one that of equals, the other of the first order, and if the first term of the latter series is subtracted from the first term of the series of equals, the second term from the second term, etc., then the differences give one-half of the total first series. However, when we add the term, the aggregates are found to be $\frac{3}{2}$ of the series of equals.[5]

For instance, let R be the arbitrary term of the series of equals and the highest term of the series of the first order. Let its infinitely small part be denoted by $a = R/\infty$, and let A be the number of all terms (or the altitude of the figure); this number will go to infinity. Then the sum of the aggregates is:

[4] Here the theorem of note 3 is explicitly formulated as "Theorema universalis."

[5] This means, in our notation, $\int_0^1 (1 - x)\,dx = \frac{1}{2}$, $\int_0^1 (1 + x)\,dx = \frac{3}{2}$.

$R - 0a$	$R + 0a$
$R - 1a$	$R + 1a$
$R - 2a$	$R + 2a$
$R - 3a$	$R + 3a$
etc.	etc.
$R - R$	$R + R$
$AR - \frac{1}{2}AR$	$AR + \frac{1}{2}AR$

The sum of all equal terms is clearly AR. The sum of the terms of the series of the first order is half of it: $\frac{1}{2}AR$. Now $AR - \frac{1}{2}AR = \frac{1}{2}AR$, $AR + \frac{1}{2}AR = \frac{3}{2}AR$. This means that the former series is to the series of equals as $\frac{1}{2}$ to 1, and the latter as $\frac{3}{2}$ to 1.

Proposition 111. *Theorem.* If from a series of equals are subtracted, term by term, the terms of a series of the second, third, fourth, etc. order, these differences give $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ of the total series of equals. If we add, the aggregates are $\frac{4}{3}$, $\frac{5}{4}$, $\frac{6}{5}$, etc. of this total sum.[6] Indeed, take the terms

	$R^2 \mp 0a^2$	$R^3 \mp 0a^3$	$R^4 \mp 0a^4$
	$R^2 \mp 1a^2$	$R^3 \mp 1a^3$	$R^4 \mp 1a^4$
	$R^2 \mp 4a^2$	$R^3 \mp 8a^3$	$R^4 \mp 16a^4$
	$R^2 \mp 9a^2$	$R^3 \mp 27a^3$	$R^4 \mp 81a^4$
	....	....	....
until	$R^2 \mp R^2$	$R^3 \mp R^3$	$R^4 \mp R^4$

Then the sums are (Prop. 44)

$$AR^2 \mp \tfrac{1}{3}AR^2, \quad AR^3 \mp \tfrac{1}{4}AR^3, \quad AR^4 \mp \tfrac{1}{5}AR^4.$$

Hence the sum of the differences gives

$$1 - \tfrac{1}{3} = \tfrac{2}{3}, \quad 1 - \tfrac{1}{4} = \tfrac{3}{4}, \quad 1 - \tfrac{1}{5} = \tfrac{4}{5}, \quad \text{etc.}$$

and the sum of the aggregates gives

$$1 + \tfrac{1}{3} = \tfrac{4}{3}, \quad 1 + \tfrac{1}{4} = \tfrac{5}{4}, \quad 1 + \tfrac{1}{5} = \tfrac{6}{5}, \quad \text{etc.}$$

Proposition 117. ...We replace the $1a$, $2a$, $3a$, etc. of previous propositions by a, b, c, etc., to show better the procedure of the operation:[7]

Series	Squares	Cubes
$R - 0$	$R^2 - 0R + 00$	$R^3 - 0R^2 + 00R - 000$
$R - a$	$R^2 - 2aR + a^2$	$R^3 - 3aR^2 + 3aR - a^3$
$R - b$	$R^2 - 2bR + b^2$	. . . .
$R - c$	$R^2 - 2cR + c^2$	
etc.	etc.	
$R - R$	$R^2 - 2RR + R^2$	$R^3 - 3RR^2 + 3R^2R - R^3$
$AR - \frac{1}{2}$	$AR^2 - \frac{2}{2}AR^2 + \frac{1}{3}AR^2$	$AR^3 - \frac{3}{2}AR^3 + \frac{3}{3}AR^3 - \frac{1}{4}AR^3$

[6] In our notation, $\int_0^1 (1 \pm x^n)\,dx = 1 \pm 1/(n+1)$, $n \geqslant 0$.

[7] In our notation, $\int_0^1 (1 - x)^k\,dx = 1/(k+1) = k!/(k+1)!$, $k \geqslant 0$.

Hence

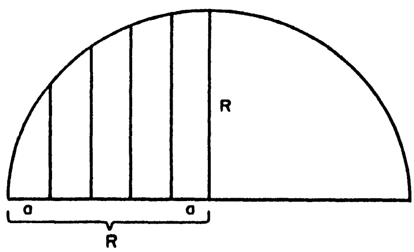
$1 - \frac{1}{2} = \frac{1}{2}$; or $\frac{1}{2}$; $\quad 1 - \frac{2}{2} + \frac{1}{3} = \frac{1}{3}$; $\frac{1 \times 2}{2 \times 3}$; $\quad 1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4} = \frac{1}{4}$; $\frac{1 \times 2 \times 3}{2 \times 3 \times 4}$;

and so on. We multiply continuously the numbers in arithmetic progression by each other (as many as agree with the value of the power), beginning with 1 and 2 and then regularly increasing with 1.

Proposition 121. *Corollary.* The ratio of the [area of a] circle to the square of the diameter (or of an ellipse to any of its circumscribed parallelograms) is as the series of square roots of the term-by-term differences of the infinite series of equals and the series of the second order to this series of equals.[8]

Indeed, if we call R [Fig. 2] the radius of the circle (of which $a = R/\infty$ is the infinitesimally small part), and if we construct an infinite number of per-

Fig. 2



pendiculars or *sinus recti* in order to complete the quadrant, then these perpendiculars are the mean proportionals between the segments of the diameters (as is well known), or

between	$R + 0$,	$R + 1a$,	$R + 2a$,	$R + 3a$, etc.
and	$R - 0$,	$R - 1a$,	$R - 2a$,	$R - 3a$, etc.
whose rectangles are	$R^2 - 00$,	$R^2 - 1a^2$,	$R^2 - 4a^2$,	$R^2 - 9a^2$, etc.
the mean prop. are	$\sqrt{R^2 - 00}$,	$\sqrt{R^2 - 1a^2}$,	$\sqrt{R^2 - 4a^2}$,	$\sqrt{R^2 - 9a^2}$, etc.

Hence, whatever the ratio of the sum of these roots is to that of their maximum (the radius), such is also the ratio of a quadrant of the circle (which consists of these roots) to the square of the radius (which consists of these maxima). Therefore it is also the ratio of the whole circle to the square of the diameter.

For this ratio Wallis writes $1 : \square$ (in our notation $\pi/4$).[9]

Proposition 132 [Fig. 3]. If we subtract term by term from the infinite series of equals the series of the first order (or, if we like, of $\frac{1}{1}$th order), of order

[8] In our notation, $\int_0^1 \sqrt{1 - x^2}\, dx = \pi/4$.

[9] The standard notation π for $4 : \square$ is due to William Jones (1675–1749), a friend of Newton, who assisted him in having some of his manuscripts published (see Selection V.4). In a textbook of 1706 he wrote π for 3.14159 etc. Euler adopted it and provided for its universal acceptance through his *Introductio in analysin infinitorum* (Lausanne, 1748).

PROP. CXXXII.

Theorema.

SI exponatur infinita ſeries æqualium mulctata analoga ſerie Primanorum (vel ſi libet, Subprimanorum, quod tantundem valet,) Subſecundanorum, Subtertianorum, &c. Reſidua ipſa, eorumque Quadrata, Cubi, Biquadrata, &c. eam rationem habebunt ad congruam ſeriem æqualium, quam habet Unitas ad numeros in ſubjecta Tabella indicatos. Nempe

Seriei Æqualium mulctatæ ſerie

	Æqualia	Reſidua	Quadrata	Cubi	Biquadrat	Superſol	Sextana	Septimana	Octavana	Nonana	Decimana
Nulla	1	1	1	1	1	1	1	1	1	1	1
Subprimanorum	1	2	3	4	5	6	7	8	9	10	11
Subſecundanorum	1	3	6	10	15	21	28	36	45	55	66
Subtertianorum	1	4	10	20	35	56	84	120	165	220	286
Subquartanorum	1	5	15	35	70	126	210	330	495	715	1001
Subquintanorum	1	6	21	56	126	252	462	792	1287	2002	3003
Subſextanorum	1	7	28	84	210	462	924	1716	3003	5005	8008
Subſeptimanorum	1	8	36	120	330	792	1716	3432	6435	11440	19448
Suboctavanorum	1	9	45	[illegible]	495	1287	3003	6435	12870	24310	43758
Subnonanorum	1	10	55	220	715	2002	5005	11440	24310	48620	92378
Subdecimanorum	1	11	66	286	1001	3003	8008	19448	43758	92378	184756

Et ſic deinceps.

Et ſic deinceps.

Sequitur ex præcedente. Numerus autem Tabellæ quilibet intermedius eſt aggregatum ex duobus ſibi proximis altero ſurſum altero ad dextram poſitis.

Notandum etiam eſt, eundem prodire rationis conſequentem in reſiduorum Quadratis ſi auferantur Subtertiana, & in Cubis ſi auferantur Subſecundana; item in Sextanis ſi auferantur Subſeptimana, & in Septimanis ſi auferantur Subſextana; & ſic ubique, reciprocatis quaſi poteſtatibus, ut ex inſpecta tabella patet.

Sed

Fig. 3

[$1/p$ =] $\frac{1}{2}$, $\frac{1}{3}$, etc., then the series of differences and the series of order [k =] 2, 3, 4, etc. formed from them have the same ratio to the series of equals as has 1 to the numbers in [Table 2].[10]

$p \backslash k$	0	1	2	3	4	⋮	10
0	1	1	1	1	1	⋮	1
1	1	2	3	4	5	⋮	11
2	1	3	6	10	15	⋮	66
3	1	4	10	20	35	⋮	
...	...	...	...	...	...	⋮	...
10	1	11	66	206	1001	⋮	184754

Table 2

$p \backslash k$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	
$-\frac{1}{2}$	∞	1	$\frac{1}{2}\square$	$\frac{1}{2}$	$\frac{1}{3}\square$	$\frac{3}{8}$	$\frac{4}{15}\square$	A ($l = k+1$)
0	1	1	1	1	1	1	1	1
$\frac{1}{2}$	$\frac{1}{2}\square$	1	$\square$	$\frac{3}{2}$	$\frac{4}{3}\square$	$\frac{15}{8}$	$\frac{8}{5}\square$	$A \times \frac{2l-1}{1}$
1	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$\frac{7}{2}$	$\frac{2l+0}{2}$
$\frac{3}{2}$	$\frac{1}{3}\square$	1	$\frac{4}{3}\square$	$\frac{5}{2}$	$\frac{8}{3}\square$	$\frac{35}{8}$	$\frac{64}{15}\square$	$A \times \frac{2l-1}{1} \cdot \frac{2l+1}{3}$
2	$\frac{3}{8}$	1	$1\frac{7}{8}$	3	$4\frac{3}{8}$	6	$\frac{63}{8}$	$\frac{2l+0}{2}\ \frac{2l+1}{2}$

Table 3

This follows from the preceding. Any intermediate number in this table is the sum of the two numbers next to it, one above and the other to the left.

Proposition 184. In the preceding table we can interpolate in the following way [Fig. 4].

Proposition 189. We can now interpolate other series in the preceding table [as in Table 3].[11]

Proposition 191. *Problem.* It is proposed to determine this term $\square$ as closely as possible in absolute numbers.

Wallis finds, by further interpolation (see the row for $p = \frac{1}{2}$ in Table 3, from which can be derived

$$\frac{\frac{15}{8}}{\frac{4}{3}\square} < \frac{\frac{4}{3}\square}{\frac{3}{2}} < \frac{\frac{3}{2}}{\square}, \quad \text{etc.}),$$

[10] In our notation, $\int_0^1 (1 - x^{1/p})^k\,dx = \frac{l(l+1)\ldots(l+p-1)}{1\cdot 2\cdots p}$, $l = k + 1$, $k, p \geqslant 0$. The listing $(2l + 0)/2$, etc. is from Proposition 184.

[11] The interpolation is by means of the expressions A, 1, $A(2l - 1)/1$, etc., with the insertion of fractional values for 1. Since

$$\int_0^1 (1 - x^{1/p})^k\,dx = p\int_0^1 (1 - y)^k y^{p-1}\,dy = pB(p, k + 1),$$

$$\int_0^1 (1 - x^{1/k})^p\,dx = k\int_0^1 (1 - y)^p y^{k-1}\,dy = kB(k, p + 1),$$

the symmetry of the table expresses the symmetry of the B-function. Both integrals are equal to

$$\frac{kp}{k+p}\frac{\Gamma(k)\Gamma(p)}{\Gamma(k+p)} = \frac{\Gamma(k+1)\Gamma(p+1)}{\Gamma(k+p+1)}.$$

The values for k and p are positive integers and multiples of $\frac{1}{2}$. If $y = z^2$, we find

$$\int_0^1 (1 - x^{1/p})^k\,dx = 2p\int_0^1 (1 - z^2)^k z^{2p-1}\,dz,$$

which is a multiple of the integral $\int_0^1 x^m y^k\,dx$, $x^2 + y^2 = 1$. We can say that Wallis computed this integral for integral values of m and k. The symbol ∞ for "infinite" is due to Wallis. See also T. P. Nunn, "The arithmetic of infinities," *Mathematical Gazette 5* (1909–1911), 345–356, 377–386, with a paraphrase of the book.

PROP. CLXXXIII.

Theorema.

LAtus numeri Figurati cujuslibet, in qualibet serie Tabellæ expositæ (prop. 132.) quousque libet continuandæ; ad suum illius numerum Figuratum; rationem habet cognitam.

Nempe eam quam indicat prop. præced.

PROP. CLXXXIV.

Theorema.

ET *propterea*, Series sequentes in præmissa Tabella quousque libet continuata, non erit difficile interpolare.

Nempe, invento per prop. 182. cujusque proprio charactere, fiat interpolatio ut in prop. 175, 178, 181.

Tabella vero, ut dictum est, interpolata sic se exhibebit.

Numeri		Monadici.		Laterales.		Triangulares.		Pyramidales.		Triang. triang.	
	∞	1		$\frac{1}{2}$		$\frac{3}{8}$		$\frac{15}{48}$		$\frac{105}{384}$	
Monadici.	1	1	1	1	1	1	1	1	1	1	1
		1	□	$1\frac{1}{2}$		$1\frac{7}{8}$		$2\frac{9}{48}$		$2\frac{177}{384}$	
Laterales.	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	l
		1		$2\frac{1}{2}$		$4\frac{3}{8}$		$6\frac{27}{48}$		$9\frac{9}{384}$	
Triangulares.	$\frac{3}{8}$	1	$1\frac{7}{8}$	3	$4\frac{3}{8}$	6	$7\frac{7}{8}$	10	$12\frac{3}{8}$	15	$\frac{l^2+l}{2}$
		1		$3\frac{1}{2}$		$7\frac{7}{8}$		$14\frac{21}{48}$		$23\frac{177}{384}$	
Pyramidales.	$\frac{15}{48}$	1	$2\frac{9}{48}$	4	$6\frac{27}{48}$	10	$14\frac{21}{48}$	20	$26\frac{39}{48}$	35	$\frac{l^3+3l^2+2l}{6}$
		1		$4\frac{1}{2}$		$12\frac{3}{8}$		$26\frac{39}{48}$		$50\frac{105}{384}$	
Trianguli-triang.	$\frac{105}{384}$	1	$2\frac{177}{384}$	5	$9\frac{9}{384}$	15	$23\frac{177}{384}$	35	$50\frac{105}{384}$	70	$\frac{l^4+6l^3+11l^2+6l}{24}$

Et sic deinceps.

Vd

Fig. 4

that $\square$ is

$$\text{less than } \frac{3 \times 3 \times 5 \times 5 \cdots 13 \times 13}{2 \times 4 \times 4 \times 6 \cdots 12 \times 14} \sqrt{1\tfrac{1}{13}}$$

and

$$\text{greater than } \frac{3 \times 3 \times 5 \times 5 \cdots 13 \times 13}{2 \times 4 \times 4 \times 6 \cdots 12 \times 14} \sqrt{1\tfrac{1}{14}},$$

and so forth to as close an approximation as we like.[12]

PROP. CLXXXIX.

Theorema.

Hinc sequitur, quod Si ex Tabellæ prop. 184. locis vacuis unus quilibet numero noto suppleatur, erunt & reliqui omnes cogniti.

Verbi gratia; si numerus hac nota $\square$ designatus supponatur cognitus, reliqui omnes etiam cognoscentur; qui nempe eam habent ad illum rationem quæ hic subtus indicatur.

∞	1	$\frac{1}{2}\square$	$\frac{1}{2}$	$\frac{1}{3}\square$	$\frac{3}{8}$	$\frac{4}{15}\square$	$\frac{15}{48}$	$\frac{8}{35}\square$	$\frac{105}{384}$	A
1	1	1	1	1	1	1	1	1	1	1
$\frac{1}{2}\square$	1	$\square$	$\frac{3}{2}$	$\frac{4}{3}\square$	$\frac{15}{8}$	$\frac{8}{5}\square$	$\frac{105}{48}$	$\frac{64}{35}\square$	$\frac{945}{384}$	$A \times \frac{2l-1}{1}$
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$l=\frac{2l\mp 0}{2}$
$\frac{1}{3}\square$	1	$\frac{4}{3}\square$	$\frac{5}{2}$	$\frac{8}{3}\square$	$\frac{35}{8}$	$\frac{64}{15}\square$	$\frac{315}{48}$	$\frac{128}{21}\square$	$\frac{3465}{384}$	$A \times \frac{4l^2-1}{3}$
$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15	$\frac{l^2+1}{2}=\frac{4l^2+4l}{8}$
$\frac{4}{15}\square$	1	$\frac{8}{5}\square$	$\frac{7}{2}$	$\frac{64}{15}\square$	$\frac{63}{8}$	$\frac{128}{15}\square$	$\frac{693}{48}$	$\frac{512}{35}\square$	$\frac{9009}{384}$	$A \times \frac{8l^3+12l^2-2l-3}{15}$
$\frac{15}{48}$	1	$\frac{105}{48}$	4	$\frac{315}{48}$	10	$\frac{693}{48}$	20	$\frac{1287}{48}$	35	$\frac{l^3+3l^2+2l}{6}=\frac{8l^3+24l^2+16l}{48}$
$\frac{8}{35}\square$	1	$\frac{64}{35}\square$	$\frac{9}{2}$	$\frac{128}{21}\square$	$\frac{99}{8}$	$\frac{512}{35}\square$	$\frac{1287}{48}$	$\frac{1024}{35}\square$	$\frac{19305}{384}$	$A \times \frac{16l^4+64l^3+56l^2-16l-15}{105}$
$\frac{105}{384}$	1	$\frac{945}{384}$	5	$\frac{3465}{384}$	15	$\frac{9009}{384}$	35	$\frac{19305}{384}$	70	$\frac{l^4+6l^3+11l^2+6l}{24}=$

$$=\frac{16l^4+96l^3+176l^2+96l}{384}$$

Totus

Totus processus demonstratur ex prop. præced.

Notandum autem & hic numerum quemvis intermedium aggregatum esse ex duobus altero sursum altero ad dextram, (non proximis, sed post unum intermissum,) positis.

Cujusque seriei characterem (quatenus per prop. 182 & 187 innotescit,) libuit etiam adjungere, quo melius perspiciat lector quousque rem perduximus.

SCHOLIUM.

Atque hactenus quidem rem perduxisse videamur satis feliciter. Verum hic tandem hæret aqua. Neque enim video quo pacto possim vel quantitatem $\square$ reperire, vel characterem seriei A. (Et propterea, nec characteres serierum imparium penitus assequi, licet eorum ad invicem rationes cognoscantur; nec imparium serierum locos impares, quamvis & horum etiam cognoscantur quas habent ad invicem rationes.) Quanquam enim si numeri laterales sint integri, puta 1, 2, 3, 4, &c. noti sint serierum illarum termini primi; puta 1, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{15}{48}$, &c. non tamen facile est deprehendere quo pacto horum numerorum ad suos respective numeros laterales ratio possit una aliqua æquatione explicari: unde & reliquis lateribus (locorum imparium) $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. accommodari possit suæ cujusque seriei primus terminus. Quanquam enim hinc spes non exigua visa est affulsisse, lubricus tamen quem præ manibus habemus Proteus tam hic quam superius non raro elapsus, spem fefellit. Quem autem & hic comprimenti vultum ostenderit, non erit fortassis ingratum apposuisse. Nempe —

PROP. CXCI.

Problema.

PRopositum sit inquirere, quantus sit terminus □ (tabellæ Prop. 189.) in numeris absolutis quam proxime.

Quo facilius res succedat, progressionis (ibidem repertæ) termini

$$\tfrac{1}{2}\square .\ 1 .\ \square .\ \tfrac{3}{2} .\ \tfrac{4}{3}\square .\ \frac{3\times 5}{2\times 4} .\ \frac{4\times 6}{3\times 5}\square .\ \frac{3\times 5\times 7}{2\times 4\times 6} .\ \&c.$$

dicantur, $\alpha . a . \beta . b . \gamma . c . \quad \delta . \quad d . \quad$ &c.

Est autem $1 . 2 :: \alpha . \beta$. Et $2 . 3 :: a . b$. Et $3 . 4 :: \beta . \gamma$. Et $4 . 5 :: b . c$. Et $5 . 6 :: \gamma . \delta$. Et $6 . 7 :: c . d$.

Hoc est, $\frac{\beta}{\alpha} = \frac{2}{1}$, $\frac{b}{a} = \frac{3}{2}$, $\frac{\gamma}{\beta} = \frac{4}{3}$, $\frac{c}{b} = \frac{5}{4}$, $\frac{\delta}{\gamma} = \frac{6}{5}$, $\frac{d}{c} = \frac{7}{6}$, &c.

Ideoque (cum rationes continue multiplicantes perpetuo decrescant,) erit

$$\frac{\beta}{a}\left\{\begin{array}{l} \text{minor duarum}\left\{\frac{a}{\alpha}\times\frac{\beta}{a} = \frac{\beta}{\alpha} = \frac{2}{1}\right. \\[2ex] \text{major duarum}\left\{\frac{\beta}{a}\times\frac{b}{\beta} = \frac{b}{a} = \frac{3}{2}.\right. \end{array}\right\}\ \text{Ideoque}\left\{\begin{array}{l} \text{minor quam, } \sqrt{\tfrac{2}{1}} = \sqrt{1\tfrac{1}{1}}. \\[2ex] \text{major quam, } \sqrt{\tfrac{3}{2}} = \sqrt{1\tfrac{1}{2}}. \end{array}\right.$$

$$\text{Et propterea } \beta = a\times\frac{\beta}{a} = \square,\left\{\begin{array}{l} \text{minor quam, } 1\sqrt{2} = 1\sqrt{1\tfrac{1}{1}}. \\[2ex] \text{major quam, } 1\sqrt{\tfrac{3}{2}} = 1\sqrt{1\tfrac{1}{2}}. \end{array}\right.$$

Nnn 2 Item

$$\text{Item } \frac{\gamma}{b}\left\{\begin{array}{l} \text{minor duarum}\left\{\frac{b}{\beta}\times\frac{\gamma}{b} = \frac{\gamma}{\beta} = \frac{4}{3}\right. \\[2ex] \text{major duarum}\left\{\frac{\gamma}{b}\times\frac{c}{\gamma} = \frac{c}{b} = \frac{5}{4}.\right. \end{array}\right\}\ \text{Ideoque}\left\{\begin{array}{l} \text{minor quam, } \sqrt{\tfrac{4}{3}} = \sqrt{1\tfrac{1}{3}}. \\[2ex] \text{major quam, } \sqrt{\tfrac{5}{4}} = \sqrt{1\tfrac{1}{4}}. \end{array}\right.$$

$$\text{Et propterea } b\times\frac{\gamma}{b} = \gamma = \tfrac{4}{3}\square,\left\{\begin{array}{l} \text{minor quam } \tfrac{3}{2}\times\sqrt{1\tfrac{1}{3}}. \\ \text{major quam } \tfrac{3}{2}\times\sqrt{1\tfrac{1}{4}}. \end{array}\right.$$

Hoc est, □ minor quam $\frac{3\times 3}{2\times 4}\times\sqrt{1\tfrac{1}{3}}$. major quam $\frac{3\times 3}{2\times 4}\times\sqrt{1\tfrac{1}{4}}$.

$$\text{Et (pari ratione) erit } \delta = c\times\frac{\delta}{c} = \frac{4\times 6}{3\times 5}\square,\left\{\begin{array}{l} \text{minor quam } \frac{3\times 5}{2\times 4}\times\sqrt{1\tfrac{1}{5}}. \\[2ex] \text{major quam } \frac{3\times 5}{2\times 4}\times\sqrt{1\tfrac{1}{6}}. \end{array}\right.$$

Hoc est, □ minor quam $\frac{3\times 3\times 5\times 5}{2\times 4\times 4\times 6}\sqrt{1\tfrac{1}{5}}$. major quam $\frac{3\times 3\times 5\times 5}{2\times 4\times 4\times 6}\sqrt{1\tfrac{1}{6}}$.

Et (continuata ejusmodi operatione juxta Tabellæ leges) invenietur

$$\square\left\{\begin{array}{l} \text{minor quam } \frac{3\times 3\times 5\times 5\times 7\times 7\times 9\times 9\times 11\times 11\times 13\times 13}{2\times 4\times 4\times 6\times 6\times 8\times 8\times 10\times 10\times 12\times 12\times 14}\times\sqrt{1\tfrac{1}{13}}. \\[2ex] \text{major quam } \frac{3\times 3\times 5\times 5\times 7\times 7\times 9\times 9\times 11\times 11\times 13\times 13}{2\times 4\times 4\times 6\times 6\times 8\times 8\times 10\times 10\times 12\times 12\times 14}\times\sqrt{1\tfrac{1}{14}}. \end{array}\right.$$

Et sic deinceps quousque libet.

Idem Aliter.

Post hanc autem nostram, ipsius quantitatis □ designationem; libet etiam aliam subjungere, quam à Nobilissimo Viro, atque acutissimo simul Geometra, Dom. Guliel. Vicecom. & Barone *Brouncker*, accepi.

Cum illi progressionum aliquot mearum proposuerim, & qua lege procederent indicaverim, id interim rogans, ut qua forma quantitatem illam commode designandam putaverit, indicaret; Nobilissimus Vir ille, re apud se perpensa, methodo item Infinitorum sibi peculiari quantitatem ad hanc formam commodissime designandam judicavit.

$$\square = 1\,\tfrac{1}{2}\,\tfrac{9}{2}\,\tfrac{25}{2}\,\tfrac{49}{2}\,\tfrac{81}{2}\ \&c.$$

Nempe si unitati adjungatur fractio, quæ denominatorem habeat continue fractum; ea lege, ut particularium fractionum Numeratores sint 1, 9, 25, &c. numeri quadratici imparium 1, 3, 5, &c. Denominator vero ubique 2 cum adjuncta fractione, & sic in infinitum.

CHRISTIANI HUGENII,
CONST. F.

DE

CIRCULI MAGNITUDINE INVENTA.

ACCEDUNT EJUSDEM

Problematum quorundam illuſtrium Conſtructiones.

Xx 5

PROBLEMA IV. PROPOS. XX.

Circumferentiæ ad diametrum rationem investigare; & ex datis inscriptis in dato circulo invenire longitudinem arcuum quibus illæ subtenduntur.

TAB. XL. Fig. 6. Esto circulus centro D, cujus diameter C B, & sit arcus B A sextans circumferentiæ, cui subtensa ducatur A B, itemque sinus A M. Positâ igitur D B semidiametro partium 100000, totidem quoque erit subtensa B A. A M vero partium 86603 non unâ minus; hoc est, si una pars sive unitas auferatur ab 86603 fiet minor debito. quippe semissis lateris trianguli æquilateri circulo inscripti.

Hinc excessus A B supra A M fit 13397 vero minor. Cujus triens 4465⅔ additus ipsi A B 100000, fiunt partes 104465⅔ minores arcu A B Et hic primus est minor terminus, quo postea alium vero propiorem inveniemus. Prius autem major quoque terminus secundum Theorema præcedens inquirendus est.

Tres nimirum sunt numeri quibus quartum proportionalem invenire oportet. Primus est partium duplæ A B & triplæ A M qui erit 459807, vero minor, (nam hoc quoque observandum ut minor sit, idemque in cæteris prout dicetur) secundus quadruplæ A B & simplæ A M qui 486603 vero maj. Et tertius triens excessus A B supra A M, 4466 vero major. Itaque quartus proportionalis erit 4727 vero maj. quo addito ad A B 100000 fit 104727, major numero partium, quas continet arcus A B, peripheriæ sextans. * Jam igitur invenimus longitudinem arcus A B secundum minorem majoremque terminum, quorum hic quidem longe propior vero est, cum vero proximus sit 104719.

* per præced.

Sed ex utroque istorum alius minor terminus habebitur priore accuratior si utamur præcepto sequenti, quod à diligentiori centrorum gravitatis inspectione dependet.

In-

Inventorum terminorum differentia sesquitertia jungatur duplæ subtensæ & sinui triplo, & quam rationem habet ex his composita ad triplam sesquitertiam seu $\frac{10}{3}$ utriusque simul, sinus, subtensæque, eandem habeat subtensæ supra sinum excessus ad aliam quandam; Hæc ad sinum addita rectam constituet arcu minorem.

Minor terminus erat 104465$\frac{1}{3}$. Major 104727. differentia horum est 261$\frac{2}{3}$. Estque rursus tribus numeris inveniendus quartus proportionalis. Primus est partium duplæ A B & triplæ A M & sesquitertiæ terminorum differentiæ, 460158 vero major. Secundus $\frac{10}{3}$ utriusque simul A B, A M, 622008 vero minor. Tertius denique excessus A B supra A M, 13397 vero min. Quibus quartus proportionalis est 18109 vero min. Hic igitur additus numero partium A M 86602$\frac{1}{2}$ vero min. fiunt 104711$\frac{1}{2}$ minores arcu A B. Quare sexcuplum earum, 628269 minus erit circumferentià totà. At quoniam 104727 majores inventæ sunt arcu A B, earum sexcuplum 628362 circumferentià majus erit. Itaque circumferentiæ ad diametrum ratio minor est quam 628362, major autem quam 628269 ad 200000. Sive minor quam 314181, major autem quam 314135 ad 100000. Unde constat minorem utique esse quam triplam sesquiseptimam, & majorem quam 3$\frac{10}{71}$. Quin etiam Longomontani error per hæc refutatur, qui scripsit peripheriam majorem esse partibus 314185 qualium rad. 100000.

Esto nunc arcus A B $\frac{1}{8}$ circumferentiæ, & erit A M, semissis lateris quadrati circulo inscripti, partium 7071068, non unâ minus, qualium radius D B 10000000. A B vero latus octanguli partium 7653668 non unâ majus. Quibus datis ad similitudinem præcedentium invenietur primus minor terminus longitudinis arcus A B 7847868. Deinde major terminus 7854066. Et ex utroque rursus terminus minor accuratior 7853885. Unde constat peripheriæ ad diametrum rationem minorem haberi quam 31416$\frac{1}{4}$, majorem autem quam 31415 ad 10000.

Et quum terminus major 7854066 à vera arcus A B longitugitudine minus distet quam partibus 85; (Est enim arcus A B, per ea quæ supra ostendimus, major quam 785398I) partes autem 85 efficiant minus quam duos scrupulos secundos, hoc est, quam $\frac{2}{1296000}$ circumferentiæ, nam tota earundem plures habet quam 60000000: Hinc manifestum est, si trianguli rectanguli angulos quæramus ex datis lateribus, eo modo quo majorem istum terminum paulò antè, nunquam duobus scrupulis secundis aberraturos; etiamsi æqualia inter se fuerint latera circa angulum rectum, veluti hic erant in triangulo D A M.

Si vero ea sit ratio lateris D M ad M A, ut angulus A D M non excedat $\frac{1}{4}$ recti; non unius tertii scrupuli error erit. Posito enim arcu A B $\frac{1}{16}$ circumferentiæ, erit A M semissis lateris octanguli æquilateri circulo inscripti partium 382683433, non unâ minus. A B vero latus sexdecanguli 390180644 non unâ amplius, qualium radius D B 1000000000. Unde primus minor terminus longitudinis arcus A B invenitur partium 392679714. Terminus autem major 392699148. Et ex his minor rursus 392699010. Constat autem ex supra demonstratis arcum A B $\frac{1}{16}$ peripheriæ, majorem esse quam 392699081, quas terminus major superat partibus 67. Hæ autem minus efficiunt uno scrupulo tertio, hoc est, $\frac{1}{77760000}$ totius circumferentiæ, quoniam ea major est utique quam 6000000000.

Porro ex novissimis terminis inventis orietur ratio circumferentiæ ad diametrum minor quam $3141593\frac{1}{5}$, major autem quam 3141592 ad 1000000.

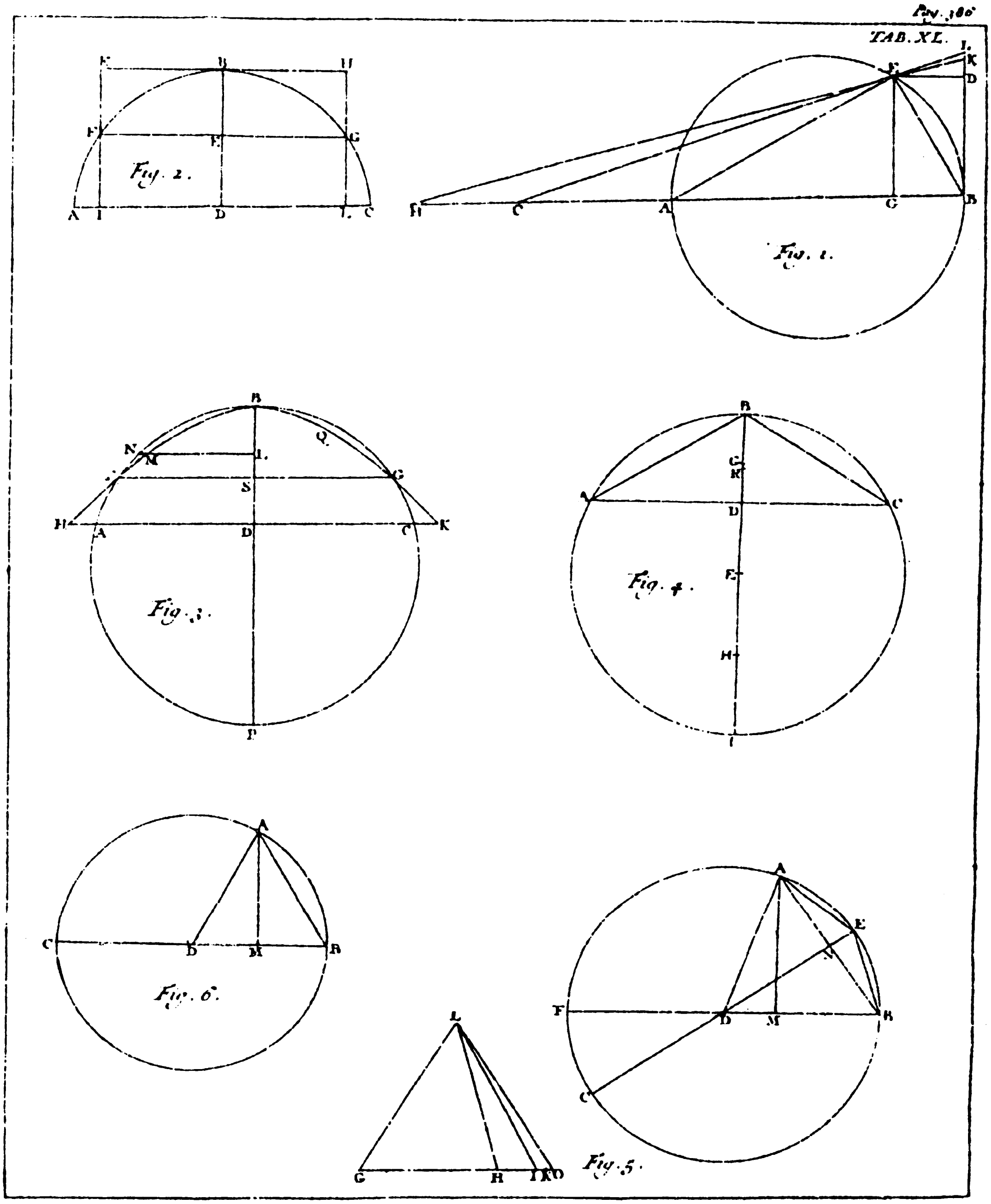
Quod si $\frac{1}{60}$ circumferentiæ ponatur arcus A B, seu partium 6 qualium tota 360: Erit A M semissis lateris trigintanguli inscripti partium 104528463267 66, non unâ minus, qualium radius 10000000000000. Et A B latus sexagintanguli inscripti 104671912485 88 non unâ amplius. Invenieturque ex his arcus A B secundum primum minorem terminum 1047197288 9195. Secundum majorem 1047197551 2584. Et ex his minor alter terminus 1047197551 1302. Unde efficitur peripheriæ ad diametrum ratio minor quam 31415926538, 31415926538, major autem quam 31415926533 ad 10000000000.

Quos terminos si ex additis inscriptorum & circumscriptorum polygonorum lateribus inquirendum esset ferè ad laterum quadringenta millia deveniendum. Nam ex sexagintangulo inscripto circumscriptoque hoc tantum probatur, minorem esse rationem peripheriæ ad diametrum quam 3145 ad 1000, majorem autem quam 3140. Adeo ut triplum & amplius verarum notarum numerum nostro ratiocinio productum appareat. Idem vero in ulterioribus polygonis si quis experiatur semper evenire cernet: non ignota nobis ratione, sed quæ longiori explicatione indigeret.

Porro autem quomodo, datis quibuscunque aliis inscriptis, arcuum quibus subtenduntur longitudo per hæc inveniri queat satis puto manifestum. Si enim quadrati inscripti latere majores sunt, longitudo arcus ad semicircumferentiam reliqui inquirenda est, cujus tum quoque subtensa datur. Sciendum autem & dimidiorum arcuum subtensas inveniri cum totius arcus subtensa data est. Atque hâc ratione si bisectionibus uti placebit, poterimus ad omnem subtensam, arcus ipsius longitudinem quamlibet veræ propinquam non difficulter cognoscere. Utile hoc ad sinuum tabulas examinandas. Imo ad componendas quoque: quia cognitâ arcus alicujus subtensâ, etiam ejus qui paulo major minorve sit satis accuratè definiri potest.

Pag. 380

TAB. XL.



39. (RS.CE.6) J. GREGORY TO COLLINS

ST ANDREWS,
15 *February* 1671.

SIR,

Since my last to you I have received three of yours, one dated December 15, another December 24, the third January 21.[1] There is no fear that any of your letters miscarrie; our post here is abundantly sure; albeit he be slow, for his ordinar is to go to Edinburgh only once in the fortnight neither durst hazard a letter with any extraordinar occasion. Not onlie this but also several other studies maturing hath made me slow this time by post in my answers: but now I shall strive to answer orderlie to all.[2] I do not question that all equations may be formed by tables, but I doubt exceedingly if all equations can be solved by the help only of the tables of logarithms and sines without serieses. Yea I judge it absolutely impossible, and have as much ground for it as convinceth myself. As for the 1, 2, 3, 4 of your discourse I do not question any of them; but as for the 5, I will hardly take Dulaurens his word nor yet Frenicle for it, as to your discourse concerning ranks of numbers your 1st, 2nd and 4th I wish ye had explicat more fully: I doubt not of the 3, but I think it hard if not impossible, to find by a certain method a series of logarithms or sines, whose first, second and third differences etc. are the same, or proportional with the respective given differences of the homogenea of an aequation whose roots are in arithmetical progression. I thank you now kindly for Mr Newton his problem * of interest: I would humbly desire the like favour (if it were not too much trouble to you) in sending me Slusius his exercitation.† for I am exceedingly pleased in what I have seen of his. I cannot express how much I think myself engaged to you for your account of new books; if it were not for you, I would be, as it were, dead to all the world. As to yours, dated 24 Dec., I can hardly beleev, till I see it, that there is any general, compendious & geometrical method for adding an harmonical progression; for if it be, it is also applicable to this following progression $\frac{100}{31}, \frac{100}{41}, \frac{100}{51}, \frac{100}{61}, \frac{100}{71}$, but it is evident that the sum of these hath no less denominator than the product of all the particular denominators, multiplied among themselves, which is got by the ordinary and tedious methods or if you will go to symbols with an harmonical progression, you will find the same difficulties: it were no hard matter to give several particular rules, as for example: let a be the first term, b the second, the first three are $=b+\frac{2a^2}{2a-b}$, the first four are $=b+\frac{6a^3-2ba^2-ba^2}{6a^2-7ba+2b^2}$ yet I question these be more compendious than the ordinar. If there be any such universal

* No. 34, p. 142.
† Sent in May, 1671: No. 42, p. 182.

method I suppose it must be taken from the first, second & third differences, etc; which I believe will little compendize the work: at present I have not leisure to examine it, but afterward I may. I admire much the pocket tube of Mr Newton,[3] being only 6 inches long to magnify 150 times: if it magnifie the diameter of the object so it is incredible; if the superficies it may sufficiently discover the Satellites of Jupiter, & consequently be extraordinarily good: if the soliditie, it must be but ordinar, & not sufficient for that effort: but that, which I think strange, is that it doth not so at a short distance: for certainlie al tubes in a shorter distance are drawn to a greater length, and consequentlie magnifieth more; and in a short distance the due figure is an Cartesian spheroid, which approacheth the more the segment of a sphere, than an hyperbolic conoid, which is the just figure for a considerable distance. I suppose the tube must be overcharged by the eyeglass, & so sufficiently discover the Satellites of Jupiter because of its great magnification (which here only is required) but it must fail in near objects (which require a distinct vision) because of the confused sight occasioned by the overcharging. As for Mr Newton's universal method,[4] I imagine I have some knowledge of it, both as to geometrick & mechanick curves, however I thank you for the series ye sent me, and send you these following in requital. Sit radius $=r$, arcus $=a$, tangens $=t$, secans $=s$, erit $a=t-\dfrac{t^3}{3r^2}+\dfrac{t^5}{5r^4}-\dfrac{t^7}{7r^6}+\dfrac{t^9}{9r^8}$

eritque $$t=a+\frac{a^3}{3r^2}+\frac{2a^5}{15r^4}+\frac{17a^7}{315r^6}+\frac{3233a^9}{181440r^8},$$

et $$s=r+\frac{a^2}{2r}+\frac{5a^4}{24r^3}+\frac{61a^6}{720r^5}+\frac{277a^8}{8064r^7}:$$

Sit nunc tangens artificialis $=t$, & secans artificialis $=s$, & integer quadrans $=q$, erit $s=\dfrac{a^2}{2r}+\dfrac{a^4}{12r^3}+\dfrac{a^6}{45r^5}+\dfrac{17a^8}{2520r^7}+\dfrac{3233a^{10}}{1814400r^9}$; Sit $2a-q=e$, erit $t=e+\dfrac{e^3}{6r^2}+\dfrac{e^5}{24r^4}+\dfrac{61e^7}{5040r^6}+\dfrac{277e^9}{72576r^8}$; si nunc secans artificialis $45°=s$, sitque $s+l$ secans artificialis ad libitum, erit

ejus arcus $$=\tfrac{1}{2}q+l-\frac{l^2}{r}+\frac{4l^3}{3r^2}-\frac{7l^4}{3r^3}+\frac{14l^5}{3r^4}-\frac{452l^6}{45r^5};$$

eritque $$2a-q=t-\frac{t^3}{6r^2}+\frac{t^5}{24r^4}-\frac{61t^7}{5040r^6}+\frac{277}{72576}\frac{t^9}{r^8}.$$

Ye shall here tak notice that the radius artificialis $=0$, and that when ye find $q>2a$, or the artificial secant of 45° to be greater than the given secant, to alter the signs and go on in the work according to the ordinary precepts of Algebra. Sit ellipsis [5] cujus alter semiaxis $=r$, alter $=c$, ex quolibet curvae ellipticae puncto demittatur in semiaxem r recta perpendicularis $=a$: erit curva elliptica perpendiculari a adjacens

$$=a+\frac{r^2a^3}{6c^4}+\frac{4r^2c^2a^5-r^4a^5}{40c^8}+\frac{8c^4r^2a^7+r^6a^7-4c^2r^4a^7}{112c^{12}}$$
$$+\frac{64c^6r^2a^9-48c^4r^4a^9+24c^2r^6a^9-5r^8a^9}{1152c^{16}}:$$

Si determinetur ellipseos species, series haec simplicior evadet. Ut si $c=2r$, foret curva predicta

$$a+\frac{a^3}{96r^2}+\frac{3a^5}{2048r^4}+\frac{113a^7}{458752r^6}+\frac{3419a^9}{75497472r^8}:$$

Reliquis vero manentibus, si curva praedicta esset hyperbola, praedicta quoque series ei inserviret, si omnium terminorum partes affirmentur; et negentur, totus terminus tertius, totus quintus, septimus, et in locis imparibus. I thank you werie heartilie for your good advice, as to the publication of my notions, and for your civil profer; I would be very sorry to put you to so much trouble. I have no inclination to publish anything, save only to reprint my quadratura of the circle, and to add some little trifle to it. As to my method of finding the roots of all equations: one series gives only one root, but for every root there may be infinite number of series: there is some industrie required to enter the series, and to know which root it relateth to; but it is like I may entertain you at more length with this matter hereafter. Ye need not be so closehanded of anything I send you: ye may communicate them to whom ye will, for I am little concerned if they be published under others' name or not. I pray you thank Dr Barrow in my name for the pains he hath been at for my satisfaction [6]: I do werie much admire the fertility of his wit. If ye please, ye may communicate this following unto him. In primo praeclarissimi D. Barrow problemate $\mathrm{HMDO}=\frac{\mathrm{DE}^2}{2}$, etiam posito rectam THO eandem esse cum curva KXL in 2do problemate: nescio an ex hoc capite ulla possit dari 2di problematis solutio. Vereor plenam tertii problematis solutionem a secundo non dependere: si enim EZF fuerit recta rectae AD parallela; ex secundo problemate (quod in hoc casu idem est cum 1^{mo}) integre soluto, datur tantummodo, sicut ego percipio, una tertii problematis solutio, nempe VADB rectangulum, cujus latus $VA = AE$ vel DF: si vero infinitae non dentur, sicut dubito in omnibus hisce, datur saltem altera, nimirum VADB semicirculus, cujus radius est aequalis ipsi AE vel DF. Subtilissima sunt et acerrima ingenii specimina, quae subjungit vir doctissimus, existimo theoremata illa non solum maximas quantitates sed etiam minimas quandoque determinare [7]; ex. gr. (in 1^{mo} theoremate) si cycloformis potestatis m centro D, diametro DA (si ita loqui liceat) descripta, per punctum L, tota intra curvam BLA cadat; erit $DG^m + GL^m$ absolute minimum. Si vero dicta cycloformis alibi occurrat curvae BLA, dantur plura minima. Si cycloformis tota extra curvam BLA cadat; erit $DG^m + GL^m$ absolute maximum, si alibi occurrat, dantur plura maxima. Idem etiam dicimus in 2^{do} theoremate, Si supponatur BLA extendi in rectam, et super eadem in punctis suis, respectivis ordinatas LG perpendiculariter erigi; item centro B, diametro BLA; per punctum G cycloformem describi. Quae hic censemus, tertio et quarto theoremati applicamus, ponendo hyperboliformem potestatibus debitam loco cycloformis, in hoc solo est discrimen, quod (cum hyperboliformis sit figura interminabilis) in figura terminata BDA non detur absolute minimum. Si fuerit TG semper ad GL sicut determinata R ad aliam ex puncto G ipsi AD perpendiculariter erectam) et ex hisce aliis descripta figura vel illi analoga ponatur loco cycloformis, eadem adhuc quinto applicat. Denique si super AD ex punctis G erigantur perpendiculares, quarum quadrata aequentur spatii semper respectivi ADGL duplo, et figura ex his conflata ponatur vice cycloformis, dicta quoque sexto applicantur. No more at present but rests

Your humble servant

J. GREGORIE.

The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha

RANJAN ROY
Beloit College
Beloit, WI 53511

1. Introduction

The formula for π mentioned in the title of this article is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \tag{1}$$

One simple and well-known modern proof goes as follows:

$$\begin{aligned}\arctan x &= \int_0^x \frac{1}{1+t^2}\,dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1}\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt.\end{aligned}$$

The last integral tends to zero if $|x| \leqslant 1$, for

$$\left|\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt\right| \leqslant \left|\int_0^x t^{2n+2}\,dt\right| = \frac{|x|^{2n+3}}{2n+3} \to 0 \quad \text{as } n \to \infty.$$

Thus, arctan x has an infinite series representation for $|x| \leqslant 1$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots . \tag{2}$$

The series for $\pi/4$ is obtained by setting $x = 1$ in (2). The series (2) was obtained independently by Gottfried Wilhelm Leibniz (1646–1716), James Gregory (1638–1675) and an Indian mathematician of the fourteenth century or probably the fifteenth century whose identity is not definitely known. Usually ascribed to Nilakantha, the Indian proof of (2) appears to date from the mid-fifteenth century and was a consequence of an effort to rectify the circle. The details of the circumstances and ideas leading to the discovery of the series by Leibniz and Gregory are known. It is interesting to go into these details for several reasons. The infinite series began to play a role in mathematics only in the second half of the seventeenth century. Prior to that, particular cases of the infinite geometric series were the only ones to be used. The arctan series was obtained by Leibniz and Gregory early in their study of infinite series and, in fact, before the methods and algorithms of calculus were fully developed. The history of the arctan series is, therefore, important because it reveals early ideas on series and their relationship with quadrature or the process of finding the area under a curve. In the case of Leibniz, it is possible to see how he used and transformed older ideas on quadrature to develop his methods. Leibniz's work, in fact, was primarily concerned with quadrature; the $\pi/4$ series resulted (in 1673) when he applied his method to the circle. Gregory, by comparison, was interested in finding an infinite series representation of any given function and discovered the relationship between this and the successive derivatives of the given function. Gregory's discovery, made in 1671, is none other than the Taylor series; note that Taylor was not born until 1685. The ideas of calculus, such as integration by parts, change of variables, and higher derivatives, were not completely understood in the early 1670s. Some particular cases were known, usually garbed in geometric language. For example, the fundamental theorem of calculus was stated as a geometric theorem in a work of Gregory's written in 1668. Similar examples can also be seen in a book by Isaac Barrow, Newton's mentor, published in 1670. Of course, very soon after this transitional period, Leibniz began to create the techniques, algorithms and notations of calculus as they are now known. He had been preceded by Newton, at least as far as the techniques go, but Newton did not publish anything until considerably later. It is, therefore, possible to see how the work on arctan was at once dependent on earlier concepts and a transitional step toward later ideas.

Finally, although the proofs of (2) by Leibniz, Gregory and Nilakantha are very different in approach and motivation, they all bear a relation to the modern proof given above.

2. Gottfried Wilhelm Leibniz (1646–1716)

Leibniz's mathematical background[1] at the time he found the $\pi/4$ formula can be quickly described. He had earned a doctor's degree in law in February 1667, but had studied mathematics on his own. In 1672, he was a mere amateur in mathematics. That year, he visited Paris and met Christiaan Huygens (1629–1695), the foremost physicist and mathematician in continental Europe. Leibniz told the story of this meeting in a 1679 letter to the mathematician Tschirnhaus, "at that time... I did not know the correct definition of the center of gravity. For, when by chance I spoke of it to Huygens, I let him know that I thought that a straight line drawn through the center of gravity always cut a figure into two equal parts,... Huygens laughed when he heard this, and told me that nothing was further from the truth. So I, excited by this stimulus, began to apply myself to the study of the more intricate geometry, although as a matter of fact I had not at that time really studied the *Elements* [Euclid]... Huygens, who thought me a better geometer than I was, gave me to read the letters of Pascal, published under the name of Dettonville; and from these I gathered the method of indivisibles and centers of gravity, that is to say the well-known methods of Cavalieri and Guldinus."[2]

[1]For further information about Leibniz's mathematical development, the reader may consult: J. E. Hofmann, *Leibniz in Paris 1672–1676* (Cambridge: The Cambridge University Press, 1974) and its review by A. Weil, *Collected Papers* Vol. 3 (New York: Springer-Verlag, 1979). An English translation of Leibniz's own account, *Historia et origo calculi differentialis*, can be found in J. M. Child, *The Early Mathematical Manuscripts of Leibniz* (Chicago: Open Court, 1920). An easily available synopsis of Leibniz's work in calculus is given in C. H. Edwards, Jr., *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979).

[2]*The Early Mathematical Manuscripts*, p. 215.

Bonaventura Cavalieri (1598–1647) published his *Geometria Indivisibilibus* in 1635. This book was very influential in the development of calculus. Cavalieri's work indicated that

The study of Pascal played an important role in Leibniz's development as a mathematician. It was from Pascal that he learned the ideas of the "characteristic triangle" and "transmutation." In order to understand the concept of transmutation, suppose A and B are two areas (or volumes) which have been divided up into indivisibles usually taken to be infinitesimal rectangles (or prisms). If there is a one-to-one correspondence between the indivisibles of A and B and if these indivisibles have equal areas (or volumes), then B is said to be obtained from A by transmutation and it follows that A and B have equal areas (or volumes). Pascal had also considered infinitesimal triangles and shown their use in finding, among other things, the area of the surface of a sphere. Leibniz was struck by the idea of an infinitesimal triangle and its possibilities. He was able to derive an interesting transmutation formula, which he then applied to the quadrature of a circle and thereby discovered the series for π. To obtain the transmutation formula, consider two neighboring points $P(x, y)$, and $Q(x + dx, y + dy)$ on a curve $y = f(x)$. First Leibniz shows that area $(\Delta OPQ) = (1/2)$ area (rectangle $(ABCD)$). See FIGURE 1. Here PT is tangent to $y = f(x)$ at P and OS is perpendicular to PT. Let p denote the length of OS and z that of $AC = BD =$ ordinate of T.

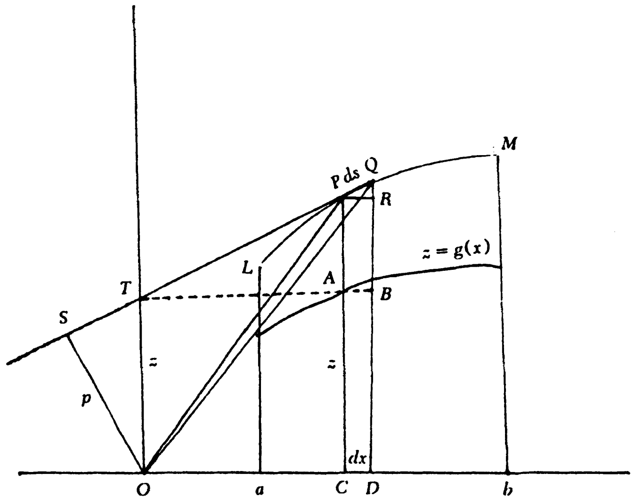


FIGURE 1

$$\int_0^a x^n\, dx = \frac{a^{n+1}}{n+1},$$

when n is a positive integer.

Blaise Pascal (1623–1662) made important and fundamental contributions to projective geometry, probability theory and the development of calculus. The work to which Leibniz refers was published in 1658 and contains the first statement and proof of

$$\int_{\theta_0}^{\theta} \sin\phi\, d\phi = \cos\theta_0 - \cos\theta.$$

This proof is presented in D. J. Struik's *A Source Book in Mathematics 1200–1800* (Cambridge: Harvard University Press, 1969), p. 239.

Paul Guldin (1577–1643), a Swiss mathematician of considerable note, contributed to the development of calculus, and his methods were generally more rigorous than those of Cavalieri.

Since ΔOST is similar to the characteristic ΔPQR,

$$\frac{dx}{p}=\frac{ds}{z},$$

where ds is the length of PQ. Thus,

$$\text{area}\,(OPQ)=\frac{1}{2}pds=\frac{1}{2}zdx. \tag{3}$$

Now, observe that for each point P on $y=f(x)$ there is a corresponding point A. Thus, as P moves from L to M, the points A form a curve, say $Z=g(x)$. If sector OLM denotes the closed region formed by $y=f(x)$ and the straight lines OL and OM, then (3) implies that

$$\text{area}\,(\text{sector } OLM)=\frac{1}{2}\int_a^b g(x)\,dx. \tag{4}$$

This is the transmutation formula of Leibniz. From (4), it follows that the area under $y=f(x)$ is

$$\begin{aligned}\int_a^b y\,dx &= \frac{b}{2}f(b)-\frac{a}{2}f(a)+\text{area}\,(\text{sector } OLM)\\ &=\frac{1}{2}\left([xy]_a^b+\int_a^b z\,dx\right).\end{aligned} \tag{5}$$

This is none other than a particular case of the formula for integration by parts. For it is easily seen from FIGURE 1 that

$$z=y-x\frac{dy}{dx}. \tag{6}$$

Substituting this value of z in (5), it follows that

$$\int_a^b y\,dx=[xy]_a^b-\int_{f(a)}^{f(b)}x\,dy,$$

which is what one gets on integration by parts.

Now consider a circle of radius 1 and center $(1,0)$. Its equation is $y^2=2x-x^2$. In this case, (6) implies that

$$z=\sqrt{2x-x^2}-\frac{x(1-x)}{\sqrt{2x-x^2}}=\sqrt{\frac{x}{2-x}}=\frac{x}{y}, \tag{7}$$

so that

$$x=\frac{2z^2}{1+z^2}. \tag{8}$$

In FIGURE 2, let $A\hat{O}B=2\theta$. Then the area of the sector $AOB=\theta$ and

$$\theta=\text{area}\,(\,\triangle\, AOB)+\text{area}\,(\text{region between arc } AB \text{ and line } AB). \tag{9}$$

By the transmutation formula (4), the second area is $\frac{1}{2}\int_0^x z\,dt$ where z is given by (7). Now, from FIGURE 3 below it is seen that

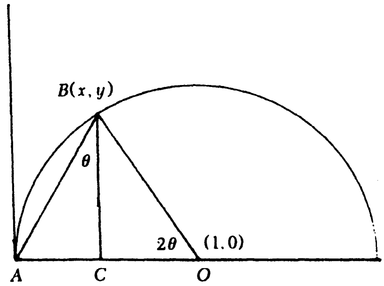


FIGURE 2

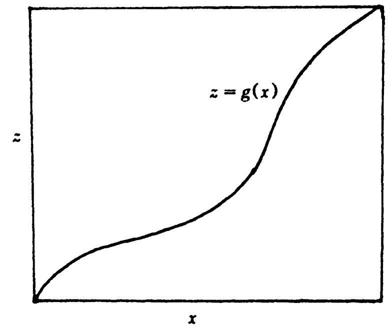


FIGURE 3

$$\frac{1}{2}\int_0^x z\,dt = \frac{1}{2}\left(xz - \int_0^z x\,du\right). \tag{10}$$

Using (8) and (10), it is now possible to rewrite (9) as

$$\begin{aligned}\theta &= \tfrac{1}{2}y + \tfrac{1}{2}xz - \int_0^z \frac{t^2}{1+t^2}\,dt \\ &= \tfrac{1}{2}[z(2-x) + xz] - \int_0^z \frac{t^2}{1+t^2}\,dt \quad (\text{since } y = z(2-x)) \\ &= z - \int_0^z \frac{t^2}{1+t^2}\,dt.\end{aligned}$$

At this point, Leibniz was able to use a technique employed by Nicolaus Mercator (1620–1687). The latter had considered the problem of the quadrature of the hyperbola $y(1+x) = 1$. Since it was already known that

$$\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1},$$

he solved the problem by expanding $1/(1+x)$ as an infinite series and integrating term by term. He simultaneously had the expansion for $\log(1+x)$. Mercator published this result in 1668, though he probably had obtained it a few years earlier. A year later, John Wallis (1616–1703) determined the values of x for which the series is valid. Thus, Leibniz found that

$$\theta = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots. \tag{11}$$

In FIGURE 2, $A\hat{B}C = \theta$ and $z = x/y = \tan\theta$. Therefore, (11) is the series for $\arctan z$.

Of course, Leibniz did not invent the notation for the integral and differential used above until 1675, and his description of the procedures is geometrical but the ideas are the same.

The discovery of the infinite series for π was Leibniz's first great achievement. He communicated his result to Huygens, who congratulated him, saying that this remarkable property of the circle will be celebrated among mathematicians forever. Even Isaac Newton (1642–1727) praised Leibniz's discovery. In a letter of October 24, 1676, to Henry Oldenburg, secretary of the Royal Society of London, he writes, "Leibniz's method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else."[3] Of course, for Leibniz this was only a first step to greater things as he himself says in his "Historia et origo calculi differentialis."

3. James Gregory (1638–1675)

Leibniz had been anticipated in the discovery of the series for arctan by the Scottish mathematician, James Gregory, though the latter did not note the particular case for $\pi/4$.[4] Since Gregory did not publish most of his work on infinite series and also because he died early and worked in isolation during the last seven years of his life, his work did not have the influence it deserved. Gregory's early scientific interest was in optics about which he wrote a masterly book at the age of twenty-four. His book, the *Optica Promota*, contains the earliest description of a reflecting telescope. It was in the hope, which ultimately remained unfulfilled, of constructing such an instrument that he travelled to London in 1663 and made the acquaintance of John Collins (1624–1683), an accountant and amateur mathematician. This friendship with Collins was to prove very important for Gregory when the latter was working alone at St. Andrews University in Scotland. Collins kept him abreast of the work of the English mathematicians such as Isaac Newton, John Pell (1611–1685) and others with whom Collins was in contact.[5]

Gregory spent the years 1664–1668 in Italy and came under the influence of the Italian school of geometry founded by Cavalieri. It was from Stefano degli Angeli (1623–1697), a student of Cavalieri, that Gregory learned about the work of Pierre de Fermat (1601–1665), Cavalieri, Evangelista Torricelli (1608–1647) and others. While in Italy, he wrote two books: *Vera Circuli et Hyperbolae Quadratura* in 1667, and *Geometriae Pars Universalis* in 1668. The first book contains some highly original ideas. Gregory attempted to show that the area of a general sector of an ellipse, circle or hyperbola could not be expressed in terms of the areas of the inscribed and circumscribed triangle and quadrilateral using arithmetical operations and root extraction. The attempt failed but Gregory introduced a number of important ideas such as convergence and algebraic and transcendental functions. The second book contains the first published statement and proof of the fundamental theorem of calculus in geometrical form. It is known that Newton had discovered this result not later than 1666, although he did not make it public until later.

Gregory returned to London in the summer of 1668; Collins then informed him of the latest discoveries of mathematicians working in England, including Mercator's recently published proof of

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

[3]See H. W. Turnbull (ed.), *The Correspondence of Isaac Newton* (Cambridge: The University Press, 1960), Vol. 2, p. 130.

[4]Peter Beckmann has persuasively argued that Gregory must have known the series for $\pi/4$ as well. See Beckmann's *A History of Pi* (Boulder, Colorado: The Golem Press, 1977), p. 133.

[5]The reader might find it of interest to consult: H. W. Turnbull (ed.), *James Gregory Tercentenary Memorial Volume* (London: G. Bell, 1939). This volume contains Gregory's scientific correspondence with John Collins and a discussion of the former's life and work.

Meditation on these discoveries led Gregory to publish his next book, *Exercitationes Geometricae*, in the winter of 1668. This is a sequel to the *Pars Universalis* and is mainly about the logarithmic function and its applications. It contains, for example, the first evaluations of the indefinite integrals of $\sec x$ and $\tan x$.[6] The results are stated in geometric form.

In the autumn of 1668, Gregory was appointed to the chair in St. Andrews and he took up his duties in the winter of 1668/1669. He began regular correspondence with Collins soon after this, communicating to him his latest mathematical discoveries and requesting Collins to keep him informed of the latest activities of the English mathematicians. Thus, in a letter of March 24, 1670, Collins writes, "Mr. Newtone of Cambridge sent the following series for finding the Area of a Zone of a Circle to Mr. Dary, to compare with the said Dary's approaches, putting R the radius and B the parallel distance of a Chord from the Diameter the Area of the space or Zone between them is =

$$2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5} - \frac{5B^9}{576R^7}."^{7}$$

This is all Collins writes about the series but it is, in fact, the value of the integral $2\int_0^B (R^2 - x^2)^{1/2}\,dx$ after expanding by the binomial theorem and term by term integration. Newton had discovered the binomial expansion for fractional exponents in the winter of 1664/1665, but it was first made public in the aforementioned letter of 1676 to Oldenburg.

There is evidence that Gregory had rediscovered the binomial theorem by 1668.[8] However, it should be noted that the expansion for $(1-x)^{1/2}$ does not necessarily

---

[6] A proof of the formula

$$\int_0^{\theta} \sec\phi\, d\phi = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

was of considerable significance and interest to mathematicians in the 1660's due to its connection with a problem in navigation. Gerhard Mercator (1512–1594) published his engraved "Great World Map" in 1569. The construction of the map employed the famous Mercator projection. Edward Wright, a Cambridge professor of mathematics, noted that the ordinate on the Mercator map corresponding to a latitude of θ^0 on the globe is given by $c\int_0^{\theta} \sec\phi\, d\phi$, where c is suitably chosen according to the size of the map. In 1599, Wright published this result in his *Certaine Errors in Navigation Corrected*, which also contained a table of latitudes computed by the continued addition of the secants of 1′, 2′, 3′, etc. This approximation to $\int_0^{\theta} \sec\phi\, d\phi$ was sufficiently exact for the mariner's use. In the early 1640's, Henry Bond observed that the values in Wright's table could be obtained by taking the logarithm of $\tan(\pi/4 + \theta/2)$. This observation was published in 1645 in Richard Norwood's *Epitome of Navigation*. A theoretical proof of this observation was very desirable and Nicolaus Mercator had offered a sum of money for its demonstration in 1666. John Collins, who had himself written a book on navigation, drew Gregory's attention to this problem and, as we noted, Gregory supplied a proof. For more details, one may consult the following two articles by F. Cajori: "On an Integration ante-dating the Integral Calculus," *Bibliotheca Mathematica* Vol. 14 (1913/14), pp. 312–19, and "Algebra In Napier's Day and Alleged Prior Invention of Logarithms," in C. G. Knott (ed.), *Napier Memorial Volume* (London: Longmans, Green & Co., 1915), pp. 93–106. More recently, J. Lohne has established that Thomas Harriot (1560–1621) had evaluated the integral $\int_0^{\theta} \sec\phi\, d\phi$ in 1594 by a stereographic projection of a spherical loxodrome from the south pole into a logarithmic spiral. This work was unpublished and remained unknown until Lohne brought it to light. See J. A. Lohne, "Thomas Harriot als Mathematiker," *Centaurus*, Vol. 11, 1965–66, pp. 19–45. Thus it happened that, although $\int \sec\theta\, d\theta$ is a relatively difficult trigonometric integral, it was the first one to be discovered.

[7] *James Gregory*, p. 89.

[8] See *The Correspondence of Isaac Newton*, Vol. 1, p. 52, note 1.

imply a knowledge of the binomial theorem. Newton himself had proved the expansion by applying the well-known method for finding square roots of numbers to the algebraic expression $1-x$. Moreover, it appears that the expansion of $(1-x)^{1/2}$ was discovered by Henry Briggs (1556–1630) in the 1620's, while he was constructing the log tables.[9] But there is no indication that Gregory or Newton knew of this. In any case, for reasons unknown, Gregory was unable to make anything of the series—as evidenced by his reply of April 20, "I cannot understand the series you sent me of the circle, if this be the original, I take it to be no series."[10] However, by September 5, 1670, he had discovered the general interpolation formula, now called the Gregory-Newton interpolation formula, and had made from it a number of remarkable deductions. He now knew how "to find the sinus having the arc and to find the number having the logarithm." The latter result is precisely the binomial expansion for arbitrary exponents. For, if we take x as the logarithm of y to the base $1+d$, then $y=(1+d)^x$ and Gregory gives the solution as

$$(1+d)^x = 1 + xd + \frac{x(x-1)}{1\cdot 2}d^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}d^3 + \cdots .^{11}$$

It is possible that Newton's series in Collins' letter had set Gregory off on the course of these discoveries, but he did not even at this point see that he could deduce Newton's result. Soon after, he did observe this and wrote on December 19, 1670, "I admire much my own dullness, that in such a considerable time I had not taken notice of this."[12] All this time, he was very eager to learn about Newton's results on series and particularly the methods he had used. Finally on December 24, 1670, Collins sent him Newton's series for $\sin x$, $\cos x$, $\sin^{-1}x$ and $x\cot x$, adding that Newton had a universal method which could be applied to any function. Gregory then made a concentrated effort to discover a general method for himself. He succeeded. In a famous letter of February 15, 1671 to Collins he writes, "As for Mr. Newton's universal method, I imagine I have some knowledge of it, both as to geometrick and mechanick curves, however I thank you for the series ye sent me and send you these following in requital."[13] Gregory then gives the series for $\arctan x$, $\tan x$, $\sec x$, $\log\sec x$, $\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, $\operatorname{arcsec}(\sqrt{2}\,e^x)$, and $2\arctan\tanh x/2$. However, what he had found was not Newton's method but rather the Taylor expansion more than forty years before Brook Taylor (1685–1731). Newton's method consisted of reversion of series, expansion by the binomial theorem, long division by series and term by term integration.[14] Thinking that he had rediscovered Newton's method, Gregory did not

[9] See D. T. Whiteside, "Henry Briggs: The Binomial theorem Anticipated," *The Mathematical Gazette*, Vol. 15, (1962), p. 9. Whiteside shows how the expansion of $(1+x)^{1/2}$ arose out of Brigg's work on logarithms.

[10] *James Gregory*, p. 92.

[11] In their review of the Gregory Memorial Volume, M. Dehn and E. Hellinger explain how the binomial expansion comes out of the interpolation formula. See *The American Mathematical Monthly*, Vol. 50, (1943), p. 149.

[12] *James Gregory*, p. 148.

[13] *Ibid.*, p. 170.

[14] It should be mentioned that Newton himself discovered the Taylor series around 1691. See D. T. Whiteside (ed.), *The Mathematical Papers of Isaac Newton*, Vol. VII (Cambridge: The Cambridge University Press, 1976), p. 19. In fact, Taylor was anticipated by at least five mathematicians. However, the

publish his results. It is only from notes that he made on the back of a letter from Gedeon Shaw, an Edinburgh stationer, dated January 29, 1671, that it is possible to conclude that Gregory had the idea of the Taylor series. These notes contain the successive derivatives of tan x, sec x, and the other functions whose expansions he sent to Collins. The following extract from the notes gives the successive derivatives of tan θ; here m is successively y, $\frac{dy}{d\theta}$, $\frac{d^2y}{d\theta^2}$, etc., and $q = r \tan\theta$. Gregory writes[15]:

1st	2nd	3rd	4th
$m = q$	$m = r + \frac{q^2}{r}$	$m = 2q + \frac{2q^3}{r^2}$	$m = 2r + \frac{8q^2}{r} + \frac{6q^4}{r^3}$

5th

$$m = 16q + \frac{40q^3}{r^2} + \frac{24q^5}{r^4}$$

6th

$$m = 16r + \frac{136q^2}{r} + \frac{240q^4}{r^3} + \frac{120q^6}{r^5}$$

7th

$$m = 272q + 987\frac{q^3}{r^2} + 1680\frac{q^5}{r^4} + 720\frac{q^7}{r^6}$$

8th

$$m = 272r + 3233\frac{q^2}{r} + 11361\frac{q^4}{r^3} + 13440\frac{q^6}{r^5} + 5040\frac{q^8}{r^7}.$$

It is clear from the form in which the successive derivatives are written that each one is formed by multiplying the derivative with respect to q of the preceding term by $r + \frac{q^2}{r}$. Now writing $a = r\theta$, Gregory gives the series in the letter to Collins as follows:

$$r\tan\theta = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{3233a^9}{181440r^8} + \cdots.$$

The reasons for supposing that these notes were written not much before he wrote to Collins and were used to construct the series are (i) the date of Gedeon Shaw's letter and (ii) Gregory's error in computing the coefficient of $\frac{q^3}{r^2}$ in the 7th m, which should be 1232 instead of 987 and which, in turn, leads to the error in the 8th m, where the coefficient of $\frac{q^2}{r}$ should be 3968 instead of 3233. This error is then repeated in the series showing the origin of the series. Moreover, in the early parts of the notes, Gregory is unsure about how he should write the successive derivatives. For example, he attempts to write the derivative of sec θ as a function of sec θ but then abandons the idea. He comes back to it later and sees that it is easier to work with m^2 instead of m since the m^2's can be expressed as polynomials in tan θ. This is, of course, sufficient to give him the series for sec θ. The series for log sec θ and log tan$(\pi/4 + \theta)$ he then obtains by term by term integration of the series for tan θ and sec θ

Taylor series is not unjustly named after Brook Taylor who published it in 1715. He saw the importance of the result and derived several interesting consequences. For a discussion of these matters see: L. Feigenbaum, "Brook Taylor and the Method of Increments," *Archive for History of Exact Sciences*, Vol. 34, (1985), pp. 1–140.

[15]*James Gregory*, p. 352.

respectively. Naturally, the 3233 error is repeated. He must have obtained the series for arc tan x from the 2nd m which can be written as

$$\frac{da}{dq} = \frac{r^2}{r^2 + q^2} = 1 - \frac{q^2}{r^2} + \frac{q^4}{r^4} - \cdots .$$

The arctan series follows after term by term integration. Clearly, Gregory had made great progress in the study of infinite series and the calculus and, had he lived longer and published his work, he might have been classed with Newton and Leibniz as a co-discoverer of the calculus. Unfortunately, he was struck by a sudden illness, accompanied with blindness, as he was showing some students the satellites of Jupiter. He did not recover and died soon after in October, 1675, at the age of thirty-seven.

4. Kerala Gargya Nilakantha (c.1450–c.1550)

Another independent discovery of the series for arctan x and other trigonometric functions was made by mathematicians in South India during the fifteenth century. The series are given in Sanskrit verse in a book by Nilakantha called *Tantrasangraha* and a commentary on this work called *Tantrasangraha-vakhya* of unknown authorship. The theorems are stated without proof but a proof of the arctan, cosine and sine series can be found in a later work called *Yuktibhasa*. This was written in Malayalam, the language spoken in Kerala, the southwest coast of India, by Jyesthadeva (c.1500–c.1610) and is also a commentary on the *Tantrasangraha*. These works were first brought to the notice of the western world by an Englishman named C. M. Whish in 1835. Unfortunately, his paper on the subject had almost no impact and went unnoticed for almost a century when C. Rajagopal[16] and his associates began publishing their findings from a study of these manuscripts. The contributions of medieval Indian mathematicians are now beginning to be recognized and discussed by authorities in the field of the history of mathematics.[17]

It appears from the astronomical data contained in the *Tantrasangraha* that it was composed around the year 1500. The *Yuktibhasa* was written about a century later. It is not completely clear who the discoverer of these series was. In the *Aryabhatiya-bhasya*, a work on astronomy, Nilakantha attributes the series for sine to Madhava. This mathematician lived between the years 1340–1425. It is not known whether

[16]Rajagopal's work may be found in the following papers: (with M. S. Rangachari) "On an Untapped Source of Medieval Keralese Mathematics," *Archive for History of Exact Sciences*, Vol. 18, (1977), pp. 89–102; "On Medieval Kerala Mathematics," *Archive for History of Exact Sciences*, Vol. 35, (1986), pp. 91–99. These papers give the Sanskrit verses of the *Tantrasangrahavakhya* which describe the series for the arctan, sine and cosine. An English translation and commentary is also provided. A commentary on the proof of arctan series given in the *Yuktibhasa* is available in the two papers: "A Neglected Chapter of Hindu Mathematics," *Scripta Maghematica*, Vol. 15, (1949), pp. 201–209; "On the Hindu Proof of Gregory's Series," *Ibid.*, Vol. 17, (1951), pp. 65–74. A commentary on the *Yuktibhasa's* proof of the sine and cosine series is contained in C. Rajagopal and A. Venkataraman, "The sine and cosine power series in Hindu mathematics," *Journal of the Royal Asiatic Society of Bengal, Science*, Vol. 15, (1949), pp. 1–13.

[17]See J. E. Hofmann, "Über eine alt indische Berechnung von π und ihre allgemeine Bedeutung," *Mathematische-Physikalische Semester Berichte*, Bd. 3, H. 3/4, Hamburg (1953). See also D. T. Whiteside, "Patterns of Mathematical Thought in the later Seventeenth Century," *Archive for History of Exact Sciences*, Vol. 1, (1960–1962), pp. 179–388. For a discussion of medieval Indian mathematicians and the *Tantrasangraha* in particular, one might consult: A. P. Jushkevich, *Geschichte der Mathematik in Mittelalter* (German translation Leipzig, 1964, of the Russian original, Moscow, 1961).

Madhava found the other series as well or whether they are somewhat later discoveries.

Little is known about these mathematicians. Madhava lived near Cochin in the very southern part of India (Kerala) and some of his astronomical work still survives. Nilakantha was a versatile genius who wrote not only on astronomy and mathematics but also on philosophy and grammar. His erudite expositions on the latter subjects were well known and studied until recently. He attracted several gifted students, including Tuncath Ramanujan Ezuthassan, an early and important figure in Kerala literature. About Jyesthadeva, nothing is known except that he was a Brahmin of the house of Parakroda.

In the *Tantrasangraha-vakhya*, the series for arctan, sine and cosine are given in verse which, when converted to mathematical symbols may be written as

$$r\arctan\frac{y}{x}=\frac{1}{1}\cdot\frac{ry}{x}-\frac{1}{3}\cdot\frac{ry^3}{x^3}+\frac{1}{5}\cdot\frac{ry^5}{x^5}-\cdots,\ \text{where } \frac{y}{x}\leqslant 1,$$

$$y=s-s\cdot\frac{s^2}{(2^2+2)r^2}+s\cdot\frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2}-\cdots\quad\text{(sine)}$$

$$r-x=r\cdot\frac{s^2}{(2^2-2)r^2}-r\cdot\frac{s^2}{(2^2-2)r^2}\cdot\frac{s^2}{(4^2-4)r^2}+\cdots\quad\text{(cosine)}.$$

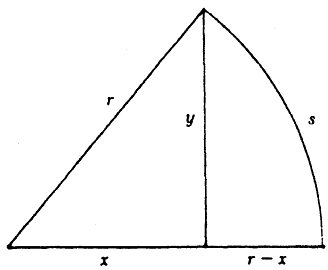


FIGURE 4

There are also some special features in the *Tantrasangraha's* treatment of the series for $\pi/4$ which were not considered by Leibniz and Gregory. Nilakantha states some rational approximations for the error incurred on taking only the first n terms of the series. The expression for the approximation is then used to transform the series for $\pi/4$ into one which converges more rapidly. The errors are given as follows:

$$\frac{\pi}{4}\approx 1-\frac{1}{3}+\frac{1}{5}-\cdots\mp\frac{1}{n}\pm f_i(n+1)\qquad i=1,2,3,\tag{12}$$

where

$$f_1(n)=\frac{1}{2n},\ f_2(n)=\frac{n/2}{n^2+1}\text{ and } f_3(n)=\frac{(n/2)^2+1}{(n^2+5)n/2}.$$

The transformed series are as follows:

$$\frac{\pi}{4}=\frac{3}{4}+\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\cdots\tag{13}$$

and

$$\frac{\pi}{4}=\frac{4}{1^5+4\cdot 1}-\frac{4}{3^5+4\cdot 3}+\frac{4}{5^5+4\cdot 5}-\cdots.$$

Leibniz's proof of the formula for $\pi/4$ was found by the quadrature of a circle. The proof in Jyesthadeva's book is by a direct rectification of an arc of a circle. In the diagram given below, the arc AC is a quarter circle of radius one with center O and $OABC$ is a square. The side AB is divided into n equal parts of length δ so that $n\delta=1$, $P_{r-1}P_r=\delta$. EF and $P_{r-1}D$ are perpendicular to OP_r. Now, the triangles OEF and $OP_{r-1}D$ are similar, which gives

$$\frac{EF}{OE}=\frac{P_{r-1}D}{OP_{r-1}},\quad \text{that is, } EF=\frac{P_{r-1}D}{OP_{r-1}}.$$

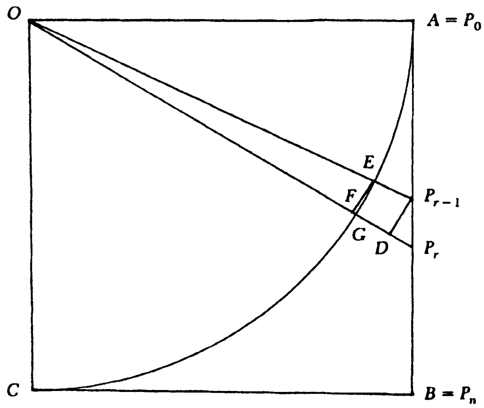


FIGURE 5

The similarity of the Δs $P_{r-1}P_rD$ and OAP_r gives

$$\frac{P_{r-1}P_r}{OP_r}=\frac{P_{r-1}D}{OA}\text{ or } P_{r-1}D=\frac{P_{r-1}P_r}{OP_r}.$$

Thus,

$$EF=\frac{P_{r-1}P_r}{OP_{r-1}OP_r}\simeq\frac{P_{r-1}P_r}{OP_r^2}=\frac{\delta}{1+AP_r^2}=\frac{\delta}{1+r^2\delta^2}.$$

Since arc $EG\simeq EF\simeq\dfrac{\delta}{1+r^2\delta_r^2}$, $\dfrac{1}{8}$ arc of circle is

$$\frac{\pi}{4}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\delta}{1+r^2\delta^2}. \tag{14}$$

Of course, a clear idea of limits did not exist at that time so that the relation was understood in an intuitive sense only. To evaluate the limit, Jyesthadeva uses two lemmas. One is the geometric series

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots.$$

Jyesthadeva says that the expansion is obtained on iterating the following procedure:

$$\frac{1}{1+x} = 1 - x\left(\frac{1}{1+x}\right) = 1 - x\left(1 - x\left(\frac{1}{1+x}\right)\right).$$

The other result is that

$$S_n^{(p)} \equiv 1^p + 2^p + \cdots + n^p \sim \frac{n^{p+1}}{p+1} \quad \text{for large } n. \tag{15}$$

A sketch of a proof is given by Jyesthadeva. He notes first that

$$nS_n^{(p-1)} = S_n^{(p)} + S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_{n-1}^{(p-1)}. \tag{16}$$

This is easy to verify. Relation (16) is also contained in the work of the tenth century Arab mathematician Alhazen, who gives a geometrical proof in the Greek tradition[18]. He uses it to evaluate $S_n^{(3)}$ and $S_n^{(4)}$ which occur in a problem about the volume of a certain solid of revolution. *Yuktibhasa* shows that for $p = 2, 3$

$$S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_{n-1}^{(p-1)} \sim \frac{S_n^{(p)}}{p}, \tag{17}$$

and then suggests that by induction the result will be true for all values of p. Once this is granted, it follows that if

$$S_n^{(p-1)} \sim \frac{n^p}{p},$$

then by (16) and (17),

$$nS_n^{(p-1)} \sim S_n^{(p)} + \frac{S_n^{(p)}}{p} \quad \text{or} \quad S_n^{(p)} \sim \frac{n^{p+1}}{p+1},$$

and (15) is inductively proved.

We now note that (14) can be rewritten, after expanding $1/(1 + r^2\delta^2)$ into a geometric series, as

$$\begin{aligned}\frac{\pi}{4} &= \lim_{n\to\infty}\left[\delta\sum_{r=1}^{n} 1 - \delta^3\sum_{r=1}^{n} r^2 + \delta^5\sum_{r=1}^{n} r^4 - \cdots\right]\\ &= \lim_{n\to\infty}\left[1 - \frac{1}{n^3}\sum_{r=1}^{n} r^2 + \frac{1}{n^5}\sum_{r=1}^{n} r^4 - \cdots\right]\\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,\end{aligned}$$

where we have used relation (15) and the fact that $\delta = 1/n$. Now consider the approximation (12) and its application to the transformation of series. Suppose that

$$\sigma_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \pm \frac{1}{n} \mp f(n+1),$$

where $f(n+1)$ is a rational function of n which will make σ_n a better approximation of $\pi/4$ than the nth partial sum S_n. Changing n to $n-2$ we get

[18] See *The Historical Development of the Calculus* (mentioned in footnote 1), p. 84. Alhazen is the latinized form of the name Ibn Al-Haytham (c. 965–1039).

$$\sigma_{n-2}=1-\frac{1}{3}+\frac{1}{5}-\cdots\mp\frac{1}{n-2}\pm f(n-1).$$

Subtracting the second relation from the first,

$$\pm u_n=\sigma_n-\sigma_{n-2}=\pm\frac{1}{n}\mp f(n+1)\mp f(n-1). \tag{18}$$

Then

$$\begin{aligned}\sigma_n&=\sigma_{n-2}\pm u_n\\&=\sigma_{n-4}\mp u_{n-2}\pm u_n\\&=\cdots=\sigma_1-u_3+u_5-u_7+\cdots\pm u_n\\&=1-f(2)-u_3+u_5-u_7+\cdots u_n.\end{aligned}$$

It is clear that

$$\lim_{n\to\infty}\sigma_n=\frac{\pi}{4}$$

and therefore

$$\frac{\pi}{4}=1-f(2)-u_3+u_5-u_7+\cdots. \tag{19}$$

Thus, we have a new series for $\pi/4$ which depends on how the function $f(n)$ is chosen. Naturally, the aim is to choose $f(n)$ in such a way that (19) is more rapidly convergent than (1). This is the idea behind the series (13). Now equation (18) implies that

$$f(n+1)+f(n-1)=\frac{1}{n}-u_n. \tag{20}$$

For (19) to be more rapidly convergent than (1), u_n should be $o(1/n)$, that is, negligible compared to $1/n$. It is reasonable to assume $f(n+1)\simeq f(n-1)\simeq f(n)$. These observations together with (20) imply that $f(n)=1/2n$ is a possible rational approximation in equation (12). With this $f(n)$, the value of u_n is given by (20) to be

$$u_n=\frac{1}{n}-\frac{1}{2(n+1)}-\frac{1}{2(n-1)}=-\frac{1}{n^3-n}.$$

Substituting this in (19) gives us (13), which is

$$\frac{\pi}{4}=1-\frac{1}{4}+\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\cdots.$$

The other series

$$\frac{\pi}{4}=\frac{4}{1^5+4\cdot 1}-\frac{4}{3^5+4\cdot 3}+\frac{4}{5^5+4\cdot 5}-\cdots$$

is obtained by taking $f(n)=\dfrac{n/2}{n^2+1}$ in (19).

It should be mentioned that Newton was aware of the correction $f_1(n)=1/2n$. For in the letter to Oldenburg, referred to earlier, he says, "By the series of Leibniz also if half the term in the last place be added and some other like devices be employed, the computation can be carried to many figures." However, he says nothing about transforming the series by means of this correction.

It appears that Nilakantha was aware of the impossibility of finding a finite series of rational numbers to represent π. In the *Aryabhatiya-bhasya* he writes, "If the diameter, measured using some unit of measure, were commensurable with that unit, then the circumference would not likewise allow itself to be measured by means of the same unit; so likewise in the case where the circumference is measurable by some unit, then the diameter cannot be measured using the same unit."[19]

The *Yuktibhasa* contains a proof of the arctan series also and it is obtained in exactly the same way except that one rectifies only a part of the 1/8 circle.

It can be shown that if $\pi/4 = S_n + f(n)$, where S_n is the nth partial sum, then $f(n)$ has the continued fraction representation

$$f(n) = \frac{1}{2}\left[\frac{1}{n+}\,\frac{1^2}{n+}\,\frac{2^2}{n+}\,\frac{3^2}{n+}\cdots\right]. \tag{21}$$

Moreover, the first three convergents are

$$f_1(n) = \frac{1}{2n}, \qquad f_2(n) = \frac{n/2}{n^2+1} \quad \text{and} \quad f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2},$$

which are the values quoted in (13). Clearly, Nilakantha was using some procedure which gave the successive convergents of the continued fraction (21) but the text contains no suggestion that (20) was actually known to him. This continued fraction implies that

$$\frac{2}{4-\pi} = 2 + \frac{1^2}{2+}\,\frac{2^2}{2+}\,\frac{3^2}{2+}\cdots,$$

which may be compared with the continued fraction of the seventeenth century English mathematician, William Brouncker (1620–1684), who gave the result

$$\frac{4}{\pi} = 1 + \frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots.$$

The third approximation

$$f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2}$$

is very effective in obtaining good numerical values for π without much calculation. For example

$$1 - \frac{1}{3} + \cdots - \frac{1}{19} + f_3(20)$$

gives the value of π correct up to eight decimal places.[20] Nilakantha himself gives 104348/33215 which is correct up to nine places. It is interesting that the Arab mathematician Jamshid-al-Kasi, who also lived in the fifteenth century, had obtained the same approximation by a different method.

---

[19] See *Geschichte der Mathematik*, p. 169.

[20] These observations concerning the continued fraction expansion of $f(n)$ and its relation to the Indian work and that of Brouncker, and concerning the decimal places in f (20), are due to D. T. Whiteside. See "On Medieval Kerala Mathematics" of footnote 13.

5. Independence of these discoveries.

The question naturally arises of the possibility of mutual influence between or among the discoverers of power series, in particular the series for the trigonometric functions. Because of the lively trade relations between the Arabs and the west coast of India over the centuries, it is generally accepted that mathematical ideas were also exchanged. However, there is no evidence in any existing mathematical works of the Arabs that they were aware of the concept of a power series. Therefore, we may grant the Indians priority in the discovery of the series for sine, cosine and arctangent. Moreover, historians of mathematics are in agreement that the European mathematicians were unaware of the Indian discovery of infinite series.[21] Thus, we may conclude that Newton, Gregory and Leibniz made their discoveries independently of the Indian work. In fact, it appears that yet another independent discovery of an infinite series giving the value of π was made by the Japanese mathematician Takebe Kenko (1664–1739) in 1722. His series is

$$\pi^2 = 4\left[1 + \sum_{n=1}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+2)!}\right].^{22}$$

This series was not obtained from the arctan series and its discussion is therefore not included. However, the independent discovery of the infinite series by different persons living in different environments and cultures gives us insight into the character of mathematics as a universal discipline.

Acknowledgement. I owe thanks to Phil Straffin for encouraging me to write this paper and to the referees for their suggestions.

---

[21] See "Patterns of Mathematical Thought in the later Seventeenth Century" of footnote 17. See also A. Weil, "History of Mathematics: Why and How" in *Collected Papers*, Vol. 3 (New York: Springer-Verlag, 1979), p. 435.

[22] See D. E. Smith and Y. Mikami, *A History of Japanese Mathematics* (Chicago: Open Court, 1914). This series was also obtained by the French missionary Pierre Jartoux (1670–1720) in 1720. He worked in China and was in correspondence with Leibniz, but the present opinion is that Takebe's discovery was independent. Leonhard Euler (1707–1783) rediscovered the same series in 1737. A simple evaluation of it can be given using Clausen's formula for the square of a hypergeometric series.

WILLIAM JONES

The First Use of π for the Circle Ratio

(Selections Made by David Eugene Smith from the Original Work.)

William Jones (1675–1749) was largely a self-made mathematician. He had considerable genius and wrote on navigation and general mathematics. He edited some of Newton's tracts. The two passages given below are taken from the *Synopsis Palmariorum Matheseos: or, a New Introduction to the Mathematics*, London, 1706. The work was intended "for the Vse of some Friends, who had neither Leisure, Conveniency, nor, perhaps, Patience, to search into so many different Authors, and turn over so many tedious volumes, as is unavoidably required to make but a tolerable Progress in the Mathematics." It was a very ingenious compendium of mathematics as then known. The symbol π first appears on page 243, and again on p. 263. The transcendence of π was proved by Lindemann in 1882. For the transcendence of e, which proved earlier (1873), see page 99.

Taking a as an arc of 30°, aud t as a tangent in a figure given, he states (p. 243):

$$6a, \text{ or } 6 \times t - \frac{1}{3}t^2 + \frac{1}{5}t^5, \text{ \&c.} = \frac{1}{2} \text{ Periphery } (\pi)\ldots$$

Let

$$\alpha = 2\sqrt{3},\ \beta = \frac{1}{3}\alpha,\ \gamma = \frac{1}{3}\beta,\ \delta = \frac{1}{3}\gamma, \text{ \&c.}$$

Then

$$\alpha - \frac{1}{3}\beta + \frac{1}{5}\gamma - \frac{1}{7}\delta + \frac{1}{9}\epsilon, \text{ \&c.} = \frac{1}{2}\pi,$$

or

$$\alpha - \frac{1}{3}\frac{3\alpha}{9} + \frac{1}{5}\frac{\alpha}{9} - \frac{1}{7}\frac{3\alpha}{9^2} + \frac{1}{9}\frac{\alpha}{9^2} - \frac{1}{11}\frac{3\alpha}{9^3} + \frac{1}{13}\frac{\alpha}{9^3}, \text{ \&c.}$$

Theref. the (Radius is to ½ Periphery, or) Diameter is to the Periphery, as 1,000, &c to 3.141592653 . 5897932384 . 6264338327 . 9502884197 . 1693993751 . 0582097494 . 4592307816 . 4062862089 . 9862803482 . 5342117067. 9+ True to above a 100 Places; as Computed by the accurate and Ready Pen of the Truly Ingenious Mr. *John Machin.*

On p. 263 he states:

There are various other ways of finding the *Lengths*, or *Areas* of particular *Curve Lines*, or *Planes*, which may very much facilitate the Practice; as for Instance, in the *Circle*, the *Diameter* is to *Circumference* as 1 to

$$\frac{16}{3} - \frac{4}{239} - \frac{1}{3}\frac{16}{5^3} - \frac{4}{239^3} + \frac{1}{5}\frac{16}{5^5} - \frac{4}{239^5} -, \text{\&c.} = 3.14159, \text{\&c.} = \pi \ldots$$

Whence in the *Circle*, any one of these three, α, c, d, being given, the other two are found, as, $d = c \div \pi = \overline{\alpha \div \frac{1}{4}\pi}\Big|^{\frac{1}{2}}$, $c = d \times \pi = \overline{\alpha \times 4\pi}\Big|^{\frac{1}{2}}$, $\alpha = \frac{1}{4}\pi d^2 = c^2 \div 4\pi$.

Area $bdDB = \frac{1}{2}X^2 - \frac{5}{3}X - \boxed{\frac{2}{9X}} - \frac{7}{81X}$, &c. $-\frac{1}{2}xx + \frac{1}{3}x + \boxed{\frac{2}{9x}} + \frac{7}{81x}$, &c. *i. e.* $= \frac{1}{2}X^2 - \frac{1}{3}X - \frac{7}{81}X$, &c. $-\frac{1}{2}x^2 + \frac{1}{3}x + \frac{7}{81}x$, &c. $-\frac{4y}{9e} - \frac{4y^3}{27e^3} - \frac{4y^5}{45e^5}$, &c.

But this Hyperbolick term for the moſt part may be very commodiouſly avoided, by altering the beginning of the Abſciſs; that is, by increaſing or diminiſhing it by ſome given quantity. As in the former Example, where $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ was the equation to the Curve; if I would make b to be the beginning of the Abſciſs, ſuppoſing Ab to be of any determinate length, *viz.* $\frac{1}{2}a$, for the remainder of the Abſciſs bB, I ſhall now write x: ſo that, if I diminiſh the Abſciſs by $\frac{1}{2}a$, by writing $x + \frac{1}{2}a$ inſtead of x, it will become $\frac{\frac{1}{2}a^3 - a^2x}{\frac{3}{4}a^2 + 2ax + x^2} = \dot{z}$; and by diviſion $\dot{z} = \frac{2}{3}a - \frac{28}{9}x + \frac{200x^2}{27a}$, &c. whence ariſes $z = \frac{2}{3}ax - \frac{14}{9}x^2 + \frac{200x^3}{81a}$, &c. $= bdDB$.

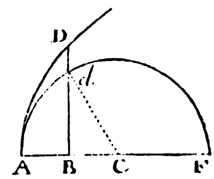
Alſo the equation $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ might have been reſolved into the Two infinite ſeries, $\dot{z} = \frac{a^3}{x^2} - \frac{a^4}{x^3} + \frac{a^5}{x^4}$, &c. $-a + x - \frac{xx}{a} + \frac{x^3}{a^2}$, &c. where there is found no term divided by the firſt power of x.

But ſuch kind of ſeries, where the powers of x aſcend infinitely in the numerators of one, and in the denominators of the other, are not ſo proper to derive the value of z from by Arithmetical Computation, when the ſpecies are to be changed into Numbers.

Scarce

Scarce any difficulty can occur to any one, who is to undertake ſuch a computation in Numbers, after the value of the Area is obtained in ſpecies. Yet for the more compleat illuſtration of the foregoing doctrine, I ſhall add an Example or Two.

Let the Hyperbola AD be propoſed, whoſe equation is $\sqrt{x + xx} = z$, its vertex being at A, and each of its Axes equal to unity; from what goes before, its Area $ADB = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} + \frac{1}{72}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}}$, &c. that is, $x^{\frac{1}{2}}$ into $\frac{2}{3}x + \frac{1}{5}x^2 - \frac{1}{28}x^3 + \frac{1}{72}x^4 - \frac{5}{704}x^5$, &c. which ſeries may be infinitely produced by multiplying the laſt term continually by the ſucceeding terms of this progreſſion, $\frac{1.3}{2.5}x \,.\, \frac{-1.5}{4.7}x \,.\, \frac{-3.7}{6.9}x \,.\, \frac{-5.9}{8.11}x \,.\, \frac{-7.11}{10.13}x$, &c. that is, the



firſt term $\frac{2}{3}x^{\frac{3}{2}}$ multiplied by $\frac{1.3}{2.5}x$, makes the ſecond term $\frac{1}{5}x^{\frac{5}{2}}$; which multiplied by $\frac{-1.5}{4.7}x$, makes the third term $\frac{-1}{28}x^{\frac{7}{2}}$; which multiplied by $\frac{-3.7}{6.9}$, makes the fourth term $+\frac{1}{72}x^{\frac{9}{2}}$. And ſo on *ad infinitum.*

Now let AB be aſſumed of any length, ſuppoſe $\frac{1}{4}$, and writing this number for x, and its root $\frac{1}{2}$ for $x^{\frac{1}{2}}$, the firſt term $\frac{2}{3}x^{\frac{3}{2}}$ or $\frac{2}{3} \times \frac{1}{8}$ being reduced to a decimal fraction, becomes 0.083333333, &c. this into $\frac{1.3}{2.5.4}$ makes 0 . 00625 the ſecond term; this into $\frac{-1.5}{4.7.4}$ makes 0 . 0002790178, &c. the third term. And ſo on for ever. But

S the

the terms thus reduced by degrees, I difpofe into Two Tables; the affirmative terms in One, and the Negative in Another, and add them up as you fee here.

+ 0 . 0833333333333333	— 0 . 0002790178571429
62500000000000	34679066051
271267361111	834465027
5135169396	26285354
144628917	961296
4954581	38676
190948	1663
7963	75
352	4
16	
1	0 . 0002825719389575
0 . 0896109885646618	+ 0 . 0896109885646618
	0 . 0893284166257043

Then from the fum of the affirmative, I take the fum of the negative terms, and there remains 0.0893284166257043 for the quantity of the Hyperbolick Area A*d*B which was to be found.

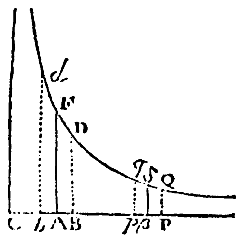
Let the Circle A*d*F [*See the fame Fig.*] be propofed, which is expreffed by the equation $\sqrt{x-xx}=\dot{z}$, whofe diameter is unity; and from what goes before its Area A*d*B will be $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$, *&c.* in which feries, fince the terms do not differ from the terms of the feries which above expreffed the Hyperbolick Area, except in the figns + and —; nothing elfe remains to be done, than to connect the fame numeral terms with their figns; that is, by fubtracting the connected fums of both the forementioned Tables, 0.0898935605036193, from the firft term doubled 0.1666666666666666, *&c.* and the remainder 0.0767731061630473 will be the portion A*d*B of the Circular Area, fuppofing AB to be a fourth part of the Diameter.

And hence we may obferve, that though the Areas of the Circle and Hyperbola are not expreffed in a Geometrical confideration, yet each of them is difcovered by the fame Arithmetical computation.

The portion of the Circle A*d*B being found, from thence the whole Area may be derived. For the radius *d*C being drawn, multiply B*d* or $\frac{1}{4}\sqrt{3}$ into BC or $\frac{1}{4}$, and one half of the product $\frac{1}{32}\sqrt{3}$, or 0.0541265877365275 will be the value of the Triangle C*d*B; which added to the Area A*d*B, will give the Sector AC*d*, 0.1308996938995747, the Sextuple of which 0.7853981633974482 is the whole Area.

And hence (by the way) the length of the Circumference will be 3.1415926535897928, which is found by dividing the Area by a fourth part of the diameter.

To this we fhall add the calculation of the Area comprehended between the Hyperbola *d*FD and its Afymptote CA, let C be the center of the Hyperbola, and putting CA $=a$, AF $=b$, and AB $=$ A$b=x$; it will be $\frac{ab}{a+x}=$ BD, and $\frac{ab}{a-x}=bd$; whence the Area AFDB $=bx-\frac{bxx}{2a}+\frac{bx^3}{3a^2}-\frac{bx^4}{4a^3}$, *&c.* And the



Area AF$db=bx+\frac{bx^2}{2a}+\frac{bx^3}{3a^2}+\frac{bx^4}{4a^3}$, *&c.* And the fum bdDB $=2bx+\frac{2bx^3}{3a^2}+\frac{2bx^5}{5a^4}+\frac{2bx^7}{7a^6}$, *&c.* Now let us fuppofe CA $=$ AF $=1$, and Ab or AB $=\frac{1}{10}$, Cb being $=0.9$, and CB $=1.1$. then fubftitut-

S 2 ing

CHAPTER X

On the Use of the Discovered Factors to Sum Infinite Series.

165. If $1 + Az + Bz^2 + Cz^3 + Dz^4 + \cdots$ $= (1 + \alpha z)(1 + \beta z)(1 + \gamma z)(1 + \delta z) \cdots$, then these factors, whether they be finite or infinite in number, must produce the expression $1 + Az + Bz^2 + Cz^3 + Dz^4 + \cdots$, when they are actually multiplied. It follows then that the coefficient A is equal to the sum $\alpha + \beta + \gamma + \delta + \epsilon + \cdots$. The coefficient B is equal to the sum of the products taken two at a time. Hence $B = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta + \cdots$. Also the coefficient C is equal to the sum of products taken three at a time, namely $C = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta + \cdots$. We also have D as the sum of products taken four at a time, and E is the sum of products taken five at a time, etc. All of this is clear from ordinary algebra.

166. Since the sum $\alpha + \beta + \gamma + \delta + \cdots$ is given along with the sum of products taken two at a time, we can find the sum of the squares $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \cdots$, since this is equal to the square of the sum diminished by two times the sum of the products taken two at a time. In a similar way the sums of the cubes, biquadratics, and higher powers can be found. If we let $P = \alpha + \beta + \gamma + \delta + \epsilon + \cdots$

$$\begin{aligned}
Q &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \cdots \\
R &= \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \cdots \\
S &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4 + \cdots \\
T &= \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \epsilon^5 + \cdots \\
V &= \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \epsilon^6 + \cdots .
\end{aligned}$$

Then P, Q, R, S, T, V, etc. can be found in the following way from A, B, C, D, etc. $P = A$, $Q = AP - 2B$, $R = AQ - BP + 3C$, $S = AR - BQ + CP - 4D$, $T = AS - BR + CQ - DP + 5E$, $V = AT - BS + CR - DQ + EP - 6F$, etc. The truth of these formulas is intuitively clear, but a rigorous proof will be given in the differential calculus.

167. Since we found above, in section 156, that

$$\begin{aligned}
\frac{e^x - e^{-x}}{2} &= x\left(1 + \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdots 7} + \cdots\right) \\
&= x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\left(1 + \frac{x^2}{16\pi^2}\right)\left(1 + \frac{x^2}{25\pi^2}\right)\cdots ,
\end{aligned}$$

it follows that

$$\begin{aligned}
1 &+ \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdots 7} + \cdots \\
&= \left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\left(1 + \frac{x^2}{16\pi^2}\right)\cdots .
\end{aligned}$$

If we let $x^2 = \pi^2 z$,

$$\begin{aligned}
1 &+ \frac{\pi^2}{1\cdot 2\cdot 3}z + \frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}z^2 + \frac{\pi^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}z^3 + \cdots \\
&= (1 + z)(1 + z/4)(1 + z/9)(1 + z/16)(1 + z/25)\cdots .
\end{aligned}$$

We use the rules stated above where $A = \frac{\pi^2}{6}$, $B = \frac{\pi^4}{120}$, $C = \frac{\pi^6}{5040}$, $D = \frac{\pi^8}{362880}$, etc., and we also have

$$P = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots ,$$

$$Q = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \frac{1}{36^2} + \cdots ,$$

$$R = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \frac{1}{36^3} + \cdots ,$$

$$S = 1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \frac{1}{36^4} + \cdots ,$$

$$T = 1 + \frac{1}{4^5} + \frac{1}{9^5} + \frac{1}{16^5} + \frac{1}{25^5} + \frac{1}{36^5} + \cdots .$$

From the values of A, B, C, D, etc. we see that $P = \frac{\pi^2}{6}$, $Q = \frac{\pi^4}{90}$, $R = \frac{\pi^6}{945}$, $S = \frac{\pi^8}{9450}$, $T = \frac{\pi^{10}}{93555}$, etc.

168. It is clear that any infinite series of the form $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$, provided n is an even integer, can be expressed in terms of π, since it always has a sum equal to a fractional part of a power of π. In order that the values of these sums can be seen even more clearly, we set down in a convenient form some more sums of these series.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{2^0}{1 \cdot 2 \cdot 3} \frac{1}{1} \pi^2$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} \cdots = \frac{2^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \frac{1}{3} \pi^4$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \cdots = \frac{2^4}{1 \cdot 2 \cdots 7} \frac{1}{3} \pi^6$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \cdots = \frac{2^6}{1 \cdot 2 \cdot 3 \cdots 9} \frac{3}{5} \pi^8$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \cdots = \frac{2^8}{1 \cdot 2 \cdot 3 \cdots 11} \frac{5}{3} \pi^{10}$$

$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \cdots = \frac{2^{10}}{1 \cdot 2 \cdot 3 \cdots 13} \frac{691}{105} \pi^{12}$$

$$1 + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \frac{1}{4^{14}} + \frac{1}{5^{14}} + \cdots = \frac{2^{12}}{1\cdot2\cdot3\cdots15}\,\frac{35}{1}\pi^{14}$$

$$1 + \frac{1}{2^{16}} + \frac{1}{3^{16}} + \frac{1}{4^{16}} + \frac{1}{5^{16}} + \cdots = \frac{2^{14}}{1\cdot2\cdot3\cdots17}\,\frac{3617}{15}\pi^{16}$$

$$1 + \frac{1}{2^{18}} + \frac{1}{3^{18}} = \frac{1}{4^{18}} + \frac{1}{5^{18}} + \cdots = \frac{2^{16}}{1\cdot2\cdot3\cdots19}\,\frac{43867}{21}\pi^{18}$$

$$1 + \frac{1}{2^{20}} + \frac{1}{3^{20}} + \frac{1}{4^{20}} + \frac{1}{5^{20}} + \cdots = \frac{2^{18}}{1\cdot2\cdot3\cdots21}\,\frac{1222277}{55}\pi^{20}$$

$$1 + \frac{1}{2^{22}} + \frac{1}{3^{22}} + \frac{1}{4^{22}} + \frac{1}{5^{22}} + \cdots = \frac{2^{20}}{1\cdot2\cdot3\cdots23}\,\frac{854513}{3}\pi^{22}$$

$$1 + \frac{1}{2^{24}} + \frac{1}{3^{24}} + \frac{1}{4^{24}} + \frac{1}{5^{24}} + \cdots = \frac{2^{22}}{1\cdot2\cdot3\cdots25}\,\frac{1181820455}{273}\pi^{24}$$

$$1 + \frac{1}{2^{26}} + \frac{1}{3^{26}} + \frac{1}{4^{26}} + \frac{1}{5^{26}} + \cdots = \frac{2^{24}}{1\cdot2\cdot3\cdots27}\,\frac{76977927}{1}\pi^{26}.$$

We could continue with more of these, but we have gone far enough to see a sequence which at first seems quite irregular, $1, \frac{1}{3}, \frac{1}{3}, \frac{3}{5}, \frac{5}{3}, \frac{691}{105}, \frac{35}{1}, \ldots$, but it is of extraordinary usefulness in several places.

169. We now treat in the same manner the equation found in section 157. There we saw that

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{1\cdot2} + \frac{x^4}{1\cdot2\cdot3\cdot4} + \frac{x^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6} + \cdots$$

$$= \left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{9\pi^2}\right)\left(1 + \frac{4x^2}{25\pi^2}\right)\left(1 + \frac{4x^2}{49\pi^2}\right)\cdots$$. We let $x^2 = \frac{\pi^2 z}{4}$,

then $1 + \frac{\pi^2}{1\cdot2\cdot4}z + \frac{\pi^4}{1\cdot2\cdot3\cdot4\cdot4^2}z^2 + \frac{\pi^6}{1\cdot2\cdots6\cdot4^3}z^3 + \cdots$

$= (1 + z)(1 + z/9)(1 + z/25)(1 + z/49)\cdots$. We now use the formulas, where $A = \frac{\pi^2}{1\cdot2\cdot4}$, $B = \frac{\pi^4}{1\cdot2\cdot3\cdot4\cdot4^2}$, $C = \frac{\pi^6}{1\cdot2\cdot3\cdots6\cdot4^3}$, etc., and

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \cdots$$

$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \cdots$$

$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \cdots$$

$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \cdots .$$

It follows that $P = \frac{1}{1}\frac{\pi^2}{2^3}$, $Q = \frac{2}{1\cdot 2\cdot 3}\frac{\pi^4}{2^5}$, $R = \frac{16}{1\cdot 2\cdot 3\cdot 4\cdot 5}\frac{\pi^6}{2^7}$, $S = \frac{272}{1\cdot 2\cdot 3\cdots 7}\frac{\pi^8}{2^9}$, $T = \frac{7936}{1\cdot 2\cdot 3\cdots 9}\frac{\pi^{10}}{2^{11}}$, $V = \frac{353792}{1\cdot 2\cdot 3\cdots 11}\frac{\pi^{12}}{2^{13}}$, $W = \frac{22368256}{1\cdot 2\cdot 3\cdots 13}\frac{\pi^{14}}{2^{15}}$.

170. The same sums of powers of odd numbers can be found from the preceding sums in which all numbers occur. If we let $M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \cdots$ and multiply both sides by $\frac{1}{2^n}$, we obtain $\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \cdots$. This series contains only even numbers, which, when subtracted from the previous series, leaves the series with only odd numbers. Hence, $M - \frac{M}{2^n} = \frac{2^n - 1}{2^n}M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \cdots$. If 2 times the series $\frac{M}{2^n}$ is subtracted from M an alternating series is produced: $M - \frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}}M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \cdots$. In this way we can sum the series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \cdots$$

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \cdots .$$

If n is an even number and the sum is $A\pi^n$, then A will be a rational number.

171. Furthermore, the expressions found in section 164 supply in the same way sums of series which are worthy of note. Since

$$\cos\frac{v}{2} + \tan\frac{g}{2}\sin\frac{v}{2} = \left(1 + \frac{v}{\pi - g}\right)\left(1 - \frac{v}{\pi + g}\right)\left(1 + \frac{v}{3\pi - g}\right)\cdots,$$

if we let $v = \frac{x}{n}\pi$ and $g = \frac{m}{n}\pi$, then

$$\left(1 + \frac{x}{n - m}\right)\left(1 - \frac{x}{n + m}\right)\left(1 + \frac{x}{3n - m}\right)\left(1 - \frac{x}{3n + m}\right)\left(1 + \frac{x}{5n - m}\right)$$
$$\left(1 - \frac{x}{5n + m}\right)\cdots = \cos\frac{x\pi}{2n} + \tan\frac{m\pi}{2n}\sin\frac{x\pi}{2n}$$
$$= 1 + \frac{\pi x}{2n}\tan\frac{m\pi}{2n} - \frac{\pi^2 x^2}{2\cdot 4n^2} - \frac{\pi^3 x^3}{2\cdot 4\cdot 6n^3}\tan\frac{m\pi}{2n}$$
$$+ \frac{\pi^4 x^4}{2\cdot 4\cdot 6\cdot 8n^4} + \cdots .$$

Using the expression in section 165, we have $A = \frac{\pi}{2n}\tan\frac{m\pi}{2n}$, $B = \frac{-\pi^2}{2\cdot 4n^2}$,

$C = \frac{-\pi^3}{2\cdot 4\cdot 6n^3}\tan\frac{m\pi}{2n}$, $D = \frac{\pi^4}{2\cdot 4\cdot 6\cdot 8n^4}$, $E = \frac{\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5}\tan\frac{m\pi}{2n}$, etc.

Further, since $\alpha = \frac{1}{n - m}$, $\beta = -\frac{1}{n + m}$, $\gamma = \frac{1}{3n - m}$,

$\delta = -\frac{1}{3n + m}$, $\epsilon = \frac{1}{5n - m}$, $\zeta = -\frac{1}{5n + m}$, etc.

172. When we follow the procedure given in section 166, we obtain the following.

$$P = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \frac{1}{5n-m} - \frac{1}{5n+m} + \cdots$$

$$Q = \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \frac{1}{(5n-m)^2} + \cdots$$

$$R = \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} + \frac{1}{(3n-m)^3} - \frac{1}{(3n+m)^3} + \frac{1}{(5n-m)^3} - \cdots$$

$$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} + \frac{1}{(5n-m)^4} + \cdots$$

$$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} + \frac{1}{(5n-m)^5} - \cdots$$

$$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} + \frac{1}{(5n-m)^6} + \cdots .$$

When we let $\tan \frac{m\pi}{2n} = k$, we obtain, as we have shown,

$$P = A = \frac{k\pi}{2n} = \frac{1}{2}\,\frac{k\pi}{n}$$

$$Q = \frac{(k^2+1)\pi^2}{4n^2} = \frac{(2k^2+2)\pi^2}{2\cdot 4n^2}$$

$$R = \frac{(k^3+k)\pi^3}{8n^3} = \frac{(6k^3+6k)\pi^3}{2\cdot 4\cdot 6n^3}$$

$$S = \frac{(3k^4+4k^2+1)\pi^4}{48n^4} = \frac{(24k^4+32k^2+8)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4}$$

$$T = \frac{(3k^5+5k^3+2k)\pi^5}{96n^5} = \frac{(120k^5+200k^3+80k)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5}.$$

173. Likewise from the last form in section 164, we obtain

$$\cos\frac{v}{2} + \cot\frac{g}{2}\sin\frac{v}{2} = \left(1 + \frac{v}{g}\right)\left(1 - \frac{v}{2\pi - g}\right)\left(1 + \frac{v}{2\pi + g}\right)$$
$$\left(1 - \frac{v}{4\pi - g}\right)\left(1 + \frac{v}{4\pi + g}\right)\cdots .$$

If we let $v = \frac{x}{n}\pi$, $g = \frac{m}{n}\pi$, and $\tan\frac{m\pi}{2n} = k$, so that $\cot\frac{g}{2} = \frac{1}{k}$ and

$$\cos\frac{\pi x}{2n} + \frac{1}{k}\sin\frac{\pi x}{2n} = 1 + \frac{\pi x}{2nk} - \frac{\pi^2 x^2}{2\cdot 4n^2} - \frac{\pi^3 x^3}{2\cdot 4\cdot 6n^3 k} + \frac{\pi^4 x^4}{2\cdot 4\cdot 6\cdot 8n^4}$$
$$+ \frac{\pi^5 x^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5 k} - \cdots = \left(1 + \frac{x}{m}\right)\left(1 - \frac{x}{2n - m}\right)\left(1 + \frac{x}{2n + m}\right)$$
$$\left(1 - \frac{x}{4n - m}\right)\left(1 + \frac{x}{4n + m}\right)\cdots .$$

When we compare this with the general formula given in section 165, we find

$$A = \frac{\pi}{2nk},\ B = \frac{-\pi^2}{2\cdot 4n^2},\ C = \frac{-\pi^3}{2\cdot 4\cdot 6n^3 k},\ D = \frac{\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},\ E = \frac{\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5 k},$$

etc. From the factors we obtain $\alpha = \frac{1}{m}$, $\beta = \frac{-1}{2n - m}$, $\gamma = \frac{1}{2n + m}$, $\delta = \frac{-1}{4n - m}$, $\epsilon = \frac{1}{4n + m}$, etc.

174. Again we follow the procedure given in section 166 in order to obtain the sums of the following series.

$$P = \frac{1}{m} - \frac{1}{2n - m} + \frac{1}{2n + m} - \frac{1}{4n - m} + \frac{1}{4n + m} - \cdots$$

$$Q = \frac{1}{m^2} + \frac{1}{(2n - m)^2} + \frac{1}{(2n + m)^2}$$
$$+ \frac{1}{(4n - m)^2} + \frac{1}{(4n + m)^2} + \cdots$$

$$R = \frac{1}{m^3} - \frac{1}{(2n-m)^3} + \frac{1}{(2n+m)^3} - \frac{1}{(4n-m)^3} + \frac{1}{(4n+m)^3} - \cdots$$

$$S = \frac{1}{m^4} + \frac{1}{(2n-m)^4} + \frac{1}{(2n+m)^4} + \frac{1}{(4n-m)^4} + \frac{1}{(4n+m)^4} + \cdots$$

$$T = \frac{1}{m^5} - \frac{1}{(2n-m)^5} + \frac{1}{(2n+m)^5} - \frac{1}{(4n-m)^5} + \frac{1}{(4n+m)^5} - \cdots .$$

We obtain the following sums:

$$P = A = \frac{\pi}{2nk} = \frac{1\pi}{2nk}$$

$$Q = \frac{(k^2+1)\pi^2}{4n^2k^2} = \frac{(2+2k^2)\pi^2}{2\cdot 4n^2k^2}$$

$$R = \frac{(k^2+1)\pi^3}{8n^3k^3} = \frac{(6+6k^2)\pi^3}{2\cdot 4\cdot 6n^3k^3}$$

$$S = \frac{(k^4+4k^2+3)\pi^4}{48n^4k^4} = \frac{(24+32k^2+3k^4)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4k^4}$$

$$T = \frac{(2k^4+5k^2+3)\pi^5}{96n^5k^5} = \frac{(120+200k^2+80k^4)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k^5}$$

$$V = \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} = \frac{(720+1440k^2+816k^4+96k^6)\pi^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12n^6k^6}$$

175. These general series deserve to be particularized by giving special values to m and n. If $m = 1$ and $n = 2$, then $k = \tan\frac{\pi}{4} = 1$, and both of the series become the same:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \cdots$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \cdots$$

$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \cdots .$$

The first of these series was seen before in section 140. The other series, which have equal exponents were discussed in section 169. The remaining series, in which the exponents are odd, we see here for the first time. It is clear that each of these series

$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \cdots$ has a sum which is some function of π.

176. Now we let $m = 1$, $n = 3$, then $k = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and the series in section 172 become

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \cdots$$

$$\frac{\pi^2}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \cdots$$

$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \cdots$$

etc., or

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$\frac{4\pi^2}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \cdots$$

$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \cdots$$

In these series there is no term which is divisible by $\frac{1}{3}$. We can find the series which contain these terms, at least those series with even exponents, as follows. Since

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots ,$$

it follows that

$$\frac{\pi^2}{6\cdot 9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \cdots = \frac{\pi^2}{54}.$$

This last series contains only those terms which are divisible by $\frac{1}{3}$, and if it is subtracted from the previous series, there remains a series which contains all terms not divisible by $\frac{1}{3}$. Then

$$\frac{8\pi}{54} = \frac{4\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$, as we have already seen.

177. With the same hypothesis, that is, $m = 1$, $n = 3$ and $k = \frac{1}{\sqrt{3}}$, from section 174 we obtain

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \cdots$$

$$\frac{\pi^2}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \cdots$$

$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} - \cdots$$

In these series, the denominators are all odd numbers, and the terms divisible by $\frac{1}{3}$ are missing. The sum of the even powers of these missing terms can be found from what we already know. Since

$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$, it follows that

$\frac{\pi^2}{8 \cdot 9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \cdots = \frac{\pi^2}{72}$. If this series, which contains all the terms with odd denominators divisible by three, is subtracted from the series above it, there remains the series of squares of odd numbers not divisible by three, so that

$$\frac{\pi^2}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots .$$

178. If the series found in sections 172 and 174 are either added or subtracted, we obtain other series which are worthy of note. We have

$$\frac{k\pi}{2n} + \frac{\pi}{2nk} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \cdots$$
$$= \frac{(k^2+1)\pi}{2nk}.$$

If we let $k = \tan \frac{m\pi}{2n} = \frac{\sin \frac{m\pi}{2n}}{\cos \frac{m\pi}{2n}}$, then $1 + k^2 = \frac{1}{\left(\cos \frac{m\pi}{2n}\right)^2}$, so that

$\frac{2k}{1+k^2} = 2 \sin \frac{m\pi}{2n} \cos \frac{m\pi}{2n} = \sin \frac{m\pi}{n}$. When we substitute these values, we obtain

$$\frac{\pi}{n \sin \frac{m\pi}{n}} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m}$$

$+ \frac{1}{3n-m} - \frac{1}{3n+m} - \cdots$. In a similar way, by subtraction, we obtain

$$\frac{\pi}{2nk} - \frac{k\pi}{2n} = \frac{(1-k^2)\pi}{2nk} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m}$$

$+ \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} - \cdots$. If we let

$$\frac{2k}{1-k^2} = \tan\frac{2m\pi}{2n} = \tan\frac{m\pi}{n} = \frac{\sin\frac{m\pi}{n}}{\cos\frac{m\pi}{n}}, \text{ then}$$

$$\frac{\pi\cos\frac{m\pi}{n}}{n\sin\frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m}$$

$+ \frac{1}{2n+m} - \frac{1}{3n-m} + \cdots$. Series with squares and higher powers which arise in this way are more easily derived through differentiation, which we will do later.

179. Since we have already considered the results when $m = 1$ and $n = 2$, or 3, we now let $m = 1$ and $n = 4$. In this case $\sin\frac{m\pi}{n} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. It follows that

$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \cdots$ and

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots$. If $m = 1$ and $n = 8$, then $\frac{m\pi}{n} = \frac{\pi}{8}$, $\sin\frac{\pi}{8} = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)^{\frac{1}{2}}$, $\cos\frac{\pi}{8} = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^{\frac{1}{2}}$,

and $\frac{\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}} = 1 + \sqrt{2}$. From these we have

$$\frac{\pi}{4\left(2-\sqrt{2}\right)^{\frac{1}{2}}} = 1 + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \cdots$$

$$\frac{\pi}{8\left(\sqrt{2}-1\right)} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \cdots$$

Now we let $m = 3$ and $n = 8$, then $\frac{m\pi}{n} = \frac{3\pi}{8}$, $\sin\frac{3\pi}{8} = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^{\frac{1}{2}}$,

$\cos\frac{3\pi}{8} = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)^{\frac{1}{2}}$, and $\frac{\cos\frac{3\pi}{8}}{\sin\frac{3\pi}{8}} = \frac{1}{\sqrt{2}+1}$. It follows that

$$\frac{\pi}{4\left(2+\sqrt{2}\right)^{\frac{1}{2}}} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \cdots$$

$$\frac{\pi}{8\left(\sqrt{2}+1\right)} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \cdots .$$

180. Through combinations of the above series we obtain

$$\frac{\pi\left(2+\sqrt{2}\right)^{\frac{1}{2}}}{4} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \cdots$$

$$\frac{\pi\left(2-\sqrt{2}\right)^{\frac{1}{2}}}{4} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \cdots$$

$$\frac{\pi\left(\left(4+2\sqrt{2}\right)^{\frac{1}{2}} + \sqrt{2} - 1\right)}{8} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \cdots$$

$$\frac{\pi\left(\left(4+2\sqrt{2}\right)^{\frac{1}{2}} - \sqrt{2} + 1\right)}{8} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \cdots$$

$$\frac{\pi\left(\sqrt{2}+1+\left(4-2\sqrt{2}\right)^{\frac{1}{2}}\right)}{8}=1+\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}$$
$$-\frac{1}{13}-\frac{1}{15}+\frac{1}{17}+\frac{1}{19}+\cdots$$

$$\frac{\pi\left(\sqrt{2}+1-\left(4-2\sqrt{2}\right)^{\frac{1}{2}}\right)}{8}=1-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}$$
$$+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}-\cdots.$$

In the same way we could let $n = 16$ and $m = 1,3,5$, or 7 which would show the sums of series in which the terms are $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots$ and in which the various changes of positive and negative signs are different from those already seen.

181. If in the series discussed in section 178, the terms are combined two by two, we obtain the following:

$$\frac{\pi}{n \sin \frac{m\pi}{n}} = \frac{1}{m} + \frac{2m}{n^2 - m^2} - \frac{2m}{4n^2 - m^2} + \frac{2m}{9n^2 - m^2}$$
$$- \frac{2m}{16n^2 - m^2} + \cdots.$$

From this it follows that

$$\frac{1}{n^2 - m^2} - \frac{1}{4n^2 - m^2} + \frac{1}{9n^2 - m^2} - \cdots = \frac{\pi}{2mn \sin \frac{m\pi}{n}} - \frac{1}{2m^2}.$$

The other series gives us

$$\frac{\pi}{n \tan \frac{m\pi}{n}} = \frac{1}{m} - \frac{2m}{n^2 - m^2} - \frac{2m}{4n^2 - m^2} - \frac{2m}{9n^2 - m^2} - \cdots.$$

From this we have

$$\frac{1}{n^2 - m^2} + \frac{1}{4n^2 - m^2} + \frac{1}{9n^2 - m^2} + \cdots = \frac{1}{2m^2} - \frac{\pi}{2mn \tan \dfrac{m\pi}{n}}.$$

When these two series are added, we obtain

$$\frac{1}{n^2 - m^2} + \frac{1}{9n^2 - m^2} + \frac{1}{25n^2 - m^2} + \cdots = \frac{\pi \tan \dfrac{m\pi}{2n}}{4mn}.$$

If we let $n = 1$ and let m be any even number $2k$ except zero, since $\tan k\pi = 0$, we always have

$\dfrac{1}{1 - 4k^2} + \dfrac{1}{9 - 4k^2} + \dfrac{1}{25 - 4k^2} + \dfrac{1}{49 - 4k^2} + \cdots = 0.$ However, if in this series $n = 2$ and m is any odd number $2k + 1$,

since $\dfrac{1}{\tan \dfrac{m\pi}{n}} = 0$, we have

$$\frac{1}{4 - (2k + 1)^2} + \frac{1}{16 - (2k + 1)^2} + \frac{1}{36 - (2k + 1)^2} + \cdots = \frac{1}{2(2k + 1)^2}.$$

182. If we multiply the series by n^2 and let $\dfrac{m}{n} = p$,

then they take the form

$$\frac{1}{1 - p^2} - \frac{1}{4 - p^2} + \frac{1}{9 - p^2} - \frac{1}{16 - p^2} + \cdots = \frac{\pi}{2p \sin p\pi} - \frac{1}{2p^2},$$

$$\frac{1}{1 - p^2} + \frac{1}{4 - p^2} + \frac{1}{9 - p^2} + \frac{1}{16 - p^2} + \cdots = \frac{1}{p^2} - \frac{\pi}{2p \sin p\pi}.$$

If we let $p^2 = a$, then we obtain the series

$$\frac{1}{1 - a} - \frac{1}{4 - a} + \frac{1}{9 - a} - \frac{1}{16 - a} + \cdots = \frac{\pi\sqrt{a}}{2a \sin \pi\sqrt{a}} - \frac{1}{2a},$$

$$\frac{1}{1 - a} + \frac{1}{4 - a} + \frac{1}{9 - a} + \frac{1}{16 - a} + \cdots = \frac{1}{2a} - \frac{\pi\sqrt{a}}{2a \tan \pi\sqrt{a}}.$$

Provided a is not negative nor the square of an integer, then the sum of these series can be represented in terms of the circle.

183. By means of the reduction of complex exponentials to sines and cosines of circular arcs, which has been treated, we can assign negative values to a in the series just discussed. Since $e^{xi} = \cos x + i \sin x$ and $e^{-xi} = \cos x - i \sin x$, when we substitute yi for x, we obtain $\cos yi - 1 = \dfrac{e^{-y} + e^{y}}{2}$ and $\sin yi = \dfrac{e^{-y} - e^{y}}{2i}$. Now if $a = -b$ and $y = \pi\sqrt{b}$, then $\cos \pi\sqrt{-b} = \dfrac{e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}}}{2}$ and $\sin \pi\sqrt{-b} = \dfrac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{2i}$.

It follows that

$$\tan \pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{\left(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}}\right)i}.$$ Then we have

$$\frac{\pi\sqrt{-b}}{\sin \pi\sqrt{-b}} = \frac{-2\pi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$$ and

$$\frac{\pi\sqrt{-b}}{\tan \pi\sqrt{-b}} = \frac{\left(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}}\right)\pi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}.$$ From these remarks it follows that

$$\frac{1}{1+b} - \frac{1}{4+b} + \frac{1}{9+b} - \frac{1}{16+b} + \cdots$$
$$= \frac{1}{2b} - \frac{\pi\sqrt{b}}{\left(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}}\right)b},$$

$$\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \frac{1}{16+b} + \cdots$$
$$= \frac{\left(e^{\pi\sqrt{b}} + e^{-\pi\sqrt{b}}\right)\pi\sqrt{b}}{2b\left(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}}\right)} - \frac{1}{2b}.$$ These same series can be derived from section 162, using the same method which was used in this chapter. However, I have preferred to treat it in this way, since it is a nice illustration of the reduction of sines and cosines of complex arcs to real exponentials.

MÉMOIRE

SUR

QUELQUES PROPRIÉTÉS REMARQUABLES DES QUANTITÉS TRANSCENDENTES CIRCULAIRES ET LOGARITHMIQUES.

PAR M. LAMBERT. *)

§. I.

Démontrer que le diametre du cercle n'eſt point à ſa circonférence comme un nombre entier à un nombre entier, c'eſt là une choſe, dont les géometres ne ſeront gueres ſurpris. On connoit les nombres de *Ludolph*, les rapports trouvés par *Archimede*, par *Metius* etc. de même qu'un grand nombre de ſuites infinies, qui toutes ſe rapportent à la quadrature du cercle. Et ſi la ſomme de ces ſuites eſt une quantité rationelle, on doit aſſez naturellement conclure, qu'elle ſera ou un nombre entier, ou une fraction très ſimple. Car, s'il y falloit une fraction fort compoſée, quelle raiſon y auroit-il, pourquoi plutôt telle que telle autre quelconque? C'eſt ainſi, par exemple, que la ſomme de la ſuite

$$\frac{2}{1 \cdot 3} + \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \frac{2}{7 \cdot 9} + \&c.$$

eſt égale à l'unité, qui de toutes les quantités rationelles eſt la plus ſimple. Mais, en omettant alternativement les 2, 4, 6, 8 &c. termes, la ſomme des autres

$\frac{2}{1 \cdot 3}$

*) Lu en 1767.

$$\frac{2}{1.\,3} + \frac{2}{5.\,7} + \frac{2}{9.\,11} + \frac{2}{13.\,15} + \&c.$$

donne l'aire du cercle, lorsque le diametre est $= 1$. Il semble donc que, si cette somme étoit rationelle, elle devroit également pouvoir être exprimée par une fraction fort simple, telle que seroit $\frac{3}{4}$ ou $\frac{4}{5}$ &c. En effet, le diametre étant $= 1$, le rayon $= \frac{1}{2}$, le quarré du rayon $= \frac{1}{4}$, on voit bien que ces expressions étant ausi simples, elles n'y mettent point d'obstacle. Et comme il s'agit de tout le cercle, qui fait une espece d'unité, & non de quelque Secteur, qui de sa nature demanderoit des fractions fort grandes, on voit bien, qu'encore à cet égard on n'a point sujet de s'attendre à une fraction fort composée. Mais comme, après la fraction $\frac{11}{14}$ trouvée par *Archimede*, qui ne donne qu'un à peu près, on passe à celle de *Metius*, $\frac{355}{452}$, qui n'est pas non plus exacte, & dont les nombres sont considérablement plus grands, on doit être fort porté à conclure, que la somme de cette suite, bien loin d'être égale à une fraction simple, est une quantité irrationelle.

§. 2. Quelque vague que soit ce raisonnement, il y a néanmoins des cas où on ne demande pas d'avantage. Mais ces cas ne sont pas celui de la quadrature du cercle. La plûpart de ceux qui s'attachent à la chercher, le font avec une ardeur, qui les entraine quelque fois jusqu'à révoquer en doute les vérités les plus fondamentales & les mieux établies de la géométrie. Pourroit-on croire, qu'ils se trouveroient satisfaits par ce que je viens de dire? Il y faut toute autre chose. Et s'agît-il de démontrer, qu'en effet le diametre n'est pas à la circonférence comme un nombre entier à un nombre entier, cette démonstration doit être si rigide, qu'elle ne le cede à aucune démonstration géométrique. Et avec tout cela je reviens à dire, que les géométres n'en seront point surpris. Ils doivent être accoutumés depuis longtems à ne s'attendre à autre chose. Mais voici ce qui méritera plus d'attention, & ce qui fera une bonne partie de ce Mémoire. Il s'agit de faire voir, que *toutes les fois qu'un arc de cercle quelconque est commensurable au rayon, la tangente de cet arc lui est in-*

com-

commenſurable; & que réciproquement, *toute tangente commenſurable n'eſt point celle d'un arc commenſurable.* Voila de quoi être un peu plus ſurpris. Cet énoncé paroiſſoit devoir admettre une infinité d'exceptions, & il n'en admet aucune. Il fait encore voir juſqu'à quel point les quantités circulaires tranſcendentes ſont tranſcendentes, & reculées au delà de toute commenſurabilité. Comme la démonſtration que je vais donner exige toute la rigueur géométrique, & qu'en outre elle ſera un tiſſu de quelques autres theorémes, qui demandent d'être démontrés avec tout autant de rigueur, ces raiſons m'excuſeront; quand je ne me hâterai pas d'en venir à la fin, ou lorsque chemin faiſant je m'arrêterai à ce qui ſe préſentera de remarquable.

§. 3. Soit donc propoſé *un arc de cercle quelconque, mais commenſurable au rayon:* & il s'agit de trouver, *ſi cet arc de cercle ſera en même tems commenſurable à ſa tangente ou non?* Qu'on ſe figure pour cet effet une fraction telle, que ſon numérateur ſoit égal à l'arc de cercle propoſé, & que ſon dénominateur ſoit égal à la tangente de cet arc. Il eſt clair que, de quelque maniere que cet arc & ſa tangente ſoient exprimés, cette fraction doit être égale à une autre fraction, dont le numérateur & le dénominateur ſeront des nombres entiers, toutes lès fois que l'arc de cercle propoſé ſe trouvera être commenſurable à ſa tangente. Il eſt clair auſſi que cette ſeconde fraction doit pouvoir être déduite de la premiere, par la même méthode, dont on ſe ſert en arithmétique pour réduire une fraction à ſon moindre dénominateur. Cette méthode étant connue depuis *Euclide*, qui en fait la 2me prop. de ſon 7me Livre, je ne m'arrêterai pas à la démontrer de nouveau. Mais il convient de remarquer que, tandis que *Euclide* ne l'applique qu'à des nombres entiers & rationels, il faudra que je m'en ſerve d'une autre façon, lorsqu'il s'agit d'en faire l'application à des quantités, dont on ignore encore ſi elles ſeront rationelles ou non? Voici donc le procédé qui conviendra au cas dont il eſt ici queſtion.

§. 4. Soit le rayon $= 1$, un arc de cercle propoſé quelconque $= v$. Et on aura les deux ſuites infinies fort connues

Ll 2 ſin

$$\text{ſin } v = v - \frac{1}{2.3} v^3 + \frac{1}{2.3.4.5} v^5 - \frac{1}{2.3.4.5.6.7} v^7 + \&c.$$

$$\text{coſ } v = 1 - \frac{1}{2} v^2 + \frac{1}{2.3.4} v^4 - \frac{1}{2.3.4.5.6} v^6 + \&c.$$

Comme dans ce qui ſuivra je donnerai deux ſuites pour l'hyperbole qui ne différeront de ces deux qu'en ce que tous les ſignes ſont poſitifs, je différerai jusques-là de démontrer la loi de progreſſion de ces ſuites, & encore ne la démontrerai-je que pour ne rien omettre de tout ce que demande la rigueur géométrique. Il ſuffit donc d'en avoir averti les Lecteurs d'avance.

§. 5. Or comme il eſt

$$\text{tang } v = \frac{\text{ſin } v}{\text{coſ } v},$$

nous aurons, en ſubſtituant ces deux ſuites, la fraction

$$\text{tang } v = \frac{v - \frac{1}{2.3} v^3 + \frac{1}{2.3.4.5} v^5 - \&c.}{1 - \frac{1}{2} v^2 + \frac{1}{2.3.4} v^4 - \&c.}$$

Je la poſerai pour plus de briéveté

$$\text{tang } v = \frac{A}{B},$$

de ſorte qu'il ſoit

$$A = \text{ſin } v,$$
$$B = \text{coſ } v.$$

Voici maintenant le procédé que preſcrit *Euclide*.

§. 6. On diviſe B par A; ſoit le quotient $= Q'$, le réſidu $= R'$.
On diviſe A par R'; ſoit le quotient $= Q''$, le réſidu $= R''$.
On diviſe R' par R''; ſoit le quotient $= Q'''$, le réſidu $= R'''$.
On diviſe R'' par R'''; ſoit le quotient $= Q^{IV}$, le réſidu $= R^{IV}$. &c.

de

de ſorte qu'en continuant ces diviſions, on trouve ſucceſſivement

les quotiens Q', Q'', Q''' - - - - - - - - Q^n, Q^{n+1}, Q^{n+2} - - - - &c.

les réſidus R', R'', R''' - - - - - - - - R^n, R^{n+1}, R^{n+2} - - - - &c.

& il eſt clair ſans que j'en avertiſſe, que les expoſans n, $n+1$, $n+2$ &c. ne ſervent qu'à indiquer le quantieme quotient ou réſidu eſt celui où ils ſe trouvent marqués. Ce qui étant poſé, voici ce qu'il s'agit de démontrer.

§. 7. En premier lieu, *non ſeulement que la diviſion peut être continuée ſans fin, mais que les quotiens ſuivront une loi très ſimple en ce qu'il ſera*

$$\begin{aligned} Q' &= +1 : v, \\ Q'' &= -3 : v, \\ Q''' &= +5 : v, \\ Q^{IV} &= -7 : v, \text{ \&c.} \end{aligned}$$

& en général

$$Q^n = \pm (2n - 1) : v,$$

où le ſigne $+$ *eſt pour l'expoſant* n *pair, le ſigne* $-$ *pour l'expoſant* n *impair & que de la ſorte on aura pour la tangente exprimée par l'arc la fraction continue très ſimple*

$$\tang v = \cfrac{1}{1 : v - \cfrac{1}{3 : v - \cfrac{1}{5 : v - \cfrac{1}{7 : v - \cfrac{1}{9 : v - \text{\&c.}}}}}}$$

§. 8. En ſecond lieu, *que les réſidus* R', R'', R''' &c. *ſeront exprimés par les ſuites ſuivantes, dont les loix de progreſſion ſont également fort ſimples:*

Ll 3 $R' =$

$$R^{I} = -\frac{2}{2.3}v^2 + \frac{4}{2.3.4.5}v^4 - \frac{6}{2.3.4.5.6.7}v^6 + \&c.$$

$$R^{II} = -\frac{2.4}{2.3.4.5}v^3 + \frac{4.6}{2.3.4.5.6.7}v^5 - \frac{6.8}{2.3.4.5.6.7.8.9}v^7 + \&c.$$

$$R^{III} = +\frac{2.4.6}{2\cdots 7}v^4 - \frac{4.6.8}{2\cdots\cdots 9}v^6 + \frac{6.8.10}{2\cdots\cdots 11}v^8 - \&c.$$

$$R^{IV} = +\frac{2.4.6.8}{2\cdots 9}v^5 - \frac{4.6.8.10}{2\cdots\cdots 11}v^7 + \frac{6.8.10.12}{2\cdots\cdots 13}v^9 - \&c.$$

&c.

de sorte que les signes des premiers termes changent suivant l'ordre quaternaire $- \; - \; + \; +$, *& qu'en général il sera*

$$\pm R^n = -\frac{2^n(1.2\cdots n)}{1.2\cdots(2n+1)}v^{n+1} + \frac{2^{n+1}(1.2\cdots(n+1))}{1.2\cdots\cdots(2n+3)}v^{n+3} - \&c.$$

$$\pm R^{n+1} = -\frac{2^{n+1}(1.2\cdots(n+1))}{1.2\cdots\cdots(2n+3)}v^{n+2} + \frac{2^{n+2}(1.2\cdots(n+2))}{1.2\cdots\cdots(2n+5)}v^{n+4} - \&c.$$

$$\pm R^{n+2} = +\frac{2^{n+2}(1.2\cdots(n+2))}{1.2\cdots\cdots(2n+5)}v^{n+3} - \frac{2^{n+3}(1.2\cdots(n+3))}{1.2\cdots\cdots(2n+7)}v^{n+5} + \&c.$$

§. 9. Or pour donner à la démonstration de ces théoremes toute la briéveté possible, considérons que chaque résidu R^{n+2} se trouve en divisant par le résidu R^{n+1}, qui le précéde immédiatement, l'antepénultieme R^n. Cette considération fait, que la démonstration, dont il s'agit peut être partagée en deux parties. Dans la premiere il faut faire voir que; *si deux résidus* R^n, R^{n+1}, *qui se succedent immediatement, ont la forme que je leur ai donnée, le résidu* R^{n+2} *qui suit immédiatement, aura la même forme.* Ce qui étant une fois démontré, il ne reste plus que de faire voir, dans la seconde partie de la démonstration, *que la forme des deux premiers résidus est celle qu'ils doivent avoir.*

avoir. Car, de cette maniere, il eſt évident que la forme de tous les ſuivans s'établit comme d'elle-même.

§. 10. Commençons donc par diviſer le premier terme du réſidu R^n par le premier terme du réſidu R^{n+1}, afin d'avoir le quotient

$$Q^{n+2} = \frac{2^n(1.2.3\cdots n)}{1.2.3\cdots(2n+1)}v^{n+1} : \frac{2^{n+1}(1.2.3\cdots(n+1))}{1.2.3\cdots\cdots(2n+3)}v^{n+2}$$

$$= 1 : \frac{2(n+1)v}{(2n+2).(2n+3)} = (2n+3) : v.$$

Et il eſt clair que, le réſidu R^{n+1} étant multiplié par ce quotient

$$Q^{n+2} = (2n+3) : v,$$

& le produit étant ſouſtrait du réſidu R^n, il doit reſter le réſidu R^{n+2}

§. 11. Mais afin de n'avoir pas beſoin de faire cette opération pour chaque terme ſéparément & de nous borner par là à une ſimple induction, prenons le terme général de chacune des ſuites qui expriment les réſidus R^n, R^{n+1}, R^{n+2}, de ſorte qu'en prenant le $m$tieme terme des réſidus R^n, R^{n+1}, nous prenions le $(m-1)$tieme terme du réſidu R^{n+2}. Ce qui étant obſervé, ces termes ſeront

$$\pm r^n = -\frac{2^{n+m-1}(m\cdot(m+1).(m+2)\cdots(n+m-1)v^{n+2m-1}}{1.2.3.4\cdots\cdots\cdots(2n+2m-1)}$$

$$\pm r^{n+1} = -\frac{2^{n+m}.(m\cdot(m+1)(m+2)\cdots\cdots(n+m)v^{n+2m}}{1.2.3.4\cdots\cdots\cdots(2n+2m+1)}$$

$$\pm r^{n+2} = -\frac{2^{n+m}.((m-1)\cdot m\cdot(m+1)\cdots\cdots(n+m)\cdot v^{n+2m-1}}{1.2.3.4\cdots\cdots\cdots(2n+m+1)}$$

Or, puiſqu'il doit être

r^n

$$r^n - r^{n+1}.(2n+3):v = r^{n+2},$$

& qu'en effet il eſt

$$r^n - r^{n+1}(2n+3):v = -\frac{2^{n+m-1}.(m \cdots (n+m-1))\,v^{n+2m-1}}{1.2.3 \cdots\cdots (2n+2m-1)}$$

$$+\frac{2^{n+m}.(m \cdots (n+m))\,v^{n+2m}}{1.2.3 \cdots\cdots (2n+2m+1)}.\frac{2n+3}{v}$$

$$= \frac{2^{n+m-1}.(m \cdots (n+m-1))}{1.2 \cdots (2m+2m-2)}\,v^{n+2m-1}.(-1+\frac{2.(n+m).(2n+3)}{(2n+2m).(2n+2m+1)}$$

$$= -\frac{2^{n+m-1}.(m \cdots (n+m-1))}{1.2 \cdots (2n+2m-2)}\,v^{n+2m-1}.\frac{(2m-2).(2n+2m)}{(2n+2m).(2n+m+1)}$$

$$= -\frac{2^{n+m}.((m-1).m\,(m+1) \cdots\cdots (n+m)\,v^{n+2m-1}}{1.2.3 \cdots\cdots\cdots (2n+m+1)},$$

& partant

$$= \pm r^{n+2}.$$

On voit, que les réſidus R^n, R^{n+1} ayant la forme que je leur ai donnée, le réſidu R^{n+2} aura la même forme. Il ne s'agira donc plus, que de s'aſſurer de la forme des deux premiers réſidus R', R'', afin d'établir ce que cette premiere partie de notre démonſtration avoit admis comme vrai en forme d'hypotheſe. Et c'eſt ce qui fera la ſeconde partie de la démonſtration.

§. 12. Souvenons-nous pour cet effet, que le premier réſidu R' eſt celui qui reſte en diviſant le

$$\text{coſ}\, v = 1 - \frac{1}{2}v^2 + \frac{1}{2.3.4}v^4 \cdots \frac{1}{1 \cdots m}v^m \cdots\cdots \&c.$$

par le

$$\text{ſin}\, v = v - \frac{1}{2.3}v^3 + \frac{1}{2.3.4.5}v^5 \cdots \frac{1}{1 \cdots (m+1)}v^{m+1} \cdots\cdots \&c.$$

Or

Or le quotient qui résulte de la division du premier terme, étant $= 1 : v$, on voit qu'il sera

$$R' = \cos v - \frac{1}{v} \cdot \sin v.$$

Multipliant donc le terme général du diviseur,

$$\pm \frac{1}{1.2.\ldots(m+1)} v^{m+1},$$

par $1 : v$, & soustraïant le produit

$$\pm \frac{1}{1.2\ldots(m+1)} \cdot v^m,$$

du terme général du dividende

$$\pm \frac{1}{1.2\ldots m} \cdot v^m,$$

on aura le terme général du premier résidu R'

$$r' = \pm \frac{m \cdot v^m}{1\ldots(m+1)}.$$

Or $(m+1)$ étant toujours un nombre impair, m sera un nombre pair, & le premier résidu sera

$$R' = -\frac{2}{2.3} v^2 + \frac{4}{2.3.4.5} v^4 - \frac{6}{2\ldots7} v^6 + \&c.$$

tel que nous l'avons supposé.

§. 13. Le second résidu R'' résulte de la division de

$$\sin v = v - \frac{1}{2.3} v^3 + \frac{1}{2.3.4.5} v^5 - \&c.\ldots \pm \frac{1}{1.2\ldots(m-1)} v^{m-1}$$

par le premier résidu que nous venons de trouver

$$R' = -\frac{2}{2.3} v^2 + \frac{4}{2.3.4.5} v^4 - \frac{6}{2\ldots7} v^7 + \ldots \mp \frac{m v^m}{1\ldots(m+1)}$$

Or

Or le quotient qui résulte de la division du premier terme, étant $= - 3 \cdot v$, on voit qu'il sera

$$R'' = \sin v - \frac{3}{v} \cdot R'.$$

Multipliant donc le terme général du diviseur

$$\mp \frac{m v^m}{1 \cdot \cdot \cdot \cdot \cdot \cdot (m+1)},$$

par $- 3 : v$, & soustraiant le produit

$$\pm \frac{3 m v^{m-1}}{1 \cdot \cdot \cdot \cdot \cdot \cdot (m+1)},$$

du terme général du dividende

$$\pm \frac{1}{1 \cdot \cdot \cdot \cdot \cdot \cdot (m-1)} v^{m-1},$$

le terme général du second résidu sera

$$r'' = \pm \frac{v^{m-1}}{1 \cdot \cdot \cdot \cdot \cdot (m-1)} \mp \frac{3 m v^{m-1}}{1 \cdot \cdot \cdot \cdot \cdot (m+1)}$$

$$= \pm \frac{(m-2) \cdot m \cdot v^{m-1}}{1 \cdot \cdot \cdot \cdot \cdot (m+1)}.$$

Substituant donc pour m les nombres pairs, nous aurons le second résidu

$$R'' = - \frac{2 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5} v^3 + \frac{4 \cdot 6}{2 \cdot \cdot \cdot 7} v^5 - \frac{6 \cdot 8}{2 \cdot \cdot \cdot 9} v^7 + \text{\&c.}$$

encore tel que nous l'avons supposé. Ainsi la forme des deux premiers résidus étant démontrée, il s'ensuit, en vertu de la premiere partie de notre démonstration, que la forme de tous les résidus suivans l'est également.

§. 14. Maintenant il n'est plus nécessaire de démontrer séparément la loi de la progression des quotiens Q', Q'', Q''' &c. Car la loi des résidus étant démontrée, il est par là même démontré qu'un quotient quelconque sera (§. 10)

$\pm$

$$\pm Q^{n+2} = (2n+3) : v,$$

ce qui, en vertu de la théorie des fractions continues, donne

$$\operatorname{tang} v = \cfrac{1}{1 : v - \cfrac{1}{3 : v - \cfrac{1}{5 : v - \cfrac{1}{7 : v - \cfrac{1}{9 : v - \cfrac{1}{11 : v - 1 \ \&c.}}}}}}$$

d'où l'on voit en même tems, que *toutes les fois que l'arc v sera égal à une partie aliquote du rayon, tous ces quotiens seront des nombres entiers croissans dans une progression arithmétique.*

Et voilà ce quil faut observer, parce que dans le théoreme d'*Euclide* cité cy-dessus (§. 3.) tous les quotiens sont supposés être des nombres entiers. Ainsi jusques là la méthode que prescrit *Euclide*, sera applicable à tous ces cas, où l'arc v est une partie aliquote du rayon. Mais, encore dans ces cas, il s'y joint une autre circonstance qu'il convient de faire remarquer.

§. 15. Le probleme que propose *Euclide*, c'est *de trouver le plus grand commun diviseur de deux nombres entiers, qui ne sont pas premiers entre eux.* Ce probleme est résoluble toutes les fois qu'un des résidus R', R'', R''' &c. - - - R^{x} devient $= 0$, sans que le résidu précédent R^{x-1} soit égal à l'unité, ce qui suivant, la 1re Prop. du même livre n'arrive que lorsque les deux nombres proposés sont premiers entre eux, bien entendu que tous les quotiens Q', Q'', Q''' &c. sont supposés être des nombres entiers. Or nous venons de voir, que cette derniere supposition a lieu dans le cas dont il s'agit ici, toutes les fois que $\frac{1}{v}$ est un nombre entier. Mais, quant aux résidus R', R'', R''' &c. il n'y en a aucun qui devienne $= 0$. Tout au contraire, en considé-

Mm 2 rant

rant la loi de progreſſion des réſidus que nous venons de trouver, on voit, que non ſeulement ils décroiſſent ſans interruption, mais qu'ils décroiſſent même plus fortement qu'aucune progreſſion géométrique. Quoique donc cela continue à l'infini, nous pourrons néanmoins y appliquer la propoſition d'*Euclide*. Car, en vertu de cette propoſition, *le plus grand commun diviſeur de* A, B, *eſt en même tems le plus grand commun diviſeur de tous les réſidus* R, R', R'' &c. Or ces réſidus décroiſſant en ſorte qu'enfin ils deviennent plus petits qu'aucune quantité aſſignable, il s'enſuit que *le plus grand commun diviſeur de* A, B, *eſt plus petit qu'aucune quantité aſſignable;* ce qui veut dire qu'il n'y en a point, & que par conſéquent A, B, étant des quantités incommenſurables, *la*

$$\text{tang } v = \frac{A}{B}$$

ſera une quantité irrationelle toutes les fois que l'arc v *ſera une partie aliquote du rayon.*

§. 16. Voilà donc à quoi ſe borne l'uſage qu'on peut faire de la propoſition d'*Euclide*. Il s'agit maintenant de l'étendre à tous les cas où l'arc v eſt commenſurable au rayon. Pour cet effet, & pour démontrer encore quelques autres théoremes, je vais reprendre la fraction continue

$$\text{tang } v = \cfrac{1}{1:v - \cfrac{1}{3:v - \cfrac{1}{5:v - \cfrac{1}{7:v - 1 \;\&c.}}}}$$

& en faiſant $1 : v = w$, je la transformerai en

$$\text{tang } v = \cfrac{1}{w - \cfrac{1}{3w - \cfrac{1}{5w - \cfrac{1}{7w - 1 \;\&c.}}}}$$

§. 17.

17 LAMBERT. IRRATIONALITY OF π

By 1750 the number π had been expressed by infinite series, infinite products, and infinite continued fractions, its value had been computed by infinite series to 127 places of decimals (see Selection V.15), and it had been given its present symbol. All these efforts, however, had not contributed to the solution of the ancient problem of the quadrature of the circle; the question whether a circle whose area is equal to that of a given square can be constructed with the sole use of straightedge and compass remained unanswered. It was Euler's discovery of the relation between trigonometric and exponential functions that eventually led to an answer. The first step was made by J. H. Lambert, when, in 1766–1767, he used Euler's work to prove the irrationality not only of π, but also of e.

Johann Heinrich Lambert (1728–1777) was a Swiss from Mülhausen (then in Switzerland). Called to Berlin by Frederick the Great, he became a member of the Berlin Academy and thus a colleague of Euler and Lagrange. His name is also connected with the introduction of hyperbolic functions (1770), with perspective (1759, 1774), and with the so-called Lambert projection in cartography (1772).

Lambert published his proof of the irrationality of π in his "Vorläufige Kenntnisse für die, so die Quadratur und Rectification des Circuls suchen," *Beyträge zum Gebrauche der Mathematik und deren Anwendung 2* (Berlin, 1770), 140–169, written in 1766, and in more detail in the "Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques," *Histoire de l'Académie, Berlin, 1761* (1768), 265–322, presented in 1767. They have been reprinted in the *Opera mathematica*, ed. A. Speiser (2 vols.; Füssli, Zurich, 1946, 1948), I, 194–212, II, 112–159. The following text is a translation from pp. 132–138 of vol. II. Lambert writes tang where we write tan. See also F. Rudio, *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung* (Teubner, Leipzig, 1892).

---

37. *Now I say that this tangent* [$\tan \varphi/\omega$] *will never be commensurable to the radius, whatever the integers* ω, φ *may be.*[1]

[1] In the previous sections Lambert expands $\tan v$, v an arbitrary arc of a circle of radius 1, into a continued fraction, and gets for $v = 1/w$

$$\tan v = \cfrac{1}{w - \cfrac{1}{3w + \cfrac{1}{5w - 1}}} \quad \text{etc.}$$

Investigating the partial fractions and their residues, he finds infinite series like

$$\tan v = \frac{1}{w} + \frac{1}{w(3w^2 - 1)} + \frac{1}{(3w^2 - 1)(15w^3 - 6w)} + \cdots$$

and shows (in §34) that these series converge more rapidly than any decreasing geometric series. Then, if $w = \omega : \varphi$, ω, φ being relatively prime integers, he finds for the partial fractions of $\tan v$ (§36):

$$\frac{\varphi}{\omega}, \quad \frac{3\omega\varphi}{3\omega^2 - \varphi^2}, \quad \frac{15\omega^2\varphi - \varphi^3}{15\omega^3 - 6\varphi^2\omega}, \quad \frac{105\omega^3\varphi - 10\omega\varphi^3}{105\omega^4 - 45\omega^2\varphi^2 + \varphi^2}, \quad \text{etc.,}$$

and (§37):

$$\tan\frac{\varphi}{\omega} = \frac{\varphi}{\omega} + \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega^2\varphi)} + \text{etc.}$$

Then follows the text which we reproduce.

38. To prove this theorem, let us write

$$\tan\frac{\varphi}{\omega} = \frac{M}{P},$$

such that M and P are quantities expressed in an arbitrary way, even, if you like, by decimal sequences, which always can happen, even when M, P are integers, because we have only to multiply each of them by an irrational quantity. We can also, if we like, write

$$M = \sin\frac{\varphi}{\omega}, \quad P = \cos\frac{\varphi}{\omega},$$

as above. And it is clear that, even if $\tan \varphi/\omega$ were rational, this would not necessarily hold for $\sin \varphi/\omega$ and $\cos \varphi/\omega$.

39. Since the fraction M/P exactly expresses the tangent of φ/ω, it must give all the quotients w, $3w$, $5w$, etc., which in the present case are

$$+\frac{\omega}{\varphi}, \quad -\frac{3\omega}{\varphi}, \quad +\frac{5\omega}{\varphi}, \quad -\frac{7\omega}{\varphi}, \quad \text{etc.}$$

40. Hence, if the tangent of φ/ω is rational, then clearly M will be to P as an integer μ is to an integer π, such that, if μ, π are relatively prime, we shall have

$$M : \mu = P : \pi = D,$$

and D will be the greatest common divisor of M, P. And since reciprocally

$$M : D = \mu, \quad P : D = \pi,$$

we see that, since M, P are supposed to be irrational quantities, their greatest common divisor will be equally an irrational quantity, which is the smaller, the larger the quotients μ, π are.

41. Here are therefore *the two suppositions of which we must show the incompatibility*. Let us first divide P by M, and the quotient must be $\omega : \varphi$. But since $\omega : \varphi$ is a fraction, let us divide φP by M, and the quotient ω will be the φ-tuple of $\omega : \varphi$. It is clear that we could divide it by φ if we wished to do so. This is not necessary, since it will be sufficient that ω be an integer. Having thus obtained ω by dividing φP by M, let the residue be R'. This residue will equally be the φ-tuple of what it would have been, and that we have to keep in mind. Now, since $P : D = \pi$, an integer, we still have $\varphi P : D = \varphi\pi$, an integer. Finally, $R' : D$ will also be an integer. Indeed, since

$$\varphi P = \omega M + R',$$

we shall have

$$\frac{\varphi P}{D} = \frac{\omega M}{D} + \frac{R'}{D}.$$

But $\varphi P : D = \varphi\pi$, $\omega M : D = \omega\mu$; hence

$$\varphi\pi = \omega\mu + \frac{R'}{D},$$

which gives

$$\frac{R'}{D} = \varphi\pi - \omega\mu = \text{integer},$$

which we shall call r', so that $R'/D = r'$. The residue of the first division will therefore still have the divisor D, the greatest common divisor of M, P.

42. Now let us pass to the second division. The residue R' being the φ-tuple of what it would have been if we had divided P instead of φP, we must take this into account by the second division, where we divide φM, instead of M, by R' in order to obtain the second quotient, which $= 3\omega : \varphi$. However, in order to avoid the fractional quotient here also, let us divide $\varphi^2 M$ by R', in order to have the quotient 3ω, an integer. Let the residue be R'', and we shall have

$$\varphi^2 M = 3\omega R' + R'';$$

hence, dividing by D,

$$\frac{\varphi^2 M}{D} = \frac{3\omega R'}{D} + \frac{R''}{D}.$$

But

$$\frac{\varphi^2 M}{D} = \varphi^2 m = \text{integer},$$

$$\frac{3\omega R'}{D} = 3\omega r' = \text{integer};$$

hence

$$\varphi^2 m = 3\omega r' + \frac{R''}{D},$$

which gives $R''/D = \varphi^2 m - 3\omega r'$ = an integer number, which we shall write $= r''$, so that

$$\frac{R''}{D} = r''.$$

Hence the greatest common divisor of M, P, R' is still of the second residue R''.

43. Let the next residues be R''', $R^{\text{iv}}, \ldots, R^n, R^{n+1}, R^{n+2}, \ldots$ which correspond to the φ-tuple quotients $5\omega, 7\omega, \ldots, (2n-1)\omega, (2n+1)\omega, (2n+3)\omega, \ldots$, and we have to prove in general that if two arbitrary residues R^n, R^{n+1}, in

immediate succession, still have D as divisor, the next residue R^{n+2} will have it too, so that, if we write

$$R^n : D = r^n,$$

$$R^{n+1} : D = r^{n+1},$$

where r^n and r^{n+1} are integers, we shall also have

$$R^{n+2} : D = r^{n+2},$$

an integer. This is the demonstration.

We omit this proof in §44, since the reasoning follows that of §42.

---

45. Now we have seen that r', r'' are integers (§§41, 42), hence also r''', $r^{\mathrm{iv}}, \ldots, r^n, \ldots$ to infinity will be integers. Hence any one of the residues R', R'', $R''', \ldots, R^n, \ldots$ to infinity will have D as common divisor. Let us now find the value of these residues expressed in M, P.

46. Every division provides us with an equation for this purpose, since we have

$$\begin{aligned} R' &= \varphi P - \omega M, \\ R'' &= \varphi^2 M - 3\omega R', \\ R''' &= \varphi^2 R' - 5\omega R'', \quad \text{etc.} \end{aligned}$$

But let us observe that in the existing case the quotients ω, 3ω, 5ω, etc. are alternately positive and negative and that the signs of the residues succeed each other in the order $- \; - \; + \; +$. These equations can therefore be changed into

$$\begin{aligned} R' &= \omega M - \varphi P, \\ R'' &= 3\omega R' - \varphi^2 M, \\ R''' &= 5\omega R'' - \varphi^2 R', \\ &\cdots\cdots\cdots \end{aligned}$$

or in general

$$R^{n+2} = (2n - 1)R^{n+1}\omega - \varphi^2 R^n.$$

From this we see that every residue is related to the two preceding in the same way as the numerators and denominators of the fractions that approximate the value of tan φ/ω (§36).

47. Let us make the substitutions indicated by these equations in order to express all these residues by M, P. We shall have

$$R' = \omega M - \varphi P,$$
$$R'' = (3\omega^2 - \varphi^2)M - 3\omega\varphi P,$$
$$R''' = (15\omega^3 - 6\omega\varphi^2)M - (15\omega^2\varphi - \varphi^3)P, \quad \text{etc.}$$

And since these coefficients of M, P are the denominators and numerators of the fractions we found above for $\tan \varphi/\omega$ (§36), we see also that we shall have

$$\frac{M}{P} - \frac{\varphi}{\omega} = \frac{R'}{\omega P},$$
$$\frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} = \frac{R''}{(3\omega^2 - \varphi^2)P},$$
$$\frac{M}{P} - \frac{15\omega^2\varphi - \varphi^3}{15\omega^3 - 6\omega\varphi^2} = \frac{R'''}{(15\omega^3 - 6\omega\varphi^2)P}, \quad \text{etc.}$$

48. But we have

$$\frac{M}{P} = \tan\frac{\varphi}{\omega};$$

hence (§§37, 34)

$$\frac{M}{P} - \frac{\varphi}{\omega} = \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^2 - 6\omega\varphi^2)} + \text{etc.},$$

$$\frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} = \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)} + \text{etc.};$$

hence

$$\frac{R'}{\omega P} = \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)} + \text{etc.},$$
$$\frac{R''}{(3\omega^2 - \varphi^2)P} = \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)} + \text{etc.},$$
$$\frac{R'''}{(15\omega^3 - 6\omega\varphi^2)P} = \frac{\varphi^7}{(15\omega^3 - 6\omega\varphi^2)(105\omega^4 - 45\omega^2\varphi^2 + \varphi^4)} + \text{etc.}$$

Thus all the residues can be found by means of the sequence of differences (§37)

$$\tan\frac{\varphi}{\omega} = \frac{\varphi}{\omega} + \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)}$$
$$+ \frac{\varphi^7}{(15\omega^3 - 6\omega\varphi^2)(105\omega^4 - 45\omega^2\varphi^2 + \varphi^4)} + \text{etc.}$$

by omitting 1, 2, 3, 4, etc. of the first terms and multiplying the sum of the following terms by the first factor of the denominator of the first term that is retained and by P.

49. Now, this sequence of differences is more convergent than a decreasing geometric progression (§§34, 35). Hence the residues R', R'', R''', etc. decrease in such a way that they become smaller than any assignable quantity. And as every one of these residues, having D as common divisor, is a multiple of D, it follows that this common divisor D is smaller than any assignable quantity, which makes $D = 0$. Consequently $M : P$ is a quantity incommensurable with unity, hence irrational.

50. Hence *every time that a circular arc* $= \varphi/\omega$ *is commensurable with the radius* $= 1$, *hence rational, the tangent of this arc will be a quantity incommensurable with the radius, hence irrational. And conversely, every rational tangent is the tangent of an irrational arc.*

51. Now, since the tangent of 45° is rational, and equal to the radius, the arc of 45°, and hence also the arc of 90°, 180°, 360°, is incommensurable with the radius. Hence *the circumference of the circle does not stand to the diameter as an integer to an integer.* Thus we have here this theorem in the form of a corollary to another theorem that is infinitely more universal.

52. Indeed, it is precisely this absolute universality that may well surprise us.

Lambert then goes on to draw consequences from his theorem concerning arcs with rational values of the tangent. Then he draws an analogy between hyperbolic and trigonometric functions and proves from the continued fraction for $e^u + 1$ that e and all its powers with integral exponents are irrational, and that all rational numbers have irrational natural logarithms. He ends with the sweeping conjecture that "no circular or logarithmic transcendental quantity into which no other transcendental quantity enters can be expressed by any irrational radical quantity," where by "radical quantity" he means one that is expressible by such numbers as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt{2 + \sqrt{3}}$, and so forth. Lambert does not prove this; if he had, he would have solved the problem of the quadrature of the circle. The proof of Lambert's conjecture had to wait for the work of C. Hermite (1873), and F. Lindemann (1882). See, for instance, H. Weber and J. Wellstein, *Encyklopädie der Elementar-Mathematik* (3rd ed.; Teubner, Leipzig, 1909), I, 478–492; G. Hessenberg, *Transzendenz von e und π* (Teubner, Leipzig, Berlin, 1912); U. G. Mitchell and M. Strain, "The number e," *Osiris 1* (1936), 476–496.

CONTRIBUTIONS TO MATHEMATICS,

COMPRISING CHIEFLY

THE

RECTIFICATION OF THE CIRCLE

TO 607 PLACES OF DECIMALS.

BY

WILLIAM SHANKS,

HOUGHTON-LE-SPRING, DURHAM.

LONDON:

G. BELL, 186, FLEET-STREET; MACMILLAN & Co., CAMBRIDGE;
ANDREWS, DURHAM.

1853.

PRINTED FOR THE AUTHOR.

TO

WILLIAM RUTHERFORD, ESQUIRE, L.L.D., F.R.A.S.,

ROYAL MILITARY ACADEMY, WOOLWICH.

MY DEAR SIR,

I know of no person to whom I can, with so great pleasure, so much propriety, and such deep gratitude, inscribe this small volume as to you, from whom I received my earliest lessons in numbers; and I earnestly wish it had been something more worthy of notice that the Pupil was presenting to the Master, than the present "CONTRIBUTIONS TO MATHEMATICS" can pretend to be. Still, I venture to indulge and to express the thought, that whatever proves the firm determination of any of Great Britain's Mathematicians not to allow themselves to be outstripped by their Continental neighbours, even in calculation, redounds, in its measure, to their own credit, and also affords pleasure and satisfaction to all who feel interested in excellence of this kind.

You, Sir, have laboured long, ardently, and most successfully in the Study of Mathematics, and have evinced, in nearly every branch of that beautiful but abstruse science, such clearness and depth of thought as are rarely met with. This however is not all. We seldom indeed find profundity united with great facility of computation;—but you happily combine both in a very eminent degree.

I regret my inability adequately to convey to you the heart-felt sentiments of esteem, gratitude, and respect with which

I am, My dear Sir,
Your sincere and obliged Friend,

THE AUTHOR.

Houghton-le-Spring,
Feb. 28, 1853.

PREFACE.

TOWARDS the close of the year 1850, the Author first formed the design of rectifying the Circle to upwards of 300 places of decimals. He was fully aware, at that time, that the accomplishment of his purpose would add little or nothing to his fame as a Mathematician, though it might as a Computer; nor would it be productive of anything in the shape of pecuniary recompense at all adequate to the labour of such lengthy computations. He was anxious to fill up scanty intervals of leisure with the achievement of something original, and which, at the same time, should not subject him either to great tension of thought, or to consult books. He is aware that works on nearly every branch of Mathematics are being published almost weekly, both in Europe and America; and that it has therefore become no easy task to ascertain what really is original matter, even in the pure science itself. Beautiful speculations, especially in both Plane and Curved

Geometry, have, even of late years, been pushed to a very great extent in our own country, without much regard being paid, as to the probability of their ultimate and direct utility. The Doctrine of Impossible or Imaginary Quantities, which certainly long perplexed Mathematicians, has also lately received a proper share of attention from men of genius. The Integration of Quantities seems to merit farther labour and research; and no doubt this important and abstruse branch will, by and by, obtain due consideration, and we shall have important simplifications of, and additions to, our already large stock of knowledge.

Holding sentiments somewhat similar to the above, and having, as before stated, occasionally had spare moments from the duties of an arduous profession, the Writer entertains the hope, that Mathematicians will look with indulgence on his present "Contributions" to their favourite science, and also induce their Friends and Patrons of Mathematical Studies, to accord him their generous support, by purchasing copies of the work.

Dr. Rutherford, of the Royal Military Academy, Woolwich, a distinguished Mathematician, has kindly co-operated with the Author in separately calculating and then collating the value of π to 441 places of decimals, so that accuracy is ensured at least to that extent; and it is confidently hoped no error has crept into the remaining 86 places.

It is proper to state, that the talented writer just mentioned lately sent a "Paper on determining the value of π," to the Royal Society of London, an Abstract of which has just been published in the Proceedings of that Learned Body, wherein are given the values of $\tan^{-1}\frac{1}{5}$, of $\tan^{-1}\frac{1}{239}$, and of π, to 441 places of decimals: also, the extension of the same, by the present writer, to 527 places.

In the following pages, the ratio of the circumference of a circle to its diameter, is determined to the great extent of 527 places of decimals.

The values, of the base of Napier's Logarithms, of the Napierian logarithms of 2, 3, 5, and 10, and of the Modulus of the Common System, are given to 137 places of decimals, except that of M, which extends only to 136.

A few of the higher powers of 2, as far as 2^{721}, having been obtained in the calculation of $\tan^{-1}\frac{1}{5}$, conclude the volume.

It only remains to add, that Machin's formula, viz., $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$, was employed in finding π:—and that the values of $\tan^{-1}\frac{1}{5}$, and of $\tan^{-1}\frac{1}{239}$ are found and given separately; as also the *value of each term of the series* employed in determining these two arcs.

Houghton-le-Spring,
Feb. 28, 1853.

Since the above date, and while the following sheets were in the Press, the Author has extended the values of $\tan^{-1}\frac{1}{5}$ and of $\tan^{-1}\frac{1}{239}$ to 609, and the value of π to 607 decimals; which extensions are given in the proper place. Should Mathematicians evince a wish to possess the *extended values of each term of the series used in finding these arcs,* a few supplementary sheets might soon be furnished.

April 30, 1853.

LIST OF SUBSCRIBERS.

	No. of Copies.
W. Whewell, D.D., Master of Trinity College, Cambridge - - -	1
M. A. C., Houghton-le-Spring - - - - -	1
W. Rutherford, Esq., L.L.D., F.R.A.S., Royal Military Academy, Woolwich -	2
Rev. Temple Chevallier, B.D., Professor of Mathematics, University, Durham -	1
Hon. & Rev. John Grey, M.A., Rector of Houghton-le-Spring - -	1
Augustus De Morgan, Esq., Professor of Mathematics, University College, London - - - - - - -	1
Rev. P. Kelland, M.A., F.R.S.E., Professor of Mathematics, University, Edinburgh - - - - - - -	1
G. B. Airy, Esq., F.R.S., Astronomer Royal, Royal Observatory, Greenwich -	2
John Stephenson, Esq., Colt Crag, Northumberland - - -	2
Nathaniel Ellison, Esq., Morton House, Durham - - - -	1
S. Fenwick, Esq., F.R.A.S., Mathematical Master, Royal Military Academy, Woolwich - - - - - - -	1
Rev. Charles Whitley, M.A., Hon. Canon, & Vice-Master of University College, Durham - - - - - - -	1
Robert Rawson, Esq., Highland House, Portsea, Portsmouth - -	1
R. Tatham, D.D., Master of St. John's College, Cambridge - -	1
Sir J. F. W. Herschel, Bart., Master of the Mint, &c., &c., London - -	1

	No. of Copies.
John Narrien, Esq., F.R.S., Professor of Mathematics, Royal Military College, Sandhurst - - - - - - -	1
Rev. J. R. Young, M.A., Rector of Whitnash, Warwickshire - -	2
James Laurie, Esq., Billiter-Street, London - - - -	1
W. Goodricke, Esq., Houghton-le-Spring - - - -	2
Joseph Whitehead, Esq., Wash Brook Cottage, near Oldham, Lancashire -	1
H. W. Cookson, D.D., Master of St. Peter's College, Cambridge - -	1
Rev. Bartholomew Price, M.A., Fellow and Tutor of Pembroke College, Oxford	1
W. Finley, Esq., Durham - - - - - - -	1
Rev. Robert Walker, M.A., F.R.S., Reader in Experimental Philosophy, Oxford	1
George Bell, Esq., Hexham - - - - - - -	1
Wilkinson Mathews, Esq., Q.C., 64, Brook-Street, Hanover Square, London -	1
W. S. B. Woolhouse, Esq., F.R.A.S., Alwyne Lodge, Islington, London	2
William Winship, Jun., Esq., Morpeth - - - - -	1
Thomas Dixon Walker, Esq., Darlington - - - -	1
Mr. George Andrews, Durham - - - - -	1

TABLE OF CONTENTS.

INTRODUCTORY REMARKS AND HISTORICAL NOTICES.

The Problem, "to find the ratio of the circumference of a circle to its diameter," has engaged the attention of Mathematicians from very remote times. The Greek Geometers and Arithmeticians, chiefly from their clumsy scale of notation, could not be expected to make any considerable progress in ascertaining the ratio in question: and indeed we may remark, by the way, that even in the Geometry of Curve lines at large, they did not and could not proceed to any very great extent. Until the time of Newton and Leibnitz, the *limits* of Mathematics were but contracted; and though since, and by, their very extraordinary labours, great additions indeed have been made to the previous stock of Mathematical knowledge, yet we must admit that not very much has lately been done, or perhaps remains to be done, towards extending the *boundaries* of that department of human knowledge and research. It would not be of any service to recapitulate in this place what is already duly recorded as matter of history respecting the various Mathematicians, and the different processes employed by them, who have applied themselves to the solution of this striking and curious Problem. Nor can we well afford space even to glance at the ignorant, presumptuous, and abortive attempts made at different periods "to square the circle!" In later times, Mathematicians—and they alone are competent to deal with such enquiries—have pointed out satisfactorily how to find the value of π *ultimately* by means of infinite series of various degrees of convergency, as will be seen in what follows: but no one, so far as we know, has hitherto been able—and we are of opinion that it can never be accomplished—to ascertain the *limit*, strictly speaking, of the ratio under consideration. In other words, the circumference and diameter of a circle are incommensurable.* We proceed, then, to take up the history of the Problem since the year 1831. Previous to 1831, the value of π, as the late Professor Thomson of the University of Glasgow writes, in his work on the Differential and Integral Calculus, had been calculated "to the extraordinary extent of 140 figures!" We may here be permitted to indulge a smile at the learned writer's words, now that the ratio has been found to 607 places of decimals!

In the year 1841, Dr. Rutherford, Royal Military Academy, Woolwich, a distinguished Mathematician, presented a paper to the Royal Society of London, which was published in their transactions, in which he gave the value of π to 208 decimals.

* For full and interesting particulars, the reader is here referred to the History of this curious Problem, written by the profound Mathematician Professor De Morgan, and which is given in the Penny Cyclopædia.

The formula he employed is $\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$. Subsequently it was found that the last 56 figures were incorrect, an error having crept into the value of one of the terms of the series employed. But of this talented writer we shall have to speak afterwards.

The Mathematician who seems next to have engaged in the solution of this Problem is M. Dase, about the year 1846, then a young man of great promise, who used the formula $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$. He found the value of π correctly, as was afterwards shown by Dr. Clausen, to 200 places of decimals, and communicated his result to Professor Schumacher of Altona.

Somewhere about 1847, the eminent Mathematician Dr. Clausen, of Dorpat, above-mentioned, turned his attention to this subject, and employing Machin's formula, $\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$, and also the formula $\frac{\pi}{4} = 2 \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$, he calculated the value of π from each of these formulæ, and found it correctly to 250 decimals. The values of the four arcs and of π are given in No. 589 of the "Astronomiche Nachrichten," published in 1847.

In the earlier portions of the year 1851, the Author of this Volume, not then aware of M. Dase's, or of Dr. Clausen's labours, employed Machin's formula, given above, and calculated the value of π to 315 decimals. Dr. Rutherford, in the same year, verified the Author's results, using also Machin's formula; and moreover pushed his calculations to 350 places of decimals. During the latter part of 1851, and the early months of 1852, Dr. Rutherford and the present writer extended the value of π, the former to 441 places, and the latter to 527 decimals. In the months of March and April, 1853, the Author still further extended the ratio in question; and the value of π, given in this Volume, is, it is presumed, more extensive than has hitherto been determined, consisting of no less than 607 decimals. The values of $\tan^{-1}\frac{1}{5}$, and of $\tan^{-1}\frac{1}{239}$ are given to 609 places. These values, then, have been carefully collated, as far as 441 decimals, with Dr. Rutherford's results, and may be pronounced free from errors, inasmuch as each party worked independently of the other. The Author, therefore, is alone responsible for the accuracy of the additional 166 and 168 respective places of decimals.

Whether any other Mathematician will appear, possessing sufficient leisure, patience, and facility of computation, to calculate the value of π to a still greater extent, remains to be seen: all that the Author can say is, he takes leave of the subject for the present, and deems the farther extension of this ratio a work of considerable difficulty, notwithstanding the assertions of some writers, who have evidently had little or no experience in such matters.

EXPLANATORY REMARKS AND FORMULÆ.

FROM Machin's well known formula, viz. $\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$, the value of π, given afterwards, has been determined.

It may also as well be stated here, that from Newton's series we have,

$$\tan^{-1}\frac{1}{5} = \frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} + \frac{1}{9.5^9} - \frac{1}{11.5^{11}} +, \&c.\ ;$$

$$\text{And, } \tan^{-1}\frac{1}{239} = \frac{1}{239} - \frac{1}{3.239^3} + \frac{1}{5.239^5} - \frac{1}{7.239^7} + \frac{1}{9.239^9} -, \&c.$$

The value of each term of the above series employed in finding the two arcs is given separately, so that its accuracy may readily be tested; and no difficulty can possibly arise to Mathematicians, for whose perusal chiefly the following pages are intended, in comprehending all that follows.

$\frac{1}{5}$								
	888888	888888	888888	888888	888888	888888	888888	888888
	846153	846153	846153	846153	846153	846153	846153	846153
	588235	294117	647058	823529	411764	705882	352941	176470
	809523	809523	809523	809523	809523	809523	809523	809523
$\frac{1}{25.5^{25}}$								
	965517	241379	310344	827586	206896	551724	137931	034482
	424242	424242	424242	424242	424242	424242	424242	424242
	054054	054054	054054	054054	054054	054054	054054	054054
	780487	804878	048780	487804	878048	780487	804878	048780
	777777	777777	777777	777777	777777	777777	777777	777777
$\frac{1}{49.5^{49}}$	530612	244897	959183	673469	387755	102040	816326	530612
	490566	037735	849056	603773	584905	660377	358490	566037
	771929	824561	403508	771929	824561	403508	771929	824561
	786885	245901	639344	262295	081967	213114	754098	360655
	769230	769230	769230	769230	769230	769230	769230	769230
	942028	985507	246376	811594	202898	550724	637681	159420
$\frac{1}{73.5^{73}}$	602739	726027	397260	273972	602739	726027	397260	273972
	623376	623376	623376	623376	623376	623376	623376	623376
	876543	209876	543209	876543	209876	543209	876543	209876
	470588	235294	117647	058823	529411	764705	882352	941176
	808988	764044	943820	224719	101123	595505	617977	528089
	860215	053763	440860	215053	763440	860215	053763	440860
$\frac{1}{97.5^{97}}$	587628	865979	381443	298969	072164	948453	608247	422680
	801980	198019	801980	198019	801980	198019	801980	198019
	761904	761904	761904	761904	761904	761904	761904	761904
	247706	422018	348623	853211	009174	311926	605504	587155
	513274	336283	185840	707964	601769	911504	424778	761061
	760683	760683	760683	760683	760683	760683	760683	760683
$\frac{1}{121.5^{121}}$	165289	256198	347107	438016	528925	619834	710743	801652
	620155	038759	689922	480620	155038	759689	922480	620155
	406015	037593	984962	406015	037593	984962	406015	037593
	401459	854014	598540	145985	401459	854014	598540	145985
	886524	822695	035460	992907	801418	439716	312056	737588
$\frac{1}{145.5^{145}}$	862068	965517	241379	310344	827586	206896	551724	137931
	838926	174496	644295	302013	422818	791946	308724	832214
	934640	522875	816993	464052	287581	699346	405228	758169
	044585	987261	146496	815286	624203	821656	050955	414012
	993788	819875	776397	515527	950310	559006	211180	124223
	939393	939393	939393	939393	939393	939393	939393	939393

SUR

LA FONCTION EXPONENTIELLE.

Comptes rendus de l'Académie des Sciences, t. LXXVII, 1873, p. 18-24, 74-79, 226-233, 285-293.

I. Étant donné un nombre quelconque de quantités numériques $\alpha_1, \alpha_2, \ldots, \alpha_n$, on sait qu'on peut en approcher simultanément par des fractions de même dénominateur, de telle sorte qu'on ait

$$\alpha_1 = \frac{A_1}{A} + \frac{\delta_1}{A\sqrt[n]{A}},$$

$$\alpha_2 = \frac{A_2}{A} + \frac{\delta_2}{A\sqrt[n]{A}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\alpha_n = \frac{A_n}{A} + \frac{\delta_n}{A\sqrt[n]{A}},$$

$\delta_1, \delta_2, \ldots, \delta_n$ ne pouvant dépasser une limite qui dépend seulement de n. C'est, comme on voit, une extension du mode d'approximation résultant de la théorie des fractions continues, qui correspondrait au cas le plus simple de $n = 1$. Or, on peut se proposer une généralisation semblable de la théorie des fractions continues algébriques, en cherchant les expressions approchées de n fonctions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ par des fractions rationnelles $\frac{\Phi_1(x)}{\Phi(x)}, \frac{\Phi_2(x)}{\Phi(x)}, \ldots, \frac{\Phi_n(x)}{\Phi(x)}$, de manière que les développements en série suivant les puissances croissantes de la variable coïncident jusqu'à une puissance déterminée x^M. Voici d'abord, à cet égard, un premier résultat qui s'offre immédiatement. Supposons que les fonctions $\varphi_1(x)$, $\varphi_2(x)$, ..., $\varphi_n(x)$ soient toutes développables en séries de la forme $\alpha + \beta x + \gamma x^2 + \ldots$ et faisons

$$\Phi(x) = \mathrm{A}x^m + \mathrm{B}x^{m-1} + \ldots + \mathrm{K}x + \mathrm{L}.$$

On pourra, en général, disposer des coefficients A, B, ..., L de manière à annuler dans les n produits $\varphi_i(x)\Phi(x)$ les termes en

$$x^{\mathrm{M}}, \quad x^{\mathrm{M}-1}, \quad \ldots, \quad x^{\mathrm{M}-\mu_i+1},$$

μ_i étant un nombre entier arbitraire. Nous poserons ainsi un nombre d'équations homogènes de premier degré égal précisément à μ_i, et l'on aura

$$\varphi_i(x)\Phi(x) = \Phi_i(x) + \varepsilon_1 x^{\mathrm{M}+1} + \varepsilon_2 x^{\mathrm{M}+2} + \ldots,$$

ε_1, ε_2, ... étant des constantes, $\Phi_i(x)$ un polynome entier de degré $\mathrm{M} - \mu_i$. Or, cette relation donnant

$$\varphi_i(x) = \frac{\Phi_i(x)}{\Phi(x)} + \frac{\varepsilon_1 x^{\mathrm{M}+1} + \varepsilon_2 x^{\mathrm{M}+2} + \ldots}{\Phi(x)},$$

on voit que les développements en série de la fraction rationnelle et de la fonction seront, en effet, les mêmes jusqu'aux termes en x^{M}, et, comme le nombre total des équations posées est $\mu_1 + \mu_2 + \ldots + \mu_n$, il suffit d'assujettir à la seule condition

$$\mu_1 + \mu_2 + \ldots + \mu_n = m$$

les entiers μ_i restés jusqu'ici absolument arbitraires. C'est cette considération si simple qui a servi de point de départ à l'étude de la fonction exponentielle que je vais exposer, me proposant d'en faire l'application aux quantités

$$\varphi_1(x) = e^{ax}, \qquad \varphi_2(x) = e^{bx}, \qquad \ldots, \qquad \varphi_n(x) = e^{hx}.$$

II. Soit, pour abréger, $\mathrm{M} - m = \mu$; je compose avec les constantes a, b, ..., h le polynome

$$\mathrm{F}(z) = z^{\mu}(z-a)^{\mu_1}(z-b)^{\mu_2}\ldots(z-h)^{\mu_n},$$

de degré $\mu + \mu_1 + \ldots + \mu_n = \mathrm{M}$, et j'envisage les n intégrales définies

$$\int_0^a e^{-zx}\,\mathrm{F}(z)\,dz, \qquad \int_0^b e^{-zx}\,\mathrm{F}(z)\,dz, \qquad \ldots, \qquad \int_0^h e^{-zx}\,\mathrm{F}(z)\,dz,$$

qu'il est facile d'obtenir sous forme explicite. Faisant, en effet,

$$\mathfrak{F}(z) = \frac{F(z)}{x} + \frac{F'(z)}{x^2} + \ldots + \frac{F^{(M)}(z)}{x^{M+1}},$$

nous aurons

$$\int e^{-zx} F(z)\, dz = -e^{-zx} \mathfrak{F}(z),$$

et, par conséquent,

$$\int_0^a e^{-zx} F(z)\, dz = \mathfrak{F}(0) - e^{-ax} \mathfrak{F}(a),$$

$$\int_0^b e^{-zx} F(z)\, dz = \mathfrak{F}(0) - e^{-bx} \mathfrak{F}(b),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Or l'expression de $\mathfrak{F}(z)$ donne immédiatement, sous forme de polynomes ordonnés suivant les puissances croissantes de $\frac{1}{x}$, les diverses quantités $\mathfrak{F}(0)$, $\mathfrak{F}(a)$, $\mathfrak{F}(b)$, ..., et si l'on observe qu'on a

$$F(0) = 0, \qquad F'(0) = 0, \qquad \ldots, \qquad F^{(\mu-1)}(0) = 0,$$

puis successivement,

$$\begin{array}{llll} F(a) = 0, & F'(a) = 0, & \ldots, & F^{(\mu_1-1)}(a) = 0, \\ F(b) = 0, & F'(b) = 0, & \ldots, & F^{(\mu_2-1)}(b) = 0, \\ \multicolumn{4}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,} \end{array}$$

nous en conclurons les résultats suivants

$$\mathfrak{F}(0) = \frac{\Phi(x)}{x^{M+1}}, \qquad \mathfrak{F}(a) = \frac{\Phi_1(x)}{x^{M+1}}, \qquad \ldots, \qquad \mathfrak{F}(h) = \frac{\Phi_n(x)}{x^{M+1}},$$

où le polynome entier $\Phi(x)$ est du degré $M - \mu = m$, et les autres $\Phi_1(x)$, $\Phi_2(x)$, ..., $\Phi_n(x)$, des degrés $M - \mu_1$, $M - \mu_2$, ..., $M - \mu_n$. Cela posé, nous écrirons

$$e^{ax}\Phi(x) - \Phi_1(x) = x^{M+1} e^{ax} \int_0^a e^{-zx} F(z)\, dz,$$

$$e^{bx}\Phi(x) - \Phi_2(x) = x^{M+1} e^{bx} \int_0^b e^{-zx} F(z)\, dz,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$e^{hx}\Phi(x) - \Phi_n(x) = x^{M+1} e^{hx} \int_0^h e^{-zx} F(z)\, dz;$$

or, les intégrales définies se développant en séries de la forme $\alpha + \beta x + \gamma x^2 + \ldots$, on voit que les conditions précédemment posées comme définitions du nouveau mode d'approximation des fonctions se trouvent entièrement remplies. Nous avons ainsi obtenu, dans toute sa généralité, le système des fractions rationnelles $\frac{\Phi_1(x)}{\Phi(x)}, \frac{\Phi_2(x)}{\Phi(x)}, \ldots, \frac{\Phi_n(x)}{\Phi(x)}$, de même dénominateur, représentant les fonctions $e^{ax}, e^{bx}, \ldots, e^{hx}$, aux termes près de l'ordre x^{M+1}.

III. Soit, comme application, $n = 1$, et supposons de plus $\mu = \mu_1 = m$, ce qui donnera

$$M = 2m, \qquad F(z) = z^m(z-1)^m;$$

les dérivées de $F(z)$ pour $z = 0$ se tirent sur-le-champ du développement par la formule du binome

$$F(z) = z^{2m} - \frac{m}{1} z^{2m-1} + \frac{m(m-1)}{1.2} z^{2m-2} - \ldots + (-1)^m z^m,$$

et l'on obtient

$$\frac{F^{(2m-k)}(0)}{1.2.3\ldots 2m-k} = \frac{m(m-1)\ldots(m-k+1)}{1.2.3\ldots k}(-1)^k,$$

d'où, par suite,

$$\frac{\Phi(x)}{1.2.3\ldots m} = 2m(2m-1)\ldots(m+1) - (2m-1)(2m-2)\ldots(m+1)\frac{m}{1}x$$
$$+ (2m-2)(2m-3)\ldots(m+1)\frac{m(m-1)}{1.2}x^2 - \ldots + (-1)^m x^m.$$

Pour avoir, en second lieu, les valeurs des dérivées quand on suppose $z = 1$, nous poserons $z = 1 + h$, afin de développer suivant les puissances de h le polynome $F(1+h) = h^m(h+1)^m$. Or les coefficients précédemment obtenus se reproduisant, sauf le signe, on voit qu'on aura

$$\Phi_1(x) = \Phi(-x).$$

Ces résultats conduisent à introduire, au lieu de $\Phi(x)$ et $\Phi_1(x)$, les polynomes

$$\Pi(x) = \frac{\Phi(x)}{1.2.3\ldots m}, \qquad \Pi_1(x) = \frac{\Phi_1(x)}{1.2.3\ldots m},$$

dont les coefficients sont des nombres entiers; on aura ainsi

$$e^x \mathrm{H}(x) - \mathrm{H}_1(x) = \frac{x^{2m+1}}{1.2.3\ldots m} e^x \int_0^1 e^{-zx} z^m (z-1)^m \, dz$$
$$= (-1)^m \frac{x^{2m+1}}{1.2.3\ldots m} \int_0^1 e^{x(1-z)} z^m (1-z)^m \, dz,$$

et l'on met en évidence que le premier membre peut devenir, pour une valeur suffisamment grande de m, plus petit que toute quantité donnée. Nous savons effectivement que le facteur $\frac{x^{2m+1}}{1.2.3\ldots m}$ a zéro pour limite, et il en est de même de l'intégrale, la quantité $z^m(1-z)^m$ étant toujours inférieure à son maximum $\left(\frac{1}{2}\right)^m$ qui décroît indéfiniment quand m augmente. Il résulte de là qu'en supposant x un nombre entier, l'exponentielle e^x ne peut avoir une valeur commensurable; car si l'on fait $e^x = \frac{b}{a}$, on parvient, après avoir chassé le dénominateur, à l'égalité

$$b\,\mathrm{H}(x) - a\,\mathrm{H}_1(x) = (-1)^m \frac{a x^{2m+1}}{1.2.3\ldots m} \int_0^1 e^{x(1-x)} z^m (1-z)^m \, dz,$$

dont le second membre peut devenir moindre que toute grandeur donnée, et sans jamais s'évanouir, tandis que le premier est un nombre entier. Lambert, à qui l'on doit cette proposition, ainsi que la seule démonstration, jusqu'à ce jour obtenue, de l'irrationnalité du rapport de la circonférence au diamètre et de son carré, a tiré ces importants résultats de la fraction continue

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \ldots}}}$$

à laquelle nous parviendrons plus tard. Laissant entièrement de côté le rapport de la circonférence au diamètre, je vais maintenant tenter d'aller plus loin à l'égard du nombre e, en établissant l'impossibilité d'une relation de la forme

$$\mathrm{N} + e^a \mathrm{N}_1 + e^b \mathrm{N}_2 + \ldots + e^h \mathrm{N}_n = 0,$$

$a, b, \ldots, h$ étant des nombres entiers, ainsi que les coefficients N, $N_1, \ldots, N_n$.

IV. Je considère, à cet effet, parmi les divers systèmes de fractions rationnelles $\frac{\Phi_1(x)}{\Phi(x)}, \frac{\Phi_2(x)}{\Phi(x)}, \ldots, \frac{\Phi_n(x)}{\Phi(x)}$, celui qu'on obtient lorsqu'on suppose $\mu = \mu_1 = \ldots = \mu_n$, ce qui donne

$$m = n\mu, \qquad M = (n+1)\mu \qquad \text{et} \qquad F(z) = f^{\mu}(z),$$

en faisant

$$f(z) = z(z-a)(z-b)\ldots(z-h).$$

Soit alors, comme tout à l'heure,

$$H(x) = \frac{\Phi(x)}{1.2.3\ldots\mu}, \qquad H_1(x) = \frac{\Phi_1(x)}{1.2.3\ldots\mu}, \qquad \ldots,$$
$$H_n(x) = \frac{\Phi_n(x)}{1.2.3\ldots\mu};$$

ces nouveaux polynomes auront encore, pour leurs coefficients, des nombres entiers, et conduiront aux relations suivantes :

$$\text{(A)} \qquad \begin{cases} e^{ax}H(x) - H_1(x) = \varepsilon_1, \\ e^{bx}H(x) - H_2(x) = \varepsilon_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ e^{hx}H(x) - H_n(x) = \varepsilon_n, \end{cases}$$

en écrivant, pour abréger,

$$\varepsilon_1 = \frac{x^{M+1}e^{ax}}{1.2.3\ldots\mu}\int_0^a e^{-zx}F(z)\,dz = \int_0^a e^{x(a-z)}\frac{f^{\mu}(z)x^{(n+1)\mu+1}}{1.2.3\ldots\mu}\,dz,$$
$$\varepsilon_2 = \frac{x^{M+1}e^{bx}}{1.2.3\ldots\mu}\int_0^b e^{-zx}F(z)\,dz = \int_0^b e^{x(b-z)}\frac{f^{\mu}(z)x^{(n+1)\mu+1}}{1.2.3\ldots\mu}\,dz,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Cela posé, j'observe en premier lieu que $\varepsilon_1, \varepsilon_2, \ldots$ deviennent, pour une valeur suffisamment grande de μ, plus petits que toute quantité donnée; car, le polynome $f(z)$ ne dépassant jamais une certaine limite λ dans l'intervalle parcouru par la variable, le facteur $\frac{f^{\mu}(z)x^{(n+1)\mu+1}}{1.2.3\ldots\mu}$, qui multiplie l'exponentielle sous le signe d'intégration, est constamment inférieur à la quantité $\frac{(\lambda x^{n+1})^{\mu}x}{1.2.3\ldots\mu}$, qui a zéro pour limite.

Je suppose maintenant $x=1$ dans les équations (A), et, désignant alors par P_i la valeur correspondante de $\Pi_i(x)$ qui sera un nombre entier dans l'hypothèse admise à l'égard de $a, b, \ldots, h$, elles deviendront

$$\begin{aligned} e^a P - P_1 &= \varepsilon_1, \\ e^b P - P_2 &= \varepsilon_2, \\ &\ldots\ldots, \\ e^h P - P_n &= \varepsilon_n, \end{aligned}$$

et la relation supposée

$$N + e^a N_1 + e^b N_2 + \ldots + e^h N_n = 0$$

donnera facilement celle-ci,

$$NP + N_1 P_1 + \ldots + N_n P_n = -(N_1\varepsilon_1 + N_2\varepsilon_2 + \ldots + N_n\varepsilon_n),$$

dont le premier membre est essentiellement entier, le second, d'après ce qui a été établi relativement à $\varepsilon_1, \varepsilon_2, \ldots$ pouvant, lorsque μ augmente, devenir plus petit que toute grandeur donnée. On aura donc nécessairement, à partir d'une certaine valeur de μ et pour toutes les valeurs plus grandes,

$$NP + N_1 P_1 + \ldots + N_n P_n = 0.$$

Supposons, en conséquence, que, μ devenant successivement $\mu+1, \mu+2, \ldots, \mu+n$, P_i se change en $P'_i, P''_i, \ldots, P_i^{(n)}$; on aura de même

$$\begin{aligned} NP' &+ N_1 P'_n + \ldots + N_n P'_n = 0, \\ NP'' &+ N_1 P''_1 + \ldots + N_n P''_n = 0, \\ &\ldots\ldots\ldots\ldots, \\ NP^{(n)} &+ N_1 P_1^{(n)} + \ldots + N_n P_n^{(n)} = 0. \end{aligned}$$

Ces relations entraînent la condition suivante :

$$\begin{vmatrix} P & P_1 & \ldots & P_n \\ P' & P'_1 & \ldots & P'_n \\ P'' & P''_1 & \ldots & P''_n \\ \vdots & \vdots & \vdots & \vdots \\ P^{(n)} & P_1^{(n)} & \ldots & P_n^{(n)} \end{vmatrix} = 0.$$

En prouvant donc que ce déterminant est différent de zéro, on démontrera l'impossibilité de la relation admise

$$N + e^a N_1 + e^b N_2 + \ldots + e^h N_n = 0.$$

J'observerai dans ce but qu'on peut substituer aux termes d'une même ligne horizontale des combinaisons linéaires semblables pour toutes ces lignes, et que j'indiquerai en considérant, par exemple, la première. Elle consiste à remplacer respectivement P, P_1, P_2, ..., P_{n-1}, P_n par $P - e^{-a}P_1$, $e^{-a}P_1 - e^{-b}P_2$, ..., $e^{-g}P_{n-1} - e^{-h}P_n$, $e^{-h}P_n$; il est alors aisé de voir que, si l'on multiplie toutes ces quantités par $1.2.3\ldots\mu$, elles deviennent précisément les intégrales

$$\int_0^a e^{-z} f^\mu(z)\,dz, \quad \int_a^b e^{-z} f^\mu(z)\,dz, \quad \ldots,$$
$$\int_g^h e^{-z} f^\mu(z)\,dz, \quad \int_h^\infty e^{-z} f^\mu(z)\,dz.$$

Maintenant les autres lignes se déduisent de celle-là par le changement de μ en $\mu+1$, $\mu+2$, ..., $\mu+n$, et le déterminant transformé sur lequel nous allons raisonner est le suivant :

$$\Delta = \begin{vmatrix} \int_0^a e^{-z} f^\mu(z)\,dz, & \int_a^b e^{-z} f^\mu(z)\,dz, & \ldots, & \int_h^\infty e^{-z} f^\mu(z)\,dz \\ \int_0^a e^{-z} f^{\mu+1}(z)\,dz, & \int_a^b e^{-z} f^{\mu+1}(z)\,dz, & \ldots, & \int_h^\infty e^{-z} f^{\mu+1}(z)\,dz \\ \ldots\ldots\ldots, & \ldots\ldots\ldots, & \ldots, & \ldots\ldots\ldots \\ \int_0^a e^{-z} f^{\mu+n}(z)\,dz, & \int_a^b e^{-z} f^{\mu+n}(z)\,dz, & \ldots, & \int_h^\infty e^{-z} f^{\mu+n}(z)\,dz \end{vmatrix}.$$

V. Nous devons supposer, comme on l'a vu précédemment, que μ est un grand nombre; c'est ce qui conduit à déterminer, au moyen de la belle méthode donnée par Laplace (*De l'intégration par approximation des différentielles qui renferment des facteurs élevés à de grandes puissances* dans la *Théorie analytique des Probabilités,* p. 88), l'expression asymptotique des intégrales

$$\int_0^a e^{-z} f^\mu(z)\,dz, \quad \int_a^b e^{-z} f^\mu(z)\,dz, \quad \ldots, \quad \int_h^\infty e^{-z} f^\mu(z)\,dz,$$

afin d'en conclure pour Δ une valeur approchée, dont le rapport à la valeur exacte soit l'unité pour μ infini. Admettant, à cet effet, que les nombres entiers $a, b, \ldots, h$ soient tous positifs et rangés par ordre croissant de grandeur, de sorte que, dans chaque intégrale, la fonction $e^{-z} f^{\mu}(z)$, qui s'annule aux limites, ne présente, dans l'intervalle, qu'un seul maximum, je considérerai en premier lieu l'équation

$$\frac{f'(z)}{f(z)} = \frac{1}{\mu},$$

dont dépendent tous ces maxima. Or on sait que ses racines sont réelles et comprises, la première z_1 entre zéro et a, la seconde z_2 entre a et b, et ainsi de suite, la plus grande z_{n+1} étant supérieure à h. Envisagées comme fonctions de μ, il est aisé de voir qu'elles croissent lorsque μ augmente, et qu'en désignant par $p, q, \ldots, s$ les racines de l'équation dérivée $f'(z) = 0$, rangées par ordre croissant de grandeur, on aura, si l'on néglige $\frac{1}{\mu^2}$,

$$z_1 = p + \frac{1}{\mu}\frac{f(p)}{f''(p)}, \qquad z_2 = q + \frac{1}{\mu}\frac{f(q)}{f''(q)}, \qquad \ldots, \qquad z_n = s + \frac{1}{\mu}\frac{f(s)}{f''(s)},$$

et, en dernier lieu,

$$z_{n+1} = (n+1)\mu + \frac{a+b+\ldots+h}{n+1},$$

une approximation plus grande n'étant pas alors nécessaire. Cela posé, si l'on écrit pour un instant

$$\varphi(z) = \frac{f(z)}{\sqrt{f'^2(z) - f(z)f''(z)}},$$

les valeurs cherchées seront

$$\sqrt{\frac{2\pi}{\mu}}\, e^{-z_1} f^{\mu}(z_1)\,\varphi(z_1), \quad \sqrt{\frac{2\pi}{\mu}}\, e^{-z_2} f^{\mu}(z_2)\,\varphi(z_2), \quad \ldots,$$
$$\sqrt{\frac{2\pi}{\mu}}\, e^{-z_{n+1}} f^{\mu}(z_{n+1})\,\varphi(z_{n+1});$$

mais ces quantités se simplifient, comme on va le voir.

Considérant la première pour fixer les idées, j'observe que nous avons

$$z_1 = p + \frac{1}{\mu}\frac{f(p)}{f''(p)},$$

où p satisfait à la condition $f'(p)=0$; on en conclut $f(x_1)=f(p)$, en négligeant seulement $\frac{1}{\mu^2}$. Par conséquent, si l'on pose

$$f(z_1)=f(p)\left(1+\frac{\alpha}{\mu^2}+\frac{\alpha'}{\mu^3}+\ldots\right),$$

puis d'une manière analogue

$$\varphi(z_1)=\varphi(p)\left(1+\frac{\beta}{\mu}+\frac{\beta'}{\mu^2}+\ldots\right),$$

on aura d'abord

$$f^{\mu}(z_1)=f^{\mu}(p)\left(1+\frac{\alpha}{\mu}+\ldots\right),$$

et l'on en tire aisément

$$f^{\mu}(z_1)\varphi(z_1)=f^{\mu}(p)\varphi(p)\left(1+\frac{\gamma}{\mu}+\frac{\gamma'}{\mu^2}+\ldots\right).$$

Ainsi, en négligeant seulement des quantités infiniment petites par rapport au terme conservé, nous pouvons écrire

$$\int_0^a e^{-z}f^{\mu}(z)\,dz=\sqrt{\frac{2\pi}{\mu}}\,e^{-p}f^{\mu}(p)\varphi(p),$$

et l'on aura de même

$$\int_a^b e^{-z}f^{\mu}(z)\,dz=\sqrt{\frac{2\pi}{\mu}}\,e^{-q}f^{\mu}(q)\varphi(q),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\int_g^h e^{-z}f^{\mu}(z)\,dz=\sqrt{\frac{2\pi}{\mu}}\,e^{-s}f^{\mu}(s)\varphi(s).$$

Mais la dernière intégrale $\int_h^{\infty} e^{-z}f^{\mu}(z)\,dz$ est d'une forme analytique différente, en raison de la valeur $z_{n+1}=(n+1)\mu$ qui devient infinie avec μ. Pour y parvenir, je développerai, suivant les puissances descendantes de la variable, l'expression

$$\log[e^{-z}f^{\mu}(z)\varphi(z)],$$

en négligeant les termes en $\frac{1}{z}$, $\frac{1}{z^2}$, $\ldots$, ce qui permet d'écrire

$$\log f(z)=(n+1)\log z,\qquad \log\varphi z=\log\frac{z^{n+1}}{\sqrt{(n+1)z^{2n}+\ldots}}=\log\frac{z}{\sqrt{n+1}},$$

et, par suite,

$$\log[e^{-z}f^{\mu}(z)\varphi(z)] = (n\mu+\mu+1)\log z - z - \tfrac{1}{2}\log(n+1).$$

Après avoir substitué la valeur de z_{n+1}, une réduction facile nous donnera, en faisant, pour abréger,

$$\theta(\mu) = (n\mu+\mu+1)\log(n+1)\mu - (n+1)\mu - \tfrac{1}{2}\log(n+1),$$

cette expression semblable à celle des intégrales eulériennes de première espèce

$$\int_h^{\infty} e^{-z}f^{\mu}(z)\,dz = \sqrt{\frac{2\pi}{\mu}}\,e^{\theta(\mu)}.$$

Maintenant on va voir comment les résultats ainsi obtenus conduisent aisément à la valeur du déterminant Δ.

VI. J'effectuerai d'abord une première simplification en supprimant, dans les termes de la ligne horizontale de rang i, le facteur $\sqrt{\dfrac{2\pi}{\mu+i}}$, puis une seconde, en divisant tous les termes d'une même colonne verticale par le premier d'entre eux. Le nouveau déterminant ainsi obtenu, si l'on fait, pour abréger,

$$P=f(p),\qquad Q=f(q),\qquad \ldots,\qquad S=f(s),$$

sera évidemment

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ P & Q & S & e^{\theta(\mu+1)-\theta(\mu)} \\ P^2 & Q^2 & S^2 & e^{\theta(\mu+2)-\theta(\mu)} \\ \vdots & \vdots & \vdots & \vdots \\ P^n & Q^n & S^n & e^{\theta(\mu+n)-\theta(\mu)} \end{vmatrix}.$$

Or, on voit que μ ne figure plus que dans une colonne, dont les termes croissent d'une telle manière que le dernier $e^{\theta(\mu+n)-\theta(\mu)}$ est infiniment plus grand que tous les autres. Nous avons, en effet,

$$\begin{aligned}\theta(\mu+i) &= \theta(\mu)+i\theta'(\mu)+\frac{i^2}{2}\theta''(\mu)+\ldots\\ &= \theta(\mu)+i\left[\frac{1}{\mu}+(n+1)\log(n+1)\mu\right]\\ &\quad+\frac{i^2}{2}\left(-\frac{1}{\mu^2}+\frac{n+1}{\mu}\right)+\ldots,\end{aligned}$$

et, par conséquent, si l'on néglige $\frac{1}{\mu}, \frac{1}{\mu^2}, \cdots,$

$$\theta(\mu+i)-\theta(\mu)=i(n+1)\log(n+1)\mu,$$

d'où

$$e^{\theta(\mu+i)-\theta(\mu)}=[(n+1)\mu]^{i(n+1)}.$$

En ne conservant donc dans le déterminant que le terme en μ de l'ordre le plus élevé, il se réduit simplement à cette expression

$$[(n+1)\mu]^{n(n+1)}\begin{vmatrix} 1 & 1 & 1 \\ P & Q & S \\ P^2 & Q^2 & S^2 \\ \vdots & \vdots & \vdots \\ P^{n-1} & Q^{n-1} & S^{n-1} \end{vmatrix}.$$

Il en résulte qu'on ne peut, en général, admettre que le déterminant proposé Δ s'annule, car les quantités $P=f(p)$, $Q=f(q)$, ..., fonctions entières semblables des racines $p, q, \ldots$ de l'équation dérivée $f'(x)=0$, seront, comme ces racines, différentes entre elles. C'est ce qu'il fallait établir pour démontrer l'impossibilité de toute relation de la forme

$$N+e^aN_1+e^bN_2+\ldots+e^hN_n=0,$$

et arriver ainsi à prouver que *le nombre e ne peut être racine d'une équation algébrique de degré quelconque à coefficients entiers.*

Mais une autre voie conduira à une seconde démonstration plus rigoureuse; on peut, en effet, comme on va le voir, étendre aux fractions rationnelles

$$\frac{\Phi_1(x)}{\Phi(x)},\quad \frac{\Phi_2(x)}{\Phi(x)},\quad \ldots,\quad \frac{\Phi_n(x)}{\Phi(x)}$$

le mode de formation des réduites donné par la théorie des fractions continues, et par là mettre plus complètement en évidence le caractère arithmétique d'une irrationnelle non algébrique. Dans cet ordre d'idées, M. Liouville a déjà obtenu un théorème remarquable qui est l'objet de son travail intitulé : *Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni* *même réductible à des irrationnelles algébriques* (¹), et je rappellerai aussi que l'illustre géomètre a démontré le premier la proposition qui est le sujet de ces recherches pour les cas de l'équation du second degré et de l'équation bicarrée [*Note sur l'irrationnalité du nombre e* (*Journal de Mathématiques*, t. V, p. 192)]. Sous le point de vue auquel je me suis placé, voici la première proposition à établir :

VII. *Soient* $F(z)$, $F_1(z)$, ..., $F_{n+1}(z)$ *les polynomes déduits de l'expression*

$$z^{\mu}(z-a)^{\mu_1}(z-b)^{\mu_2}\ldots(z-h)^{\mu_n},$$

lorsqu'on attribue aux exposants μ, μ_1, ..., μ_n, $n+2$ *systèmes différents de valeurs entières et positives. En représentant, en général, par* $\frac{\Phi_i^k(x)}{\Phi^k(x)}$ *les fractions convergentes vers les exponentielles, qui correspondent à l'un quelconque d'entre eux* $F_k(z)$, *on pourra toujours déterminer les quantités* A, B, C, ..., L *par les équations suivantes :*

$$\begin{gathered}
A\Phi(x)+B\Phi^1(x)+C\Phi^2(x)+\ldots+L\Phi^{n+1}(x)=0,\\
A\Phi_1(x)+B\Phi_1^1(x)+C\Phi_1^2(x)+\ldots+L\Phi_1^{n+1}(x)=0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
A\Phi_n(x)+B\Phi_n^1(x)+C\Phi_n^2(x)+\ldots+L\Phi_n^{n+1}(x)=0.
\end{gathered}$$

Mais, au lieu de conclure de telles relations des polynomes $\Phi_i^k(x)$ supposés connus, notre objet est de les obtenir directement et *a priori;* je vais établir pour cela qu'il existe, entre les intégrales indéfinies

$$\int e^{-zx}F(z)\,dz,\quad \int e^{-zx}F_1(z)\,dz,\quad \ldots,\quad \int e^{-zx}F_{n+1}(z)\,dz,$$

une équation de la forme

$$\begin{aligned}
\mathcal{A}\int e^{-zx}F(z)\,dz+\mathcal{B}\int e^{-zx}F_1(z)\,dz+\ldots\\
+\mathcal{L}\int e^{-zx}F_{n+1}(z)\,dz &= e^{-zx}\Theta(z),
\end{aligned}$$

(¹) *Comptes rendus*, t. XVIII, p. 883 et 910.

les coefficients $\mathcal{A}$, $\mathcal{B}$, ..., $\mathcal{L}$ étant indépendants de z, et $\Theta(z)$ un polynome entier divisible par $f(z)$. Si l'on fait, en effet,

$$\mathcal{F}_k(z) = \frac{F_k(z)}{x} + \frac{F'_k(z)}{x^2} + \frac{F''_k(z)}{x^3} + \ldots,$$

on aura

$$\mathcal{A}\int e^{-zx}F(z)\,dz + \mathcal{B}\int e^{-zx}F_1(z)\,dz + \ldots + \mathcal{L}\int e^{-zx}F_{n+1}(z)\,dz$$
$$= -e^{-zx}[\mathcal{A}\mathcal{F}(z) + \mathcal{B}\mathcal{F}_1(z) + \ldots + \mathcal{L}\mathcal{F}_{n+1}(z)],$$

et il est clair que les rapports $\frac{\mathcal{B}}{\mathcal{A}}$, $\frac{\mathcal{C}}{\mathcal{A}}$, ..., $\frac{\mathcal{L}}{\mathcal{A}}$ pourront être déterminés, et d'une seule manière, par la condition supposée que le polynome

$$\Theta(z) = -[\mathcal{A}\mathcal{F}(z) + \mathcal{B}\mathcal{F}_1(z) + \ldots + \mathcal{L}\mathcal{F}_{n+1}(z)]$$

contienne comme facteur

$$f(z) = z(z-a)(z-b)\ldots(z-h).$$

Nous conclurons de là en prenant les intégrales entre les limite $z = 0$ et $z = a$, par exemple,

$$\mathcal{A}\int_0^a e^{-zx}F(z)\,dz + \mathcal{B}\int_0^a e^{-zx}F_1(z)\,dz + \ldots$$
$$+ \mathcal{L}\int_0^a e^{-zx}F_{n+1}(z)\,dz \quad = 0.$$

Maintenant, les relations

$$\int_0^a e^{-zx}F(z)\,dz = \frac{e^{ax}\Phi(x) - \Phi_1(x)}{e^{ax}x^{M+1}},$$
$$\int_0^a e^{-zx}F_1(z)\,dz = \frac{e^{ax}\Phi_1(x) - \Phi'_1(x)}{e^{ax}x^{M_1+1}},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

donneront, en égalant séparément à zéro le terme algébrique et le coefficient de l'exponentielle e^{ax}, si l'on fait, pour abréger,

$$A = \frac{\mathcal{A}}{x^{M+1}}, \qquad B = \frac{\mathcal{B}}{x^{M_1+1}}, \qquad \ldots, \qquad L = \frac{\mathcal{L}}{x^{M_{n+1}+1}},$$

les égalités suivantes :

$$A\Phi(x) + B\Phi^1(x) + \ldots + L\Phi^{n+1}(x) = 0,$$
$$A\Phi_1(x) + B\Phi_1^1(x) + \ldots + L\Phi_1^{n+1}(x) = 0.$$

Or, on aura de même, en prenant pour limites supérieures des intégrales $z = b, c, \ldots, h$,

$$A\Phi_2(x) + B\Phi_2^1(x) + \ldots + L\Phi_2^{n+1}(x) = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$A\Phi_n(x) + B\Phi_n^1(x) + \ldots + L\Phi_n^{n+1}(x) = 0,$$

et il est aisé de voir que les coefficients A, B, ..., L pourront être supposés des polynomes entiers en x. L'intégrale

$$\int_0^1 e^{-zx} z^m (z-1)^m \, dz,$$

qui figure dans la relation précédemment considérée (p. 154),

$$e^x \mathrm{H}(x) - \mathrm{H}_1(x) = \frac{x^{2m+1} e^x}{1.2.3\ldots m} \int_0^1 e^{-zx} z^m (z-1)^m \, dz,$$

nous servira d'abord d'exemple.

VIII. Dans ce cas facile, où l'on a simplement

$$f(z) = z(z-1),$$

je partirai, en supposant

$$\theta(z) = x f^{m+1}(z) + (m+1) f^m(z) f'(z),$$

de l'identité suivante :

$$\begin{aligned}\frac{d[e^{-zx}\theta(z)]}{dz} &= e^{-zx}[\theta'(z) - x\theta(z)] \\ &= e^{-zx}[-x^2 f^{m+1}(z) + (m+1) f^m(z) f''(z) \\ &\qquad + m(m+1) f^{m-1} f'^2(z)],\end{aligned}$$

et j'observerai que

$$f'^2(z) = 4z^2 - 4z + 1 = 4f(z) + 1, \qquad f''(z) = 2,$$

ce qui permet de l'écrire ainsi :

$$\begin{aligned}\frac{d[e^{-zx}\theta(z)]}{dx} &= e^{-zx}[-x^2 f^{m+1}(z) \\ &\quad + (2m+1)(2m+2) f^m(z) + m(m+1) f^{m-1}(z)].\end{aligned}$$

Nous aurons donc, en intégrant,

$$e^{-zx}\theta(z) = -x^2\int e^{-zx}f^{m+1}(z)\,dz + (2m+1)(2m+2)\int e^{-zx}f^m(z)\,dz + m(m+1)\int e^{-zx}f^{m-1}(z)\,dz,$$

et ensuite, si nous prenons pour limites $z=0$ et $z=1$,

$$x^2\int_0^1 e^{-zx}f^{m+1}(z)\,dz = (2m+1)(2m+2)\int_0^1 e^{-zx}f^m(z)\,dz + m(m+1)\int_0^1 e^{-zx}f^{m-1}(z)\,dz.$$

Soit maintenant

$$\varepsilon_m = \frac{x^{2m+1}e^x}{1.2\ldots m}\int_0^1 e^{-zx}z^m(z-1)^m\,dz,$$

et cette relation deviendra

$$\varepsilon_{m+1} = (4m+2)\varepsilon_m + x^2\varepsilon_{m-1}.$$

C'est le résultat auquel nous voulions parvenir; en y supposant successivement $m = 1, 2, 3, \ldots$, les équations qu'on en tire

$$\begin{aligned}
\varepsilon_2 &= 6\varepsilon_1 + x^2\varepsilon_0,\\
\varepsilon_3 &= 10\varepsilon_2 + x^2\varepsilon_1,\\
\varepsilon_4 &= 14\varepsilon_3 + x^2\varepsilon_2,\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

donnent aisément la fraction continue

$$\frac{\varepsilon_1}{\varepsilon_0} = -\cfrac{x^2}{6+\cfrac{x^2}{10+\cfrac{x^2}{14+\ldots}}},$$

et il suffit d'employer les valeurs

$$\varepsilon_0 = xe^x\int_0^1 e^{-zx}\,dz = e^x - 1,$$

$$\varepsilon_1 = x^3e^x\int_0^1 e^{-zx}z(z-1)\,dz = e^x(2-x) - 2 - x,$$

d'où l'on conclut

$$\frac{\varepsilon_1}{\varepsilon_0} = 2 - \frac{e^x + 1}{e^x - 1}x,$$

pour retrouver, sauf le changement de x en $\frac{x}{2}$, le résultat de Lambert (¹)

$$\frac{e^x - 1}{e^x + 1} = \cfrac{x}{2 + \cfrac{x^2}{6 + \cfrac{x^2}{10 + \cfrac{x^2}{14 + \dots}}}}$$

En abordant maintenant le cas général et me proposant d'obtenir, à l'égard des intégrales définies

$$\int_0^a e^{-z} f^m(z)\,dz, \quad \int_0^b e^{-z} f^m(z)\,dz, \quad \ldots, \quad \int_0^h e^{-z} f^m(z)\,dz,$$

un algorithme qui permette de les calculer de proche en proche, pour toutes les valeurs du nombre entier m, j'introduirai, afin de rendre les calculs plus symétriques, les modifications suivantes dans les notations précédemment admises. Je ferai

$$f(z) = (z - z_0)(z - z_1)\ldots(z - z_n),$$

au lieu de

$$f(z) = z(z - a)(z - b)\ldots(z - h),$$

de manière à considérer le polynome le plus général de degré $n + 1$; désignant ensuite par Z l'une quelconque des quantités $z_1, z_2, \ldots, z_n$, je raisonnerai sur l'intégrale

$$\int_{z_0}^{Z} e^{-z} f^m(z)\,dz,$$

qui donnera évidemment toutes celles que nous avons en vue, en faisant $z_0 = 0$. Cela étant, voici la remarque qui m'a ouvert la voie et conduit à la méthode que je vais exposer.

(¹) Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques (*Mémoires de l'Académie des Sciences de Berlin*, année 1761, p. 265). *Voir* aussi la Note IV des *Éléments de Géométrie*, de Legendre, p. 288.

IX. En intégrant les deux membres de la relation identique

$$\frac{d[e^{-z}f^m(z)]}{dz} = e^{-z}[mf^{m-1}(z)f'(z) - f^m(z)],$$

on obtient

$$e^{-z}f^m(z) = m\int e^{-z}f^{m-1}(z)f'(z)\,dz - \int e^{-z}f^m(z)\,dz,$$

et, par conséquent,

$$\int_{z_0}^{Z} e^{-z}f^m(z)\,dz = m\int_{z_0}^{Z} e^{-z}f^{m-1}(z)f'(z)\,dz,$$

ou encore

$$\int_{z_0}^{Z} e^{-z}f^m(z)\,dz = m\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_0}\,dz$$
$$+ m\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_1}\,dz + \ldots + m\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_n}\,dz,$$

d'après la formule

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_0} + \frac{1}{z-z_1} + \ldots + \frac{1}{z-z_n}.$$

Or ce sont ces nouvelles intégrales

$$\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_0}\,dz, \quad \int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_1}\,dz, \quad \ldots, \quad \int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_n}\,dz$$

qui donnent lieu à un système de relations récurrentes de la forme

$$\int_{z_0}^{Z} \frac{e^{-z}f^{m+1}(z)}{z-z_0}\,dz = (00)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_0}\,dz$$
$$+ (01)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_1}\,dz + \ldots + (0n)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_n}\,dz,$$

$$\int_{z_0}^{Z} \frac{e^{-z}f^{m+1}(z)}{z-z_1}\,dz = (10)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_0}\,dz$$
$$+ (11)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_1}\,dz + \ldots + (1n)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_n}\,dz,$$

. . . . ,

$$\int_{z_0}^{Z} \frac{e^{-z}f^{m+1}(z)}{z-z_n}\,dz = (n0)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_0}\,dz$$
$$+ (n1)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_1}\,dz + \ldots + (nn)\int_{z_0}^{Z} \frac{e^{-z}f^m(z)}{z-z_n}\,dz,$$

où les coefficients (ik), ainsi que leur déterminant, s'obtiennent d'une manière facile, comme nous verrons.

C'est donc en opérant sur les éléments au nombre de $n+1$, dans lesquels a été décomposée l'intégrale $\int_{z_0}^{Z} e^{-z} f^m(z)\,dz$, que nous parvenons à sa détermination, au lieu de chercher, comme une analogie naturelle aurait paru l'indiquer, une expression linéaire de $\int_{z_0}^{Z} e^{-z} f^{m+n+1}(z)\,dz$, au moyen de

$$\int_{z_0}^{Z} e^{-z} f^m(z)\,dz, \quad \int_{z_0}^{Z} e^{-z} f^{m+1}(z)\,dz, \quad \ldots, \quad \int_{z_0}^{Z} e^{-z} f^{m+n}(z)\,dz.$$

Mais soit, d'une manière plus générale, pour des valeurs entières quelconques des exposants,

$$F(z) = (z-z_0)^{\mu_0}(z-z_1)^{\mu_1}\ldots(z-z_n)^{\mu_n};$$

en intégrant les deux membres de l'identité

$$\frac{d[e^{-z}F(z)]}{dz} = e^{-z}[F'(z) - F(z)],$$

on aura

$$e^{-z}F(z) = \int e^{-z}F'(z)\,dz - \int e^{-z}F(z)\,dz,$$

d'où

$$\int_{z_0}^{Z} e^{-z}F(z)\,dz = \int_{z_0}^{Z} e^{-z}F'(z)\,dz.$$

Maintenant la formule

$$\frac{F'(z)}{F(z)} = \frac{\mu_0}{z-z_0} + \frac{\mu_1}{z-z_1} + \ldots + \frac{\mu_n}{z-z_n}$$

donne la décomposition suivante,

$$\int_{z_0}^{Z} e^{-z}F(z)\,dz = \mu_0 \int_{z_0}^{Z} \frac{e^{-z}F(z)\,dz}{z-z_0} + \mu_1 \int_{z_0}^{Z} \frac{e^{-z}F(z)\,dz}{z-z_1} + \ldots + \mu_n \int_{z_0}^{Z} \frac{e^{-z}F(z)\,dz}{z-z_0},$$

qui conduira pareillement au calcul des divers termes de la suite

$$\int_{z_0}^{Z} e^{-z}\,\mathrm{F}(z)\,dz, \quad \int_{z_0}^{Z} e^{-z}\,\mathrm{F}(z)\,f(z)\,dz, \quad \ldots, \quad \int_{z_0}^{Z} e^{-z}\,\mathrm{F}(z)\,f^k(z)\,dz;$$

effectivement, les éléments de décomposition de l'un quelconque d'entre eux s'expriment en fonction linéaire des quantités semblables qui se rapportent au terme précédent, ainsi qu'on va le montrer.

X. J'établirai pour cela qu'on peut toujours déterminer deux polynomes entiers de degré n, $\Theta(z)$ et $\Theta_1(z)$, tels qu'on ait, en désignant par ζ l'une des racines $z_0, z_1, \ldots, z_n$, la relation suivante :

$$\int \frac{e^{-z}\,\mathrm{F}(z)\,f(z)}{z-\zeta}\,dz = \int \frac{e^{-z}\,\mathrm{F}(z)\,\Theta_1(z)}{f(z)}\,dz - e^{-z}\,\mathrm{F}(z)\,\Theta(z).$$

En effet, si, après avoir différentié les deux membres, nous multiplions par le facteur $\dfrac{f(z)}{\mathrm{F}(z)}$, il vient

$$\frac{f(z)}{z-\zeta}f(z) = \Theta_1(z) + \left[1 - \frac{\mathrm{F}'(z)}{\mathrm{F}(z)}\right] f(z)\,\Theta(z) - f(z)\,\Theta'(z).$$

Or, $f(z)$ étant divisible par $z-\zeta$, le premier membre de cette égalité est un polynome entier de degré $2n+1$; le second est du même degré, d'après la supposition admise à l'égard de $\Theta(z)$ et $\Theta_1(z)$, et, puisque chacun de ces polynomes renferme ainsi $n+1$ coefficients indéterminés, on a bien le nombre nécessaire égal à $2n+2$ de constantes arbitraires pour effectuer l'identification. Ce point établi, j'observe qu'en supposant $z=z_i$ la fraction rationnelle $\dfrac{\mathrm{F}'(z)\,f(z)}{\mathrm{F}(z)}$ a pour valeur $\mu_i f'(z_i)$; on a, par conséquent, ces conditions

$$\begin{aligned}
\Theta_1(z_0) &= \mu_0\, f'(z_0)\,\Theta(z_0),\\
\Theta_1(z_1) &= \mu_1\, f'(z_1)\,\Theta(z_1),\\
&\ldots\ldots\ldots\ldots\ldots\ldots,\\
\Theta_1(z_n) &= \mu_n\, f'(z_n)\,\Theta(z_n),
\end{aligned}$$

qui permettent, par la formule d'interpolation, de calculer immédiatement $\Theta_1(z)$, lorsque $\Theta(z)$ sera connu. Nous avons de cette manière, en effet, l'expression suivante,

$$\frac{\Theta_1(z)}{f(z)} = \frac{\mu_0\,\Theta(z_0)}{z-z_0} + \frac{\mu_1\,\Theta(z_1)}{z-z_1} + \ldots + \frac{\mu_n\,\Theta(z_n)}{z-z_n},$$

dont nous ferons bientôt usage. Pour obtenir maintenant $\Theta(z)$, je reprends la relation proposée, en divisant les deux membres par $f(z)$, ce qui donne

$$\frac{f(z)}{z-\zeta} = \frac{\Theta_1(z)}{f(z)} + \left[1 - \frac{F'(z)}{F(z)}\right]\Theta(z) - \Theta'(z),$$

et je remarque que, la fraction $\frac{\Theta_1(z)}{f(z)}$ n'ayant pas de partie entière, on est amené à cette conséquence, que le polynome cherché doit être tel que la partie entière de l'expression

$$\left[1 - \frac{F'(z)}{F(z)}\right]\Theta(z) - \Theta'(z)$$

soit égale au quotient $\frac{f(z)}{z-\zeta}$. C'est ce qui conduit aisément à la détermination de $\Theta(z)$. Soit d'abord, à cet effet,

$$f(z) = z^{n+1} + p_1 z^n + p_2 z^{n-1} + \ldots + p_{n+1},$$

ce qui donnera

$$\frac{f(z)}{z-\zeta} = z^n + \zeta \left|\begin{array}{l} z^{n-1} + \zeta^2 \\ \quad + p_1\zeta \\ \quad + p_2 \end{array}\right| \begin{array}{l} z^{n-2} + \ldots + \zeta^n \\ \qquad + p_1\zeta^{n-1} \\ \qquad + p_2\zeta^{n-2} \\ \qquad \ldots\ldots \\ \qquad + p_n, \end{array}$$
$$\qquad\qquad + p_1$$

ou plutôt

$$\frac{f(z)}{z-\zeta} = z^n + \zeta_1 z^{n-1} + \zeta_2 z^{n-2} + \ldots + \zeta_n,$$

en écrivant, pour abréger,

$$\zeta_i = \zeta^i + p_1\zeta^{i-1} + p_2\zeta^{i-2} + \ldots + p_i.$$

Soit encore

$$\Theta(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \ldots + \alpha_n,$$

et développons la fonction $\frac{F'(z)}{F(z)}$ suivant les puissances descendantes de la variable, afin d'obtenir la partie entière du produit $\frac{F'(z)}{F(z)}\Theta(z)$. Il viendra ainsi, en posant $s_i = \mu_0 z_0^i + \mu_1 z_1^i + \mu_2 z_2^i + \ldots + \mu_n z_n^i$,

$$\frac{F'(z)}{F(z)} = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \ldots,$$

et, par conséquent,

$$\frac{F'(z)}{F(z)}\Theta(z) = \alpha_0 s_0 z^{n-1} + \left.\begin{array}{l}\alpha_1 s_0\\ +\alpha_0 s_1\end{array}\right| z^{n-2} + \left.\begin{array}{l}\alpha_2 s_0\\ +\alpha_1 s_1\\ +\alpha_0 s_2\end{array}\right| z^{n-3} + \ldots.$$

Les équations en $\alpha_0, \alpha_1, \alpha_2, \ldots$, auxquelles nous sommes amené par l'identification, sont donc

$$\begin{aligned}
1 &= \alpha_0,\\
\zeta_1 &= \alpha_1 - \alpha_0(s_0 + n),\\
\zeta_2 &= \alpha_2 - \alpha_1(s_0 + n - 1) - \alpha_0 s_1,\\
\zeta_3 &= \alpha_3 - \alpha_2(s_0 + n - 2) - \alpha_1 s_1 - \alpha_0 s_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Elles donnent

$$\begin{aligned}
\alpha_0 &= 1,\\
\alpha_1 &= \zeta_1 + s_0 + n,\\
\alpha_2 &= \zeta_2 + (s_0 + n - 1)\zeta_1 + (s_0 + n)(s_0 + n - 1) + s_1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

et montrent que $\alpha_0, \alpha_1, \alpha_2, \ldots$ sont des polynomes en ζ ayant pour coefficients des fonctions entières et à coefficients entiers de $s_0, s_1, s_2, \ldots$ et par suite des racines $z_0, z_1, \ldots, z_n$. On voit de plus que α_i est un polynome de degré i dans lequel le coefficient de ζ^i est égal à l'unité; ainsi, en posant pour plus de clarté

$$\alpha_i = \theta_i(\zeta),$$

et écrivant désormais $\Theta(z, \zeta)$ au lieu de $\Theta(z)$, afin de mettre ζ en évidence, nous aurons

$$\Theta(z, \zeta) = z^n + \theta_1(\zeta) z^{n-2} + \theta_2(\zeta) z^{n-3} + \ldots + \theta_n(\zeta).$$

De là résulte, pour le polynome $\Theta_1(z)$, la formule

$$\frac{\Theta_1(z)}{f(z)} = \frac{\mu_0\,\Theta(z_0, \zeta)}{z - z_0} + \frac{\mu_1\,\Theta(z_1, \zeta)}{z - z_1} + \ldots + \frac{\mu_n\,\Theta(z_n, \zeta)}{z - z_n},$$

et l'on en tire immédiatement le résultat que nous nous sommes proposé d'obtenir. Il suffit, en effet, de prendre les intégrales entre les limites z_0 et Z dans la relation

$$\int \frac{e^{-z}\mathrm{F}(z)f(z)}{z-\zeta}\,dz = \int \frac{e^{-z}\mathrm{F}(z)\Theta_1(z)}{f(z)}\,dz - e^{-z}\mathrm{F}(z)\,\Theta(z),$$

ce qui donne

$$\begin{aligned}\int_{z_0}^{Z} \frac{e^{-z}\mathrm{F}(z)f(z)}{z-\zeta}\,dz = \int_{z_0}^{Z} \frac{e^{-z}\mathrm{F}(z)\Theta_1(z)}{f(z)}\,dz \\ = \mu_0\,\Theta(z_0,\zeta)\int_{z_0}^{Z}\frac{e^{-z}\mathrm{F}(z)}{z-z_0}\,dz, \\ + \mu_1\,\Theta(z_1,\zeta)\int_{z_0}^{Z}\frac{e^{-z}\mathrm{F}(z)}{z-z_1}\,dz, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ + \mu_n\,\Theta(z_n,\zeta)\int_{z_0}^{Z}\frac{e^{-z}\mathrm{F}(z)}{z-z_n}\,dz.\end{aligned}$$

C'est surtout dans le cas où l'on suppose

$$\mu_0 = \mu_1 = \ldots = \mu_n = m,$$

que nous ferons usage de cette équation; si l'on fait alors

$$m\,\Theta(z_i, z_k) = (ik),$$

et qu'on prenne ζ successivement égal à $z_0, z_1, \ldots, z_n$, on en conclut, comme on voit, les relations précédemment énoncées, qui résultent de celle-ci,

$$\begin{aligned}\int_{z_0}^{Z}\frac{e^{-z}f^{m+1}(z)}{z-z_i}\,dz = (i0)\int_{z_0}^{Z}\frac{e^{-z}f^{m}(z)}{z-z_0}\,dz \\ +(i1)\int_{z_0}^{Z}\frac{e^{-z}f^{m}(z)}{z-z_1}\,dz + \ldots + (in)\int_{z_0}^{Z}\frac{e^{-z}f^{m}(z)}{z-z_n}\,dz,\end{aligned}$$

pour $i = 0, 1, 2, \ldots, n$. Je resterai encore cependant dans le cas général pour établir la proposition suivante :

X. *Soient* Δ *et* δ *les déterminants*

$$\begin{vmatrix} \Theta(z_0, z_0) & \Theta(z_1, z_0) & \ldots & \Theta(z_n, z_0) \\ \Theta(z_0, z_1) & \Theta(z_1, z_1) & \ldots & \Theta(z_n, z_1) \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ \Theta(z_0, z_n) & \Theta(z_1, z_n) & \ldots & \Theta(z_n, z_n) \end{vmatrix}$$

et

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_n \\ z_0^2 & z_1^2 & \dots & z_n^2 \\ \dots & \dots & \dots & \dots \\ z_0^n & z_1^n & \dots & z_n^n \end{vmatrix};$$

je dis qu'on a

$$\Delta = \delta^2.$$

Effectivement, l'expression de $\Theta(z, \zeta)$ sous la forme

$$\Theta(z, \zeta) = z^n + \theta_1(\zeta) z^{n-1} + \theta_2(\zeta) z^{n-2} + \ldots + \theta_n(\zeta)$$

montre que Δ est le produit des deux déterminants

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_n \\ z_0^2 & z_1^2 & \dots & z_n^2 \\ \dots & \dots & \dots & \dots \\ z_0^n & z_1^n & \dots & z_n^n \end{vmatrix}$$

et

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \theta_1(z_0) & \theta_1(z_1) & \dots & \theta_1(z_n) \\ \theta_2(z_0) & \theta_2(z_1) & \dots & \theta_2(z_n) \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ \theta_n(z_0) & \theta_n(z_1) & \dots & \theta_n(z_n) \end{vmatrix}.$$

Mais $\theta_i(\zeta)$ étant un polynome en ζ du degré i seulement, de sorte qu'on peut faire

$$\theta_i(\zeta) = \zeta^i + r\zeta^{i-1} + s\zeta^{i-2} + \ldots,$$

cette seconde quantité, d'après les théorèmes connus, se réduit simplement à la première, et l'on a bien, comme nous voulions l'établir,

$$\Delta = \delta^2.$$

Cela posé, soient

$$\varepsilon_m = \frac{1}{1.2\ldots m} \int_{z_0}^{Z} e^{-z} f^m(z)\, dz,$$

$$\varepsilon_m^i = \frac{1}{1.2\ldots m-1} \int_{z_0}^{Z} \frac{e^{-z} f^m(z)}{z - z_i}\, dz;$$

la relation établie page 167

$$\int_{z_0}^{Z} e^{-z} f^m(z)\,dz = m\int_{z_0}^{Z} \frac{e^{-z} f^m(z)}{z-z_0}\,dz$$
$$+ m\int_{z_0}^{Z} \frac{e^{-z} f^m(z)}{z-z_1}\,dz + \ldots + m\int_{z_0}^{Z} \frac{e^{-z} f^m(z)}{z-z_n}\,dz$$

deviendra plus simplement

$$\varepsilon_m = \varepsilon_m^0 + \varepsilon_m^1 + \ldots + \varepsilon_m^n;$$

et celle-ci,

$$\int_{z_0}^{Z} \frac{e^{-z} f^{m+1}(z)}{z-\zeta}\,dz = m\,\Theta(z_0,\zeta)\int_{z_0}^{Z} \frac{e^{-z} f^m(z)}{z-z_0}\,dz$$
$$+ m\,\Theta(z_1,\zeta)\int_{z_0}^{Z} \frac{e^{-z} f^m(z)}{z-z_1}\,dz + \ldots$$
$$+ m\,\Theta(z_n,\zeta)\int_{z_0}^{Z} \frac{e^{-z} f^m(z)}{z-z_n}\,dz,$$

en supposant successivement $\zeta = z_0, z_1, \ldots, z_n$, nous donnera la substitution suivante, que je désignerai par S_m, à savoir

$$\begin{aligned}
\varepsilon_{m+1}^0 &= \Theta(z_0, z_0)\varepsilon_m^0 + \Theta(z_1, z_0)\varepsilon_m^1 + \ldots + \Theta(z_n, z_0)\varepsilon_m^n,\\
\varepsilon_{m+1}^1 &= \Theta(z_0, z_1)\varepsilon_m^0 + \Theta(z_1, z_1)\varepsilon_m^1 + \ldots + \Theta(z_n, z_1)\varepsilon_m^n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\varepsilon_{m+1}^n &= \Theta(z_0, z_n)\varepsilon_m^0 + \Theta(z_1, z_n)\varepsilon_m^1 + \ldots + \Theta(z_n, z_n)\varepsilon_m^n.
\end{aligned}$$

Si l'on compose maintenant de proche $S_1, S_2, \ldots, S_{m-1}$, on en déduira les expressions de $\varepsilon_m^0, \varepsilon_m^1, \ldots, \varepsilon_m^n$ en $\varepsilon_1^0, \varepsilon_1^1, \ldots, \varepsilon_1^n$, que je représenterai ainsi :

$$\begin{aligned}
\varepsilon_m^0 &= A_0\varepsilon_1^0 + A_1\varepsilon_1^1 + \ldots + A_n\varepsilon_1^n,\\
\varepsilon_m^1 &= B_0\varepsilon_1^0 + B_1\varepsilon_1^1 + \ldots + B_n\varepsilon_1^n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\varepsilon_m^n &= L_0\varepsilon_1^0 + L_1\varepsilon_1^1 + \ldots + L_n\varepsilon_1^n,
\end{aligned}$$

et le déterminant de cette nouvelle substitution, étant égal au produit des déterminants des substitutions composantes, sera $\delta^{2(m-1)}$. Il nous reste encore à remplacer $\varepsilon_1^0, \varepsilon_1^1, \ldots, \varepsilon_1^n$ par leurs valeurs pour avoir les expressions des quantités ε_m^i sous la forme appropriée à notre objet. Ces valeurs s'obtiennent facilement, comme on va voir.

XII. J'applique à cet effet la formule générale

$$\int e^{-z} F(z)\,dz = -e^{-z} \mathcal{F}(z),$$

en supposant

$$F(z) = \frac{f(z)}{z-\zeta},$$

c'est-à-dire

$$F(z) = z^n + \begin{array}{c|}\zeta \\ + p_1\end{array}\; z^{n-1} + \begin{array}{c|}\zeta^2 \\ + p_1\zeta \\ + p_2\end{array}\; z^{n-2} + \ldots.$$

Il est aisé de voir alors que $\mathcal{F}(z)$ devient une expression entière en z et ζ, entièrement semblable à $\Theta(z, \zeta)$, de sorte que, si on la désigne par $\Phi(z, \zeta)$, on a

$$\Phi(z, \zeta) = z^n + \varphi_1(\zeta) z^{n-1} + \varphi_2(\zeta) z^{n-2} + \ldots + \varphi_n(\zeta),$$

$\varphi_i(\zeta)$ étant un polynome en ζ de degré i, dans lequel le coefficient de ζ^i est l'unité. Ainsi l'on obtient, en particulier,

$$\begin{aligned}
&\varphi_1(\zeta) = \zeta + p_1 + n,\\
&\varphi_2(\zeta) = \zeta^2 + (p_1 + n - 1)\zeta + p_2 + (n-1)p_1 + n(n-1),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

et l'analogie de forme avec $\Theta(z, \zeta)$ montre que le déterminant

$$\begin{vmatrix}
\Phi(z_0, z_0) & \Phi(z_1, z_0) & \ldots & \Phi(z_n, z_0)\\
\Phi(z_0, z_1) & \Phi(z_1, z_1) & \ldots & \Phi(z_n, z_1)\\
\ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots\\
\Phi(z_0, z_n) & \Phi(z_1, z_n) & \ldots & \Phi(z_n, z_n)
\end{vmatrix}$$

est encore égal à δ^2. Cela posé, nous tirons de la relation

$$\int_{z_0}^{Z} \frac{e^{-z} f(z)}{z-\zeta}\,dz = e^{-z_0}\Phi(z_0, \zeta) - e^{-Z}\Phi(Z, \zeta),$$

en supposant $\zeta = z_i$, la valeur cherchée

$$\varepsilon_1^i = e^{-z_0}\Phi(z_0, z_i) - e^{-Z}\Phi(Z, z_i).$$

Or, voici les expressions des quantités ε_m^i qui en résultent. Soient

$$\begin{aligned}
\mathcal{A} &= A_0\Phi(Z, z_0) + A_1\Phi(Z, z_1) + \ldots + A_n\Phi(Z, z_n),\\
\mathcal{B} &= B_0\Phi(Z, z_0) + B_1\Phi(Z, z_1) + \ldots + B_n\Phi(Z, z_n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\mathcal{L} &= L_0\Phi(Z, z_0) + L_1\Phi(Z, z_1) + \ldots + L_n\Phi(Z, z_n),
\end{aligned}$$

et convenons de représenter par $\mathcal{A}_0, \mathcal{B}_0, \ldots, \mathcal{L}_0$ les valeurs obtenues pour $Z = z_0$; on aura

$$\begin{aligned}
\varepsilon_m^0 &= e^{-z_0}\mathcal{A}_0 - e^{-Z}\mathcal{A},\\
\varepsilon_m^1 &= e^{-z_0}\mathcal{B}_0 - e^{-Z}\mathcal{B},\\
&\ldots\ldots\ldots\ldots\ldots\ldots,\\
\varepsilon_m^n &= e^{-z_0}\mathcal{L}_0 - e^{-Z}\mathcal{L}.
\end{aligned}$$

Dans ces formules, Z désigne l'une quelconque des quantités z_1, $z_2, \ldots, z_n$; maintenant, si nous voulons mettre en évidence le résultat correspondant à $Z = z_k$, nous conviendrons, en outre, de représenter, d'une part, par $\mathcal{A}_k, \mathcal{B}_k, \ldots, \mathcal{L}_k$, et de l'autre par η_{ik}^0, $\eta_{ik}^1, \ldots, \eta_{ik}^n$ les valeurs que prennent, dans ce cas, les coefficients $\mathcal{A}$, $\mathcal{B}, \ldots, \mathcal{L}$ et les quantités $\varepsilon_m^0, \varepsilon_m^1, \ldots, \varepsilon_m^n$. On obtient ainsi les équations

$$\begin{aligned}
\eta_{ik}^0 &= e^{-z_0}\mathcal{A}_0 - e^{-z_k}\mathcal{A}_k,\\
\eta_{ik}^1 &= e^{-z_0}\mathcal{B}_0 - e^{-z_k}\mathcal{B}_k,\\
&\ldots\ldots\ldots\ldots\ldots\ldots;\\
\eta_{ik}^n &= e^{-z_0}\mathcal{L}_0 - e^{-z_k}\mathcal{L}_k,
\end{aligned}$$

qui vont nous conduire à la seconde démonstration que j'ai annoncée de l'impossibilité d'une relation de la forme

$$e^{z_0}N_0 + e^{z_0}N_1 + \ldots + e^{z_n}N_n = 0,$$

les exposants $z_0, z_1, \ldots, z_n$ étant supposés entiers, ainsi que les coefficients $N_0, N_1, \ldots, N_n$.

XIII. Je dis en premier lieu que ε_m^i peut devenir plus petit que toute quantité donnée, pour une valeur suffisamment grande de m. Effectivement, l'exponentielle e^{-z} étant toujours positive, on a, comme on sait,

$$\int_{z_0}^{Z} e^{-z}\,F(z)\,dz = F(\xi)\int_{z_0}^{Z} e^{-z}\,dz = F(\xi)(e^{-z_0} - e^{-Z}),$$

$F(z)$ étant une fonction quelconque et ξ une quantité comprise entre les limites z_0 et Z de l'intégrale. Or, en supposant

$$F(z) = \frac{f^m(z)}{z - z_1},$$

on aura cette expression

$$\varepsilon'_m = \frac{f^{m-1}(\xi)}{1.2\ldots m-1}\,\frac{f(\xi)}{\xi - z_i}(e^{-z_0} - e^{-Z}),$$

qui met en évidence la propriété énoncée. Cela posé, je tire des équations

$$\begin{gathered}
\eta_1^0 = e^{-z_0}\mathcal{A}_0 - e^{-z_1}\mathcal{A}_1,\\
\eta_2^0 = e^{-z_0}\mathcal{A}_0 - e^{-z_2}\mathcal{A}_2,\\
\ldots\ldots\ldots\ldots\ldots\ldots,\\
\eta_n^0 = e^{-z_0}\mathcal{A}_0 - e^{-z_n}\mathcal{A}_n,
\end{gathered}$$

la relation suivante,

$$\begin{aligned}
&e^{z_1}\eta_1^0 N_1 + e^{z_2}\eta_2^0 N_2 + \ldots + e^{z_n}\eta_n^0 N_n\\
&\quad = e^{-z_0}(e^{z_1}N_1 + e^{z_2}N_2 + \ldots + e^{z_n}N_n)\mathcal{A}_0\\
&\qquad - (\mathcal{A}_1 N_1 + \mathcal{A}_2 N_2 + \ldots + \mathcal{A}_n N_n).
\end{aligned}$$

Si l'on introduit la condition

$$e^{z_0}N_0 + e^{z_1}N_1 + \ldots + e^{z_n}N_n = 0,$$

elle devient

$$\begin{aligned}
&e^{z_1}\eta_1^0 N_1 + e^{z_2}\eta_2^0 N_2 + \ldots + e^{z_n}\eta_n^0 N_n\\
&\quad = -(\mathcal{A}_0 N_0 + \mathcal{A}_1 N_1 + \ldots + \mathcal{A}_n N_n).
\end{aligned}$$

Or, en supposant que $z_0, z_1, \ldots, z_n$ soient entiers, il en est de même des quantités $\Theta(z_i, z_k)$, $\Phi(z_i, z_k)$, et, par conséquent, de $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$. Nous avons donc un nombre entier

$$\mathcal{A}_0 N_0 + \mathcal{A}_1 N_1 + \ldots + \mathcal{A}_n N_n,$$

qui décroît indéfiniment avec $\eta_1^0, \eta_1^1, \ldots, \eta_1^n$, lorsque m augmente; il en résulte que, à partir d'une certaine valeur de m, et pour toutes les valeurs plus grandes, on aura

$$\mathcal{A}_0 N_0 + \mathcal{A}_1 N_1 + \ldots + \mathcal{A}_n N_n = 0,$$

et, comme on obtient pareillement les conditions

$$\begin{gathered}
\mathcal{B}_0 N_0 + \mathcal{B}_1 N_1 + \ldots + \mathcal{B}_n N_n = 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\mathcal{L}_0 N_0 + \mathcal{L}_1 N_1 + \ldots + \mathcal{L}_n N_n = 0,
\end{gathered}$$

la relation

$$e^{z_0}N_0 + e^{z_1}N_1 + \ldots + e^{z_n}N_n = 0$$

a pour conséquence que le déterminant

$$\Delta = \begin{vmatrix} \mathcal{A}_0 & \mathcal{A}_1 & \dots & \mathcal{A}_n \\ \mathcal{B}_0 & \mathcal{B}_1 & \dots & \mathcal{B}_n \\ \dots & \dots & \dots & \dots \\ \mathcal{L}_0 & \mathcal{L}_1 & \dots & \mathcal{L}_n \end{vmatrix}$$

doit nécessairement être nul. Mais, d'après les expressions des quantités $\mathcal{A}_k$, $\mathcal{B}_k$, $\dots$, $\mathcal{L}_k$, Δ est le produit de ces deux autres déterminants

$$\begin{vmatrix} A_0 & A_1 & \dots & A_n \\ B_0 & B_1 & \dots & B_n \\ \dots & \dots & \dots & \dots \\ L_0 & L_1 & \dots & L_n \end{vmatrix}$$

et

$$\begin{vmatrix} \Phi(z_0, z_0) & \Phi(z_1, z_0) & \dots & \Phi(z_n, z_0) \\ \Phi(z_0, z_1) & \Phi(z_1, z_1) & \dots & \Phi(z_n, z_1) \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ \Phi(z_0, z_n) & \Phi(z_1, z_n) & \dots & \Phi(z_n, z_n) \end{vmatrix},$$

dont le premier a pour valeur $\delta^{2(m-1)}$, et le second δ^2. On a donc $\Delta = \delta^{2m}$, et il est ainsi démontré, d'une manière entièrement rigoureuse, que la relation supposée est impossible, et que, par suite, le nombre e n'est point compris dans les irrationnelles algébriques.

XIV. Il ne sera pas inutile de donner quelques exemples du mode d'approximation des quantités auquel nous avons été conduits, et je considérerai d'abord le cas le plus simple, où l'on ne considère que la seule exponentielle e^x. En faisant alors $f(z) = z(z-x)$, nous aurons

$$\varepsilon_m = \frac{1}{1.2\ldots m} \int_0^x e^{-z} z^m (z-x)^m \, dz$$

et

$$\varepsilon_m^0 = \frac{1}{1.2\ldots m-1} \int_0^x e^{-z} z^{m-1} (z-x)^m \, dz,$$

$$\varepsilon_m^1 = \frac{1}{1.2\ldots m-1} \int^x e^{-z} z^m (z-x)^{m-1} \, dz.$$

Or on obtient immédiatement

$$\Theta(z, \zeta) = z + \zeta + 2m + 1 - x,$$

d'où

$$\Theta(0, 0) = 2m + 1 - x, \qquad \Theta(x, 0) = 2m + 1,$$
$$\Theta(0, x) = 2m + 1, \qquad \Theta(x, x) = 2m + 1 + x,$$

et, par conséquent, ces relations

$$\varepsilon_{m+1}^0 = (2m + 1 - x)\varepsilon_m^0 + (2m + 1)\varepsilon_m^1,$$
$$\varepsilon_{m+1}^1 = (2m + 1)\varepsilon_m^0 + (2m + 1 + x)\varepsilon_m^1.$$

J'observerai maintenant qu'il vient, en retranchant membre à membre,

$$\varepsilon_{m+1}^1 - \varepsilon_{m+1}^0 = x(\varepsilon_m^0 + \varepsilon_m^1),$$

de sorte que, ayant

$$\varepsilon_m = \varepsilon_m^0 + \varepsilon_m^1,$$

on en conclut

$$\varepsilon_{m+1}^1 - \varepsilon_{m+1}^0 = x\varepsilon_m.$$

Joignons à cette équation la suivante :

$$\varepsilon_{m+1}^1 + \varepsilon_{m+1}^0 = \varepsilon_{m+1};$$

nous en déduirons les valeurs

$$\varepsilon_{m+1}^1 = \frac{\varepsilon_{m+1} + x\varepsilon_m}{2}, \qquad \varepsilon_{m+1}^0 = \frac{\varepsilon_{m+1} - x\varepsilon_m}{2},$$

et, si l'on y change m en $m - 1$, une simple substitution, par exemple, dans la relation

$$\varepsilon_{m+1}^0 = (2m + 1 - x)\varepsilon_m^0 + (2m + 1)\varepsilon_m^1,$$

donnera le résultat précédemment obtenu (p. 165),

$$\varepsilon_{m+1} = (4m + 2)\varepsilon_m + x^2\varepsilon_{m-1}.$$

Soient, en second lieu,

$$n = 2, \qquad z_0 = 0, \qquad z_1 = 1, \qquad z_2 = 2,$$

d'où

$$f(z) = z(z - 1)(z - 2) = z^3 - 3z^2 + 2z;$$

on trouvera

$$\Theta(z, \zeta) = z^2 + (\zeta - 1)z + (\zeta - 1)^2 + 3m(z + \zeta + 1) + 9m^2,$$

et, par conséquent,

$$\Theta(0,0)=9m^2+3m+1,\quad \Theta(0,1)=9m^2+6m,\quad \Theta(0,2)=9m^2+9m+1,$$
$$\Theta(1,0)=9m^2+6m+1,\quad \Theta(1,1)=9m^2+9m+1,\quad \Theta(1,2)=9m^2+12m+3,$$
$$\Theta(2,0)=9m^2+9m+3,\quad \Theta(2,1)=9m^2+12m+4,\quad \Theta(2,2)=9m^2+15m+7.$$

En particulier, pour $m=1$, nous aurons

$$\begin{aligned}
\varepsilon_2^0 &= 13\varepsilon_1^0+16\varepsilon_1^1+21\varepsilon_1^2,\\
\varepsilon_2^1 &= 15\varepsilon_1^0+19\varepsilon_1^1+25\varepsilon_1^2,\\
\varepsilon_2^2 &= 19\varepsilon_1^0+24\varepsilon_1^1+31\varepsilon_1^2;
\end{aligned}$$

d'ailleurs il vient facilement

$$\Phi(z,\zeta)=z^2+(\zeta-1)z+(\zeta-1)^2,$$

ce qui donne

$$\begin{aligned}
\varepsilon_1^0 &= 1-e^{-Z}(Z^2-Z+1),\\
\varepsilon_1^1 &= \quad -e^{-Z}Z^2,\\
\varepsilon_1^2 &= 1-e^{-Z}(Z^2+Z+1);
\end{aligned}$$

on en conclut

$$\begin{aligned}
\varepsilon_2^0 &= 34-e^{-Z}(50Z^2+8Z+34),\\
\varepsilon_2^1 &= 40-e^{-Z}(59Z^2+10Z+40),\\
\varepsilon_2^2 &= 50-e^{-Z}(74Z^2+12Z+50).
\end{aligned}$$

De là résulte que

$$\varepsilon_1=\varepsilon_1^0+\varepsilon_1^1+\varepsilon_1^2=2-e^{-Z}(3Z^2+2),$$
$$\varepsilon_2=\varepsilon_2^0+\varepsilon_2^1+\varepsilon_2^2=124-e^{-Z}(183Z^2+30Z+124);$$

et, si l'on fait successivement $Z=1$, $Z=2$, l'expression de ε_1 fournit les valeurs approchées

$$e=\frac{5}{2},\qquad e^2=\frac{14}{2}=7,$$

et l'expression de ε_2 les suivantes :

$$e=\frac{337}{124},\qquad e^2=\frac{916}{124},$$

où l'erreur ne porte que sur les dix-millièmes. En supposant ensuite $m = 2$, ce qui donnera [1]

$$\begin{aligned}
\varepsilon_3^0 &= 43\,\varepsilon_2^0 + 49\,\varepsilon_2^1 + 57\,\varepsilon_2^2,\\
\varepsilon_3^1 &= 48\,\varepsilon_2^0 + 55\,\varepsilon_2^1 + 64\,\varepsilon_2^2,\\
\varepsilon_3^2 &= 55\,\varepsilon_2^0 + 63\,\varepsilon_2^1 + 73\,\varepsilon_2^2,
\end{aligned}$$

nous obtiendrons

$$\begin{aligned}
\varepsilon_3^0 &= 6272 - e^{-Z}(\ 9259\,Z^2 + 1518\,Z + 6272),\\
\varepsilon_3^1 &= 7032 - e^{-Z}(10381\,Z^2 + 1702\,Z + 7032),\\
\varepsilon_3^2 &= 8040 - e^{-Z}(11869\,Z^2 + 1946\,Z + 8040),
\end{aligned}$$

d'où

$$\varepsilon_3 = 21344 - e^{-Z}(31509\,Z^2 + 5166\,Z + 21344),$$

et, par suite,

$$e = \frac{58019}{21344}, \qquad e^2 = \frac{157712}{21344},$$

l'erreur portant sur les dix-millionièmes.

---

(1) Dans le texte d'Hermite, on trouve au dernier terme du second membre de la troisième ligne le coefficient 75. M. Bourget, en refaisant les calculs, a trouvé le coefficient 73; cette rectification a amené des modifications assez importantes dans les valeurs de e et de e^2, dont l'approximation monte, de ce fait, aux dix-millionièmes. E. P.

Ueber die Zahl π.*)

Von

F. LINDEMANN in Freiburg i. Br.

Bei der Vergeblichkeit der so ausserordentlich zahlreichen Versuche**), die Quadratur des Kreises mit Cirkel und Lineal auszuführen, hält man allgemein die Lösung der bezeichneten Aufgabe für unmöglich; es fehlte aber bisher ein Beweis dieser Unmöglichkeit; nur die Irrationalität von π und von π^2 ist festgestellt. Jede mit Cirkel und Lineal ausführbare Construction lässt sich mittelst algebraischer Einkleidung zurückführen auf die Lösung von linearen und quadratischen Gleichungen, also auch auf die Lösung einer Reihe von quadratischen Gleichungen, deren erste rationale Zahlen zu Coefficienten hat, während die Coefficienten jeder folgenden nur solche irrationale Zahlen enthalten, die durch Auflösung der vorhergehenden Gleichungen eingeführt sind. Die Schlussgleichung wird also durch wiederholtes Quadriren übergeführt werden können in eine Gleichung geraden Grades, deren Coefficienten rationale Zahlen sind. Man wird sonach die Unmöglichkeit der Quadratur des Kreises darthun, wenn man nachweist, dass *die Zahl π überhaupt nicht Wurzel einer algebraischen Gleichung irgend welchen Grades mit rationalen Coefficienten sein kann.* Den dafür nöthigen Beweis zu erbringen, ist im Folgenden versucht worden.

Die wesentliche Grundlage der Untersuchung bilden die Relationen zwischen gewissen bestimmten Integralen, welche Herr Hermite angewandt hat***), um den transcendenten Charakter der Zahl e festzustellen. In § 1 sind deshalb die betreffenden Formeln zusammengestellt; § 2 und § 3 geben die Anwendung dieser Formeln zum Beweise des erwähnten Satzes; § 4 enthält weitere Verallgemeinerungen.

*) Vergl. eine Mittheilung des Hrn. Weierstrass an die Berliner Akademie, vom 22. Juni 1882.

**) Man sehe die Artikel *Cyclometrie*, *Quadratur* und *Rectification* in Klügel's mathematischen Wörterbuche.

***) Sur la fonction exponentielle, Paris 1874 (auch Comptes rendus, t. LXXVII, 1873).

§ 1.

Die Hermite'schen Formeln.

Es sei

$$f(z) = (z - z_0)(z - z_1) \cdots (z - z_n);$$

mit Z möge irgend eine der *von einander verschiedenen* Grössen $z_1, z_2, \cdots z_n$ bezeichnet werden; dann bestehen zwischen den bestimmten Integralen

$$\varepsilon_m^i = \frac{1}{1 \cdot 2 \cdots (m-1)} \int_{z_0}^{Z} \frac{e^{-z} f^m(z)}{z - z_i}\, dz$$

folgende Relationen:

$$
(1) \quad \begin{aligned}
\varepsilon_{m+1}^0 &= \Theta(z_0, z_0)\, \varepsilon_m^0 + \Theta(z_1, z_0)\, \varepsilon_m^1 + \cdots + \Theta(z_n, z_0)\, \varepsilon_m^n, \\
\varepsilon_{m+1}^1 &= \Theta(z_0, z_1)\, \varepsilon_m^0 + \Theta(z_1, z_1)\, \varepsilon_m^1 + \cdots + \Theta(z_n, z_1)\, \varepsilon_m^n, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\varepsilon_{m+1}^n &= \Theta(z_0, z_n)\, \varepsilon_m^0 + \Theta(z_1, z_n)\, \varepsilon_m^1 + \cdots + \Theta(z_n, z_n)\, \varepsilon_m^n.
\end{aligned}
$$

Die Function $\Theta(z, \zeta)$ ist eine ganze Function n^{ten} Grades ihrer beiden Argumente; in ihr sind die Coefficienten der Potenzen von z und ζ ganze symmetrische Functionen der $n+1$ Grössen z_i und enthalten ausserdem nur ganze Zahlen, *sodass sie selbst ganze Zahlen sind, sobald dies mit den Coefficienten der Potenzen von z in $f(z)$ der Fall ist;* sie sind ganze Functionen n^{ten} Grades von m; auf ihr Bildungsgesetz kommt es im Folgenden nicht weiter an, es muss nur noch bemerkt werden, dass ihre Determinante folgender Relation genügt:

$$
\begin{vmatrix}
\Theta(z_0, z_0) & \Theta(z_1, z_0) & \cdots & \Theta(z_n, z_0) \\
\Theta(z_0, z_1) & \Theta(z_1, z_1) & \cdots & \Theta(z_n, z_1) \\
\cdot & \cdot & \cdot & \cdot \\
\Theta(z_0, z_n) & \Theta(z_1, z_n) & \cdots & \Theta(z_n, z_n)
\end{vmatrix} = \delta^2,
$$

wenn

$$
\delta = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
z_0 & z_1 & \cdots & z_n \\
z_0^2 & z_1^2 & \cdots & z_n^2 \\
\cdot & \cdot & \cdot & \cdot \\
z_0^n & z_1^n & \cdots & z_n^n
\end{vmatrix}
$$

Durch die angegebenen Gleichungen sind die Integrale ε_{m+1}^i zurückgeführt auf die Integrale ε_m^i. Durch wiederholte Anwendung derselben findet man also Formeln von der Gestalt:

$$
(1\text{a})\qquad
\begin{aligned}
\varepsilon_m^0 &= A_0\varepsilon_1^0 + A_1\varepsilon_1^1 + \cdots + A_n\varepsilon_1^n,\\
\varepsilon_m^1 &= B_0\varepsilon_1^0 + B_1\varepsilon_1^1 + \cdots + B_n\varepsilon_1^n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\varepsilon_m^n &= L_0\varepsilon_1^0 + L_1\varepsilon_1^1 + \cdots + L_n\varepsilon_n^n;
\end{aligned}
$$

und man hat

$$
(2)\qquad
\begin{vmatrix}
A_0 & A_1 & \cdots & A_n\\
B_0 & B_1 & \cdots & B_n\\
\cdot & \cdot & \cdot & \cdot\\
L_0 & L_1 & \cdots & L_n
\end{vmatrix} = \delta^{2(m-1)}.
$$

Um endlich die Integrale ε_1^i auszuwerthen, werden neue ganze Functionen $\Phi(z, \zeta)$ eingeführt, welche den Functionen $\Theta(z, \zeta)$ analog gebildet sind, insbesondere hinsichtlich ihrer Coefficienten ebenfalls die bei den Θ hervorgehobene Eigenschaft zeigen und wiederum der Bedingung

$$
\begin{vmatrix}
\Phi(z_0, z_0) & \Phi(z_1, z_0) & \cdots & \Phi(z_n, z_0)\\
\Phi(z_0, z_1) & \Phi(z_1, z_1) & \cdots & \Phi(z_n, z_1)\\
\cdot & \cdot & \cdot & \cdot\\
\Phi(z_0, z_n) & \Phi(z_1, z_n) & \cdots & \Phi(z_n, z_n)
\end{vmatrix} = \delta^2
$$

genügen. Aus ihnen setzen sich die durch folgende Gleichungen definirten Grössen $\mathsf{A}, \mathsf{B}, \cdots \Lambda$ zusammen:

$$
(3)\qquad
\begin{aligned}
\mathsf{A} &= A_0\Phi(Z, z_0) + A_1\Phi(Z, z_1) + \cdots + A_n\Phi(Z, z_n),\\
\mathsf{B} &= B_0\Phi(Z, z_0) + B_1\Phi(Z, z_1) + \cdots + B_n\Phi(Z, z_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\Lambda &= L_0\Phi(Z, z_0) + L_1\Phi(Z, z_1) + \cdots + L_n\Phi(Z, z_n).
\end{aligned}
$$

Bezeichnet man endlich mit $\mathsf{A}_0, \mathsf{B}_0, \cdots \Lambda_0$ die Werthe dieser Constanten für den Fall, dass Z durch z_0 ersetzt wird, so hat man für die bestimmten Integrale ε_m^i schliesslich folgende Formeln:

$$
\begin{aligned}
\varepsilon_m^0 &= e^{-z_0}\mathsf{A}_0 - e^{-Z}\mathsf{A},\\
\varepsilon_m^1 &= e^{-z_0}\mathsf{B}_0 - e^{-Z}\mathsf{B},\\
&\cdots\cdots\cdots\\
\varepsilon_m^n &= e^{-z_0}\Lambda_0 - e^{-Z}\Lambda.
\end{aligned}
$$

Hierin bedeutet Z irgend eine der Grössen $z_1, z_2, \cdots z_n$; setzen wir insbesondere $Z = z_k$, so mögen die dann entstehenden Werthe von $\mathsf{A}, \mathsf{B}, \cdots \Lambda$ mit $\mathsf{A}_k, \mathsf{B}_k, \cdots \Lambda_k$ bezeichnet werden, und es werde

$$
\left[\varepsilon_m^i\right]_{Z=z_k} = \eta_k^i
$$

gesetzt; dann ist auch

$$\begin{aligned} &\eta_k^0 = e^{-z_0}\mathsf{A}_0 - e^{-z_k}\mathsf{A}_k, \\ (4)\qquad &\eta_k^1 = e^{-z_0}\mathsf{B}_0 - e^{-z_k}\mathsf{B}_k, \\ &\cdots\cdots\cdots\cdots \\ &\eta_k^n = e^{-z_0}\Lambda_0 - e^{-z_k}\Lambda_k. \end{aligned}$$

Die Determinante der Coefficienten $\mathsf{A}, \mathsf{B}, \cdots$ ist hier gegeben durch

$$(5)\qquad \begin{vmatrix} \mathsf{A}_0 & \mathsf{A}_1 & \cdots & \mathsf{A}_n \\ \mathsf{B}_0 & \mathsf{B}_1 & \cdots & \mathsf{B}_n \\ \cdot & \cdot & \cdot & \cdot \\ \Lambda_0 & \Lambda_1 & \cdots & \Lambda_n \end{vmatrix} = \delta^{2m}.$$

§ 2.

Ueber die symmetrischen Functionen der Grössen e^{z_i}.

Wir bemerken, dass die in § 1. zusammengestellten Relationen unabhängig davon bestehen, ob die Grössen $z, z_0, z_1, \cdots, z_n$ reell oder complex sind, denn sie beruhen einfach auf identischen Umformungen; auch kann der bei Berechnung der Integrale η_k^i gewählte Integrationsweg ein beliebiger sein; es muss nur der unendlich ferne Punkt der z-Ebene ausgeschlossen bleiben.

Wir nehmen an, dass die symmetrischen Functionen der z_i reelle oder complexe ganze Zahlen seien; auch unter $N_0, N_1, \cdots, N_n$ verstehen wir reelle oder complexe ganze Zahlen. Zunächst werde ein System von Gleichungen abgeleitet, zu denen eine Relation der Form

$$(6)\qquad N_0 e^{z_0} + N_1 e^{z_1} + \cdots + N_n e^{z_n} = 0$$

Veranlassung giebt.

Wir multipliciren die in (4) enthaltenen Gleichungen

$$\begin{aligned} &\eta_1^0 = e^{-z_0}\mathsf{A}_0 - e^{-z_1}\mathsf{A}_1, \\ (7)\qquad &\eta_2^0 = e^{-z_0}\mathsf{A}_0 - e^{-z_2}\mathsf{A}_2, \\ &\cdots\cdots\cdots\cdots \\ &\eta_n^0 = e^{-z_0}\mathsf{A}_0 - e^{-z_n}\mathsf{A}_n, \end{aligned}$$

bez. mit $N_1 e^{z_1}, \cdots, N_n e^{z_n}$, dann ergiebt sich durch Addition

$$\begin{aligned} &e^{z_1}\eta_1^0 N_1 + e^{z_2}\eta_2^0 N_2 + \cdots + e^{z_n}\eta_n^0 N_n \\ &= e^{-z_0}\left(e^{z_1}N_1 + e^{z_2}N_2 + \cdots + e^{z_n}N_n\right)\mathsf{A}_0 \\ &\qquad + \mathsf{A}_2 N_2 + \cdots + \mathsf{A}_n N_n). \end{aligned}$$

In Folge von (14) m hei die Gleichung bestehen

$$e^{z_1}\eta_1^0 N_1 + \cdots + e^{z_n}\eta_n^0 N_n = -(\mathsf{A}_0 N_0 + \mathsf{A}_1 N_1 + \cdots + \mathsf{A}_n N_n).$$

Analoge Gleichungen werden sich ergeben, wenn man A durch $\mathsf{B}, \cdots, \Lambda$ ersetzt; man wird also zu folgendem Systeme von $n+1$ Gleichungen geführt:

$$\begin{aligned}
&\mathsf{A}_0 N_0 + \mathsf{A}_1 N_1 + \cdots + \mathsf{A}_n N_n = \alpha,\\
(8)\qquad &\mathsf{B}_0 N_0 + \mathsf{B}_1 N_1 + \cdots + \mathsf{B}_n N_n = \beta,\\
&\cdots\cdots\cdots\cdots\cdots\\
&\Lambda_0 N_0 + \Lambda_1 N_1 + \cdots + \Lambda_n N_n = \lambda,
\end{aligned}$$

wo

$$\begin{aligned}
-\alpha &= e^{z_1}\eta_1^0 N_1 + e^{z_2}\eta_2^0 N_2 + \cdots + e^{z_n}\eta_n^0 N_n,\\
-\beta &= e^{z_1}\eta_1^1 N_1 + e^{z_2}\eta_2^1 N_2 + \cdots + e^{z_n}\eta_n^1 N_n,\\
&\cdots\cdots\cdots\cdots\cdots\\
-\lambda &= e^{z_1}\eta_1^n N_1 + e^{z_2}\eta_2^n N_2 + \cdots + e^{z_n}\eta_n^n N_n.
\end{aligned}$$

Dies sind die Formeln, aus welchen Herr Hermite das Transscendentsein der Zahl e direct ableitet, indem er annimmt, dass die z_i selbst ganze Zahlen sind. Für unsern Zweck sollen die Gleichungen dadurch vereinfacht werden, dass $N_1 = N_2 = \cdots = N_n$ genommen wird; *es soll ferner im Folgenden immer $z_0 = 0$ gesetzt werden.* Man hat dann an Stelle von (8):

$$\begin{aligned}
&N_0\mathsf{A}_0 + N_1(\mathsf{A}_1 + \cdots + \mathsf{A}_n) = -N_1(e^{z_1}\eta_1^0 + \cdots + e^{z_n}\eta_n^0),\\
&N_0\mathsf{B}_0 + N_1(\mathsf{B}_1 + \cdots + \mathsf{B}_n) = -N_1(e^{z_1}\eta_1^1 + \cdots + e^{z_n}\eta_n^1),\\
(9)\qquad &\cdots\cdots\cdots\cdots\cdots\\
&N_0\Lambda_0 + N_1(\Lambda_1 + \cdots + \Lambda_n) = -N_1(e^{z_1}\eta_1^n + \cdots + e^{z_n}\eta_n^n).
\end{aligned}$$

Hier stehen links ganze Functionen der Grössen $z_1, z_2, \cdots, z_n$ mit ganzzahligen Coefficienten. Vertauscht man in ihnen z_i mit z_k, so vertauscht sich nach (7) auch A_i mit A_k. Die linke Seite der ersten Gleichung ist also eine symmetrische Function der Wurzeln z_i, *folglich selbst eine ganze Zahl.*

Durch Vertauschung von z_1 mit z_2 geht B_i in Γ_i über und umgekehrt, sobald i von 1 und 2 verschieden ist; dagegen vertauscht sich gleichzeitig B_1 mit Γ_2 und B_2 mit Γ_1: es vertauschen sich also die linken Seiten der zweiten und dritten Gleichung unter einander. Analoges gilt bei beliebigen Vertauschungen der z_i; also: *Die linken Seiten der n letzten Gleichungen des Systems* (9) *sind Wurzeln einer algebraischen Gleichung von der Form*

$$(10)\qquad Y^n + M_1 Y^{n-1} + \cdots + \qquad = 0$$

mit ganzzahligen Coefficienten M_i.

Wir lassen jetzt die Zahl m unendlich gross werden. Bezeichnet M' das Maximum des absoluten Betrages von $f(z)$ für die Werthe von z, welche sich auf dem beim Integrale ε_m^i gewählten Integrationswege befinden, und ist l' die Länge dieses Weges, M_0' das Maximum des absoluten Betrages von $e^{-z}(z - z_i)^{-1}$ längs dieses Weges, so hat man

$$\text{abs}\ \varepsilon_m^i \leqq \frac{\mathsf{M}'^m \cdot \mathsf{M}_0' \cdot l'}{1 \cdot 2 \cdot 3 \cdots (m-1)}.$$

Versteht man also unter M, M_0, l die grössten derjenigen Zahlen M', M_0', l', welche bei den verschiedenen Integralen ε_m^i vorkommen, wenn man die obere Grenze Z nach einander durch $z_1, z_2, \cdots, z_n$ ersetzt und dem Index i alle möglichen Werthe beilegt, und setzt man noch $\mathsf{M}_0 l = E$, so besteht für alle Integrale η_k^i die Ungleichung:

$$\text{abs}\ \eta_k^i \leqq \frac{\mathsf{M}^m \cdot E}{1 \cdot 2 \cdot 3 \cdots (m-1)},$$

wo E und M bestimmte endliche von m unabhängige Zahlen bedeuten.

Es kann also die rechte Seite der ersten unter den Gleichungen (9) und somit die ganze Zahl $N_0 \mathsf{A}_0 + N_1(\mathsf{A}_1 + \mathsf{A}_2 + \cdots \mathsf{A}_n)$ beliebig klein gemacht werden, dadurch dass man m hinreichend gross wählt. Das ist aber nur möglich, wenn es eine endliche ganze Zahl m' giebt der Art, dass die Gleichung

$$N_0 \mathsf{A}_0 + N_1(\mathsf{A}_1 + \mathsf{A}_2 + \cdots + \mathsf{A}_n) = 0$$

genau erfüllt ist für alle ganzzahligen Werthe von m, welche nicht kleiner als m' sind.

Ebenso ergiebt sich, dass die rechten Seiten der übrigen Gleichungen (9) gleich Null werden für $m = \infty$, dass also auch die linken Seiten, d. i. die Wurzeln von (10), beliebig klein werden für hinreichend grosse Werthe von m. Dasselbe gilt folglich von den Coefficienten M_i dieser Gleichung; da letztere aber ganze Zahlen sind, so müssen sie schon für hinreichend grosse *endliche* Werthe von m genau gleich Null sein; hieraus folgt, dass auch die Wurzeln von (10) für dieselben Werthe von m sämmtlich verschwinden. Wir sind also zu folgendem Resultate gekommen: *Soll eine Gleichung von der Form*

$$N_0 + N_1 \sum e^{z_i} = 0 \tag{11}$$

bestehen, so muss es eine bestimmte positive ganze Zahl m' geben der Art, dass für alle ganzzahligen Werthe von m, die nicht kleiner als m' sind, das folgende System von Gleichungen besteht:

$$\begin{aligned} \mathsf{A}_0 N_0 + (\mathsf{A}_1 + \cdots + \mathsf{A}_n) N_1 &= 0, \\ \mathsf{B}_0 N_0 + (\mathsf{B}_1 + \cdots + \mathsf{B}_n) N_1 &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots \\ \Lambda_0 N_0 + (\Lambda_1 + \cdots + \Lambda_n) N_1 &= 0. \end{aligned} \tag{12}$$

Hieraus würde weiter folgen, dass die Determinante der Grössen A, B, $\cdots$, Λ verschwinde, da ihre aus zwei parallellen Reihen zu bildenden zweireihigen Unterdeterminanten sämmtlich Null sein müssten. Die Determinante ist aber nach (5) gleich δ^{2m}, kann also, da die z_i von einander verschieden vorausgesetzt wurden, niemals Null sein. Damit ist gezeigt, dass die Annahme einer Relation von der Form (11) zu einem Widerspruche führt, und wir haben den Satz gewonnen:

Sind $z_1, \cdots, z_n$ die von Null und von einander verschiedenen Wurzeln einer algebraischen Gleichung von der Form

$$z^n + s_1 z^{n-1} + \cdots + s_n = 0, \tag{13}$$

wo die s_i reelle oder complexe ganze Zahlen bezeichnen, und ist die Discriminante dieser Gleichung von Null verschieden, so kann eine Relation von der Form (11) *nicht bestehen, falls die N_i reelle oder complexe ganze, von Null verschiedene Zahlen bedeuten.*

Ebensowenig ist eine Relation von der Form

$$N_0 + N_1 (e^{rz_1} + e^{rz_2} + \cdots + e^{rz_n}) = 0 \tag{14}$$

möglich, wenn r eine ganze Zahl bedeutet. Denn die Grössen rz_1, $rz_2, \cdots, rz_n$ sind ebenfalls von einander und von Null verschiedene Wurzeln einer Gleichung von der Form (13), sobald dies mit $z_1, z_2, \cdots z_n$ der Fall ist; es lassen sich also auf (14) dieselben Schlüsse wie auf (11) anwenden, und es folgt:

Unter den gemachten Voraussetzungen kann keine der symmetrischen Functionen

$$e^{rz_1} + e^{rz_2} + \cdots + e^{rz_n} \tag{15}$$

gleich einer rationalen Zahl sein.

Untersuchen wir weiter, ob zwischen diesen symmetrischen Functionen eine lineare Gleichung mit ganzzahligen Coefficienten N_i, also eine Gleichung von der Gestalt

$$0 = N_0 + N_1 \sum e^{z_i} + N_2 \sum e^{2z_i} + \cdots + N_s \sum e^{sz_i} \tag{16}$$

bestehen kann, wo s irgend eine ganze positive Zahl bedeutet. Die ns Grössen $z_i, 2z_i, \cdots sz_i$, mögen zur Abkürzung mit $z_1, z_2, \cdots z_{ns}$ bezeichnet werden, so dass $z_{i+n} = 2z_i$, $z_{i+2n} = 3z_i$, etc. Dieselben sind Wurzeln einer Gleichung ns^{ten} Grades mit ganzzahligen Coefficienten von der Form (13). Aus (16) würde sich also, wie aus (6), ein System von $ns+1$ Gleichungen von der Gestalt (8) ergeben. Um dieselben einfach schreiben zu können, führen wir statt der $(n+1)^2$ Grössen $A_i, B_i, \cdots Λ_i$ jetzt $(ns+1)^2$ Grössen A ein, die mit zwei Indices versehen sein mögen, und zwar so, dass der erste Index diejenige Unterscheidung bewirkt, welche früher durch Wahl der verschiedenen Buchstaben $\mathsf{A}, \mathsf{B}, \cdots \Lambda$ angedeutet war, dass also die folgenden zu (4) analogen Gleichungen bestehen (indem $z_0 = 0$):

$$\eta_k^i = \mathsf{A}_{i0} - e^{-z_k} \mathsf{A}_{ik}, \quad \text{für } i, k \text{ gleich } 0, 1, 2, \cdots, ns.$$

Statt (8) haben wir dann:

$$(17) \quad \begin{array}{l} N_0 \mathsf{A}_{00} + N_1 \sum \mathsf{A}_{0,i} + N_2 \sum \mathsf{A}_{0,i+n} + \cdots + N_s \sum \mathsf{A}_{0,i+ns-n} = \alpha_0, \\ N_0 \mathsf{A}_{10} + N_1 \sum \mathsf{A}_{1,i} + N_2 \sum \mathsf{A}_{1,i+n} + \cdots + N_s \sum \mathsf{A}_{1,i+ns-n} = \alpha_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ N_0 \mathsf{A}_{ns,0} + N_1 \sum \mathsf{A}_{ns,i} + N_2 \sum \mathsf{A}_{ns,i+n} + \cdots + N_s \sum \mathsf{A}_{ns,i+ns-n} = \alpha_{ns}, \end{array}$$

wo die rechten Seiten gegeben sind durch

$$-\alpha_k = N_1 \sum \eta_i^{(k)} e^{z_i} + N_2 \sum \eta_{i+n}^{(k)} e^{z_{i+n}} + \cdots + N_s \eta_{i+ns-n}^{(k)} e^{z_{i+ns-n}}.$$

Alle hier vorkommenden Summen sind über i von $i = 1$ bis $i = n$ zu nehmen.

Auf den linken Seiten des Systems (17) stehen wieder ganze Functionen der z_i mit ganzzahligen Coefficienten. Vertauscht man z_i mit z_k (wo $i, k \leq n$), so vertauscht sich gleichzeitig z_{i+rn} mit z_{k+rn}, folglich auch

$$\mathsf{A}_{0,i} \text{ mit } \mathsf{A}_{0,k} \text{ und } \mathsf{A}_{0,i+rn} \text{ mit } \mathsf{A}_{0,k+rn}$$

ferner, wenn der erste Index von Null verschieden ist:

$$\mathsf{A}_{k,i+rn} \text{ mit } \mathsf{A}_{l,i+rn} \quad \text{für } l \gtrless k, i$$

$$\mathsf{A}_{i,k+rn} \text{ mit } \mathsf{A}_{k,i+rn} \quad \text{und} \quad \mathsf{A}_{i,i+rn} \text{ mit } \mathsf{A}_{k,k+rn}.$$

Die linke Seite der ersten unter den Gleichungen (17) ist sonach wieder eine ganze Zahl; die linken Seiten der übrigen Gleichungen sind Wurzeln einer algebraischen Gleichung vom Grade ns und von der Form (10). Auf alle vorkommenden ganzen Zahlen lassen sich für $m = \infty$ die bei (9) und (10) gemachten Schlüsse übertragen. Es folgt also, dass für hinreichend grosse *endliche* Werthe der ganzen Zahl m die linken Seiten der Gleichungen (17) genau gleich Null sein müssen. Daraus würde weiter folgen, dass für dieselben Werthe von m alle $(s+1)$-reihigen Determinanten genau verschwinden, welche aus den $(s+1)(ns+1)$ Coefficienten der N_i in (17) in bekannter Weise zu bilden sind. Dann müsste auch die $(ns+1)$-reihige Determinante der A_{ik} selbst gleich Null sein; diese aber ist nach (5) gleich der $2m^{\text{ten}}$ Potenz von δ, wenn man in der Determinante δ (vergl. § 1) $z_0 = 0$ macht und den Index n durch ns ersetzt.

Es ist δ gleich dem Producte aller Differenzen $\pm (pz_i - qz_k)$, welche entstehen, wenn man den Indices i, k alle Werthe $1, 2, \cdots n$, den Zahlen p, q alle Werthe $0, 1, 2, \cdots, s$ beilegt (ausgenommen

$p=q=0$). Es kann also δ nur Null sein, wenn zwischen zwei Wurzeln z_i, z_k eine Relation der Form $pz_i - qz_k = 0$ besteht. Nimmt man eine solche an, so würden die Gleichung (13) und die Gleichung

$$z^n + \frac{q}{p} s_1 z^{n-1} + \left(\frac{q}{p}\right)^2 s_2 z^{n-2} + \cdots + \left(\frac{q}{p}\right)^n s_n = 0$$

eine gemeinsame Wurzel z_i zulassen, somit beide reducibel sein. Legen wir also der Gleichung (13) die Bedingung der Irreducibilität auf, so kann δ sicher nicht verschwinden, während wir eben sahen, dass in Folge von (16) und (17) $\delta = 0$ sein müsste. Folglich:

Sind $z_1, z_2, \cdots z_n$ die Wurzeln einer irreducibeln Gleichung von der Form (13) *mit ganzzahligen Coefficienten s_i, so kann zwischen den symmetrischen Functionen* (15) *keine lineare Gleichung mit rationalen, von Null verschiedenen Coefficienten bestehen.*

§ 3.

Anwendung auf Untersuchung der Zahl π.

Die gewonnenen Sätze lassen sich übertragen auf die symmetrischen Functionen

$$(18) \qquad \begin{array}{lll} \sum e^{z_1}, & \sum e^{z_1+z_2}, & \sum e^{z_1+z_2+z_3}, \ \cdots \\ \sum e^{rz_1}, & \sum e^{r(z_1+z_2)}, & \sum e^{r(z_1+z_2+z_3)}, \cdots \end{array}$$

wo wieder r eine beliebige ganze Zahl bezeichnet. Setzen wir zunächst voraus, dass die Zahlen, welche hier als Exponenten von e auftreten, sämmtlich von Null und von einander verschieden seien. Die Annahme einer linearen Beziehung mit rationalen Coefficienten N_i zwischen den Grössen (18) würde dann zu einem Gleichungssysteme führen, welches zu (17) ganz analog ist, und dessen Unmöglichkeit in ganz derselben Weise dargethan werden kann.

Ist die gemachte Voraussetzung nicht erfüllt, so nehmen wir zunächst an, dass in einer der Functionen (18), sagen wir Σe^Z, mehrere der Exponenten Z einander gleich seien, und betrachten eine Relation von der Gestalt

$$(19) \qquad 0 = N_0 + N_1 \sum e^Z.$$

Jetzt werden sich die sämmtlichen Grössen Z in mehrere Gruppen

$$Z_1, Z_1', Z_1'', \cdots; \quad Z_2, Z_2', Z_2'', \cdots; \cdots$$

zerlegen, der Art, dass die Grössen jeder Gruppe Wurzeln einer irreducibeln Gleichung mit rationalen Coefficienten sind und sich bei Vertauschungen der z_i nur unter einander vertauschen. Es war aber bei Behandlung der in § 2. supponirten Gleichungen (11), (14), (16) allein massgebend, dass in jeder einzelnen Summe nur solche Exponenten von e vorkamen, die sich unter einander vertauschen bei den Vertauschungen der z_i. Wir werden daher jetzt unmittelbar auf die Unmöglichkeit einer Relation von der Gestalt

$$0 = N_0 + N_1 \sum e^{Z_1} + N_2 \sum e^{Z_2} + \cdots$$

schliessen dürfen; und in dieser ist die Relation (19) als besonderer Fall enthalten.

Sollte ferner eine der Grössen Z gleich Null sein, so würde dies einer Aenderung des numerischen Werthes der Zahl N_0 äquivalent sein, das Resultat also nicht beeinflussen; es sei denn, dass alle auf der rechten Seite von (19) vorkommenden Exponenten gleichzeitig verschwänden. Wenn die z_i Wurzeln einer irreducibeln Gleichung sind, wird dies nur eintreten können für die Exponenten der Functionen

$$e^Z = e^{r(z_1 + z_2 + \cdots + z_n)} \tag{20}$$

falls in (13) der Coefficient s_1 von z^{n-1} gleich Null ist. *Dieser Fall giebt im Folgenden immer eine Ausnahme* und soll nicht jedesmal wieder hervorgehoben werden.

Betrachtet man weiter eine lineare Relation, in der *mehrere* der Functionen (18) vorkommen, so wird man es beim Auftreten gleicher Exponenten wieder so einrichten, dass jede Zahl N_i multiplicirt erscheint in eine Summe, in deren Gliedern die Exponenten von e Wurzeln einer einzigen irreducibeln Gleichung sind, und die bei Gelegenheit von (19) gemachten Bemerkungen wiederholen. Als Specialfall erhält man dann wieder die zu Anfang dieses Paragraphen besprochene Relation, wo jede Zahl N_i in eine der symmetrischen Functionen (18) multiplicirt ist. — Schliesslich gelangen wir so zu folgenden Sätzen:

Es kann keine der symmetrischen Functionen (18) *gleich einer rationalen Zahl sein, ausgenommen eine Function* (20) *falls in* (13) s_1 *verschwindet.* Und allgemeiner: *Zwischen den Functionen* (18) *kann keine lineare Relation mit rationalen Coefficienten bestehen, ausgenommen den Fall, wo* s_1 *in* (13) *gleich Null ist, in welchem Falle eine solche Function der Grössen* (20) *gleich einer rationalen Zahl ist.*

Nun ist die erste Reihe der Grössen (18) identisch mit der Reihe der Coefficienten M_i derjenigen algebraischen Gleichung n^{ten} Grades

$$V^n - \mathsf{M}_1 V^{n-1} + \mathsf{M}_2 V^{n-2} - \cdots \pm \mathsf{M}_n = 0, \tag{21}$$

welche von den Zahlen e^{z_i} befriedigt wird. Wir wissen also, dass diese Coefficienten M_i nicht gleich rationalen Zahlen sind, ausgenommen M_n für $s_1 = 0$, und dass zwischen ihnen keine lineare Relation mit rationalen Coefficienten Statt hat. Eine solche aber würde nothwendig erfüllt sein müssen, wenn eine der Wurzeln von (21), d. i. der Grössen e^{z_i}, gleich einer rationalen Zahl wäre. Also:

Ist Z Wurzel einer irreducibeln algebraischen Gleichung von der Form (13), *so kann e^Z nicht gleich einer rationalen Zahl sein.*

Es ist aber die Bedingung $e^{\pi\sqrt{-1}} = -1$ erfüllt, also kann $\pi\sqrt{-1}$ nicht Wurzel einer irreducibeln Gleichung der Form (13) sein. Auf letztere Form kann jede Gleichung mit rationalen Coefficienten leicht gebracht werden, indem man pz statt z als Unbekannte auffasst, unter p eine passend gewählte ganze Zahl verstanden. Es folgt so der in der Einleitung hervorgehobene Satz:

Die Ludolph'sche Zahl π kann nicht Wurzel einer algebraischen Gleichung mit rationalen (reellen oder complexen) Coefficienten sein.

§ 4.

Verallgemeinerung der gewonnenen Resultate.

Im Vorstehenden sind alle Schlüsse nur so weit durchgeführt, als es nöthig oder nützlich schien, um zu dem zuletzt ausgesprochenen Satze zu gelangen. Dieselben sind aber sofort einer weiteren Verallgemeinerung fähig, welche dann eine bemerkenswerthe Anwendung auf die Untersuchung der natürlichen Logarithmen gestattet.

Es seien $Z_1, Z_2, Z_3, \cdots$ irgend welche ganze Functionen der Grössen z_i mit ganzzahligen Coefficienten. Die von einander verschiedenen Werthe, welche Z_k durch Vertauschungen der z_i annimmt, mögen mit $Z_k', Z_k'', Z_k''', \cdots$ bezeichnet werden. Dieselben sind dann Wurzeln einer irreducibeln Gleichung mit ganzzahligen Coefficienten, in welcher der Coefficient der höchsten Potenz der Unbekannten gleich Eins ist. Nach den zu Anfang von § 3. gemachten Erörterungen lassen sich auf die Grössen

$$(22)\qquad \sum e^{Z_1}, \quad \sum e^{Z_2}, \cdots$$

wieder die analogen Schlüsse anwenden, wenn

$$\sum e^{Z_k} = e^{Z_k} + e^{Z_k'} + e^{Z_k''} + \cdots$$

gesetzt wird. *Es kann daher keine Relation von der Gestalt bestehen:*

$$(23)\qquad 0 = N_0 + N_1 \sum e^{Z_1} + N_2 \sum e^{Z_2} + \cdots.$$

Auf die Form der rechten Seite kann jede ganze Function der Grössen (22) sofort gebracht werden; also folgt als Corollar, dass *keine ganze Function der Grössen* (22), *insbesondere der Grössen* M_i *in* (21), *welche rationale Coefficienten hat, verschwinden kann.*

Uns kommt es mehr auf andere Folgerungen an. Es ergiebt sich nämlich, dass *die als unmöglich nachgewiesenen Relationen auch noch unmöglich sind, wenn man unter den N_i beliebige algebraische Irrationalzahlen versteht.* In der That, schreiben wir (23) in der Form

$$N_0 - V = 0,$$

und bezeichnen wir mit N_0', N_0'', $\cdots$, V', V'' $\cdots$ die Grössen, welche aus N_0 bez. V entstehen, falls man die N_i mit allen Grössen vertauscht, die mit den N_i zusammen Wurzeln algebraischer Gleichungen mit rationalen Coefficienten sind, so würde auch die Gleichung

$$(N_0 - V)(N_0' - V')(N_0'' - V'') \cdots = 0$$

erfüllt sein müssen, wo links eine ganze Function der Zahlen (22) mit *rationalen* Coefficienten steht, also gerade wieder ein Ausdruck wie (23), von dem wir wissen, dass er nicht verschwinden kann. Insbesondere ergiebt sich: *Es kann keine ganze Function der* M_i *mit algebraisch irrationalen Coefficienten verschwinden*, ausgenommen (im Falle $\Sigma z_1 = 0$) eine solche Function von M_n allein.

Eine derartige Relation aber würde nach (21) erfüllt sein, wenn e^{z_i} eine algebraisch irrationale Zahl wäre. Hebt man noch, wie am Schlusse von § 3., die der Gleichung (13) zunächst auferlegte Beschränkung auf, so folgt also:

Ist z eine rationale Zahl ($\gtrless 0$) oder algebraisch irrational, so ist e^z eine transscendente Zahl.

Und durch Umkehrung:

Der natürliche Logarithmus einer rationalen Zahl (die Einheit ausgenommen) oder einer algebraisch irrationalen Zahl ist stets eine transscendente Zahl.

Auch hier haben wir die Ueberlegungen specieller gefasst als nöthig wäre, da wir zunächst die ausgesprochenen speciellen Sätze im Auge hatten. Man kann indessen Alles (mit Rücksicht auf die Erörterungen zu Anfang von § 3.) in folgendem allgemeinen Theoreme zusammenfassen:

Ist eine Reihe algebraischer, von einander verschiedener Gleichungen von der Form (13) $f_1 = 0$, $f_2 = 0$, $\cdots$, $f_s = 0$ mit beliebigen ganzzahligen Coefficienten gegeben, sind dieselben sämmtlich irreducibel, und bezeichnet man mit Z_i, Z_i', Z_i'', $\cdots$ die Wurzeln der Gleichung $f_i = 0$, mit N_k beliebige rationale oder algebraisch irrationale Zahlen (die nicht sämmtlich gleich Null sind), so kann eine Gleichung der Form

$$(24) \qquad N_0 + N_1 \sum e^{Z_1} + N_2 \sum e^{Z_2} + \cdots + N_s \sum e^{Z_s} = 0$$

nicht bestehen; es sei denn, dass eine der ganzen Functionen f_i einfach *mit z identisch ist, etwa $f_1 = z$, in welchem Falle die Gleichung* (24) *möglich wird für $N_2 = N_3 = \cdots = N_s = 0$, $N_1 = -N_0$.*

Dieser Satz wiederum führt zu folgendem Resultate:

Versteht man unter den z_i beliebige rationale oder algebraisch irrationale, von einander verschiedene Zahlen, und unter den N_i ebensolche Zahlen, die nicht sämmtlich gleich Null sind, so kann keine Gleichung der Form bestehen:

$$0 = N_0 e^{z_0} + N_1 e^{z_1} + N_2 e^{z_2} + \cdots + N_r e^{z_r}.$$

Bildet man nämlich das Product aller Ausdrücke, welche aus der rechten Seite dadurch entstehen, dass man die N_i auf alle Weisen unter einander vertauscht, die z_i aber mit denjenigen Grössen vertauscht, die mit ihnen zusammen Wurzeln irreducibler Gleichungen sind, so ist dieses Product eben ein Ausdruck, wie er auf der rechten Seite von (23) bez. (24) erscheint; es kann also keiner seiner Factoren verschwinden. An der paarweisen Verschiedenheit der z_i, die beim Beweise vorausgesetzt wird, muss festgehalten werden; denn wäre z. B. $z_1 = z_2$, so wäre die Relation

$$N_1 e^{z_1} + N_2 e^{z_2} = 0$$

erfüllt, sobald $N_1 = -N_2$.

Eine genauere Darlegung der hier nur angedeuteten Beweise behalte ich mir für eine spätere Veröffentlichung vor.

Freiburg i. Br., April und Juni 1882.

Anmerkung. Litteraturnachweise über die Zahl π findet man auch in Baltzer's Elementen der Mathematik (Arithmetik, § 31., Planimetrie, § 13.). Den älteren Beweisen für die Irrationalität von π^2 hat Herr Hermite einen neuen hinzugefügt: Borchardt's Journal, Bd. 76, p. 342, 1873.

Zu LINDEMANN's Abhandlung: „Über die LUDOLPH'sche Zahl".

Von K. WEIERSTRASS.

(Vorgetragen am 22. October [s. oben S. 919].)

Für die Ergebnisse der in der genannten Abhandlung[1] mitgetheilten Untersuchungen des Hrn. LINDEMANN, namentlich für den darin enthaltenen, bis dahin vergebens erstrebten Beweis, dass die Zahl π keine algebraische Zahl und somit die Quadratur des Kreises auf constructivem Wege unausführbar ist, hat sich in den weitesten Kreisen ein so lebhaftes Interesse kundgegeben, dass ich glaube, es werde eine möglichst elementar gehaltene, nur auf allbekannte Sätze sich stützende Begründung der LINDEMANN'schen Theoreme, wie ich sie im Nachstehenden zu geben versuche, zahlreichen Mathematikern willkommen sein.[2]

[1] Diese Abhandlung, deren Inhalt durch die Überschrift nur unvollständig angegeben wird, ist von mir am 22. Juni 1882 der Akademie vorgelegt worden und bald darauf etwas weiter ausgeführt auch in den »Mathematischen Annalen« (Bd. XX) erschienen.

[2] Ich bemerke ausdrücklich, dass die Veröffentlichung dieses im Sommer 1882 entstandenen und seinem wesentlichen Inhalte nach am 26. October desselben Jahres in der Akademie vorgetragenen Aufsatzes im Einverständnisse mit Hrn. LINDEMANN erfolgt, und dass ich damit nur bezwecke, die von demselben gegebenen oder angedeuteten Beweise seiner Sätze ohne wesentliche Modification der leitenden Grundgedanken zu vereinfachen und zu vervollständigen. Das erstere erreiche ich hauptsächlich dadurch, dass ich nicht, wie Hr. LINDEMANN es thut, bei dem Leser die vollständige Kenntniss des Inhalts der berühmten HERMITE'schen Abhandlung »Sur la fonction exponentielle« (Compt. rend. 1873) voraussetze, sondern aus derselben nur die Methode zur Herleitung des in §. 1 unter Nr. VI aufgestellten Hülfssatzes entnehme.

Der Aufsatz ist übrigens im Manuscript mehreren befreundeten Mathematikern mitgetheilt und dann unter Benutzung der Bemerkungen, die mir einige von ihnen haben zukommen lassen, umgearbeitet worden. Namentlich hat mir Hr. H. A. SCHWARZ verschiedene redactionelle Änderungen vorgeschlagen, auf die ich gern eingegangen bin. Besonderen Dank schulde ich ferner Hrn. DEDEKIND für eine auf den Beweis des im Anfange des §. 3 aufgestellten Hülfssatzes sich beziehende Mittheilung, durch welche es mir möglich geworden ist, diesen Beweis wesentlich zu vereinfachen.

§. 1.

Hülfssätze aus der Theorie der Exponentialfunction.

I. Es bedeute $|z, \lambda|$ für jeden ganzzahligen, nicht negativen Werth von λ die durch die Gleichung

$$\frac{|z,\lambda|}{\lambda!} = \sum_{\varkappa=0}^{\lambda} \frac{z^\varkappa}{\varkappa!} \tag{1}$$

definirte ganze Function λten Grades des Arguments z, ferner sei

$$g(z) = \sum_{\lambda=0}^{l} b_\lambda z^\lambda \tag{2}$$

eine beliebige ganze Function von z, und es werde gesetzt:

$$G(z) = \sum_{\lambda=0}^{l} b_\lambda |z, \lambda|; \tag{3}$$

dann besteht die Gleichung

$$g(z)e^{-z}dz = d\left(-G(z)e^{-z}\right), \tag{4}$$

und es ist $G(z)$ eine ganze Function der Veränderlichen z von demselben Grade wie die Function $g(z)$.

Dabei ist zu bemerken, dass durch die vorstehende Gleichung die Function $G(z)$ eindeutig definirt wird.

II. Es seien

$$\begin{cases} f(z) = \sum\limits_{\nu=0}^{n+1} a_\nu z^{n+1-\nu}, \\ h(z) = \sum\limits_{\nu=0}^{n} c_\nu z^\nu \end{cases} \tag{5}$$

zwei ganze Functionen von z, beziehlich vom $(n+1)$ten und vom $n$ten Grade; die Coefficienten der zweiten sollen willkürlich anzunehmende Grössen und die Coefficienten der ersten nur der Beschränkung unterworfen sein, dass a_0 nicht gleich Null sein und die Function $f(z)$ mit ihrer ersten Ableitung $f'(z)$ keinen gemeinschaftlichen Theiler besitzen darf. Man bestimme, unter m eine beliebig anzunehmende positive ganze Zahl verstehend, auf die in (I.) angegebene Weise eine Reihe von ganzen Functionen

$$H_0(z), H_1(z), \ldots H_m(z), \tag{6}$$

welche den Gleichungen

$$\begin{cases} h(z)e^{-z}dz = d\left(-H_0(z)e^{-z}\right) \\ f(z)H_{\mu-1}(z)e^{-z}dz = d\left(-H_\mu(z)e^{-z}\right) \end{cases} \quad (\mu = 1, 2, \ldots m) \tag{7}$$

genügen; dann hat man, wenn

$$h_0(z) = h(z)\,,\quad h_\mu(z) = f'(z)\,H_{\mu-1}(z) \qquad (\mu = 1, 2, \ldots m) \tag{8}$$

gesetzt wird, für $\mu = 0, 1, \ldots m-1$,

$$d\left(-\frac{H_\mu(z)}{(m-\mu)!}\overset{m-\mu}{f(z)}e^{-z}\right) = \frac{h_\mu(z)}{(m-\mu)!}\overset{m-\mu}{f(z)}e^{-z}dz - \frac{h_{\mu+1}(z)}{(m-\mu-1)!}\overset{m-\mu-1}{f(z)}e^{-z}dz, \tag{9}$$

und (für $\mu = m$)

$$d\left(-H_m(z)e^{-z}\right) = h_m(z)e^{-z}. \tag{10}$$

Aus diesen $(m+1)$ Gleichungen ergiebt sich, wenn man sie durch Addition mit einander vereinigt:

$$\frac{h(z)}{m!}\overset{m}{f(z)}e^{-z}dz = d\sum_{\mu=0}^{m}\left\{-\frac{H_\mu(z)}{(m-\mu)!}\overset{m-\mu}{f(z)}e^{-z}\right\}. \tag{11}$$

Die Coefficienten der Functionen $|z, \lambda|$ sind, wie aus der Formel (1) unmittelbar erhellt, ganze Zahlen; die Coefficienten der durch die Gleichung (3) definirten Function $G(z)$ also ganze lineare Functionen der Grössen $b_0, b_1, \ldots b_l$ mit ganzzahligen Coefficienten. Demgemäss lehren die Formeln (7), dass jede der Functionen $H_0(z), H_1(z), \ldots H_m(z)$ eine ganze Function der Grössen

$$z, a_0, a_1, \ldots a_{n+1}, c_0, c_1, \ldots c_n$$

mit ganzzahligen Coefficienten ist.

III. **Die im Vorstehenden definirte Function $H_m(z)$ ist niemals durch die Function $f(z)$ theilbar, ausgenommen in dem Falle, wo die Grössen $c_0, c_1, \ldots c_n$ sämmtlich den Werth Null haben und somit jede der Functionen $H_0(z), H_1(z), \ldots H_m(z)$ sich für jeden Werth von z auf Null reducirt.**

Es werde gesetzt

$$\overline{H}_m(z) = \sum_{\mu=0}^{m}\left\{\frac{H_\mu(z)}{(m-\mu)!}\overset{m-\mu}{f(z)}\right\}, \tag{12}$$

so ist [nach Gleichung (11)]

$$\overline{H}_m(z) - \frac{d\overline{H}_m(z)}{dz} = \frac{h(z)}{m!}\overset{m}{f(z)}.$$

Wenn daher die Coefficienten der Function $h(z)$ nicht sämmtlich den Werth Null haben, so ist $\overline{H}_m(z)$ eine ganze Function von z, die nicht für jeden Werth dieser Grösse verschwindet und deren Grad (nach I.) nicht grösser ist als $m(n+1) + n = (m+1)(n+1) - 1$. Bezeichnet man also mit $\rho + 1$ die kleinste positive ganze Zahl, für welche $\overline{H}_m(z)$ nicht durch $\overset{\rho+1}{f(z)}$ theilbar ist, und setzt

$$\overline{H}_m(z) = H_m^*(z)\overset{\rho}{f(z)},$$

so ist $H_m^*(z)$ eine ganze Function von z, die nicht für jeden Werth dieser Grösse verschwindet, und $\rho \leqq m$. Man hat dann

$$\left(H_m^*(z) - \frac{dH_m^*(z)}{dz}\right) f(z) - \rho H_m^*(z) f'(z) = \frac{h(z)}{m!} f(z)^{m-\rho+1},$$

und es ist daher $\rho H_m^*(z) f'(z)$ durch $f(z)$ theilbar. Dies aber ist, da die Functionen $f(z)$, $f'(z)$ keinen gemeinschaftlichen Theiler besitzen und die Function $H_m^*(z)$ nicht durch $f(z)$ theilbar ist, nur der Fall für $\rho = 0$; es ist also $f(z)$ kein Theiler von $H_m(z)$ und somit, der Gleichung (12) zufolge, auch kein Theiler der Function $H_m(z)$.

IV. Bezeichnet man (unter der Annahme, dass $n > 1$ sei) mit z', z'' irgend zwei derjenigen Werthe von z, für welche $f(z) = 0$ wird, so ergiebt sich aus der Gleichung (11):

$$(13) \qquad H_m(z'')e^{-z''} - H_m(z')e^{-z'} = -\int_{z'}^{z''} \frac{h(z)}{m!} f(z)^m e^{-z} dz.$$

Aus den Gleichungen (7 und 1—4) erhellt, dass die Function $H_m(z)$, bei unbestimmten Werthen der Grössen $c_0, c_1, \ldots c_n$, vom $(mn+n)$ten Grade ist und dass in derselben die Coefficienten von

$$z^{mn+n},\; z^{mn+n-1},\; \ldots\; z^{mn+n-(m-1)}$$

beziehlich durch

$$a_0^m \;,\; a_0^{m-1} \;\; \ldots\; a_0$$

theilbar sind. Daraus folgt, dass die Function $a_0^{m(n-1)} H_m(z)$ sich auf die Form

$$(14) \qquad a_0^{m(n-1)} H_m(z) = G(z, m) f(z) + g(z, m)$$

bringen lässt, in der Art, dass $G(z, m)$, $g(z, m)$ beide ganze Functionen der Grössen $z, a_0, a_1, \ldots a_{n+1}, c_0, c_1, \ldots c_n$ mit ganzzahligen Coefficienten werden und $g(z, m)$ in Beziehung auf z von nicht höherem als dem $n$ten Grade ist. Die Gleichung (13) kann also in die folgende verwandelt werden:

$$(15) \quad g(z'', m)e^{-z''} - g(z', m)e^{-z'} = -\int_{z'}^{z''} \frac{h(z)\left(a_0^{n-1} f(z)\right)^m}{m!} e^{-z} dz.$$

Die Coefficienten der Functionen $H_m(z)$, $G(z, m)$, $g(z, m)$ sind, wie für die erste aus ihrer oben angegebenen Bildungsweise, für die beiden anderen aber aus der vorstehenden Definition derselben hervorgeht, homogene ganze lineare Funtionen der willkürlich anzunehmenden Grössen $c_0, c_1, \ldots c_n$. Wenn man daher, unter ν irgend eine der Zahlen $0, 1, \ldots n$ verstehend, mit $g_\nu(z, m)$ die Function bezeichnet, in welche $g(z, m)$ dadurch übergeht, dass man $c_\nu = 1$

und die übrigen der Grössen $c_0, c_1, \ldots c_n$ sämmtlich gleich Null annimmt, so hat man

$$(16) \qquad g(z, m) = \sum_{\nu=0}^{n} c_\nu g_\nu(z, m),$$

und es gehen aus der Gleichung (15) die folgenden $(n+1)$ Gleichungen hervor:

$$(17) \quad g_\nu(z'', m)e^{-z''} - g_\nu(z', m)e^{-z'} = -\int_{z'}^{z''} \frac{z^\nu\left(a_0^{n-1} f(z)\right)^m}{m!} e^{-z} dz. \qquad (\nu=0, 1, \ldots n)$$

Die Functionen $g_\nu(z, m)$ sind ganze Functionen der Grössen $z, a_0, a_1, \ldots a_{n+1}$ mit ganzzahligen Coefficienten. Bezeichnet man ferner mit $G_\nu(z, m)$ die Function, in welche $G(z, m)$ durch die Annahme

$$h(z) = z^\nu$$

übergeht, so ist

$$(18) \quad \begin{cases} G(z, m) = \sum\limits_{\nu=0}^{n} c_\nu G_\nu(z, m), \\ a_0^{m(n-1)} H_m(z) = f(z) \sum\limits_{\nu=0}^{n} c_\nu G_\nu(z, m) + \sum\limits_{\nu=0}^{n} c_\nu g_\nu(z, m), \end{cases}$$

und es ergiebt sich aus der letzten Gleichung, da $H_m(z)$ (nach III.) nur dann durch $f(z)$ theilbar ist, wenn die Grössen c_ν sämmtlich verschwinden, dass der Ausdruck

$$\sum_{\nu=0}^{n} c_\nu g_\nu(z, m)$$

nur dann für jeden Werth von z gleich Null ist, wenn jede der Grössen $c_0, c_1, \ldots c_n$ den Werth Null hat — mit anderen Worten, dass unter den in den Gleichungen (17) vorkommenden $(n+1)$ ganzen Functionen von z

$$g_0(z, m), g_1(z, m), \ldots g_n(z, m)$$

für keinen Werth von m eine lineare Abhängigkeit stattfindet.

Aus dieser Eigenschaft der Functionen $g_\nu(z, m)$ ergiebt sich ferner:

Giebt man der Grösse z irgend $(n+1)$ bestimmte Werthe

$$z_0, z_1, \ldots z_n,$$

unter denen keine zwei gleiche sich finden, so hat die Determinante

$$|g_\nu(z_\lambda, m)| \; (\lambda, \nu = 0, 1, \ldots n)$$

stets einen von Null verschiedenen Werth.

Wäre nämlich diese Determinante gleich Null, so würden sich $(n+1)$ Grössen $c_0, c_1, \ldots c_n$ so bestimmen lassen, dass die $(n+1)$ Gleichungen

$$\sum_{\nu=0}^{n} c_\nu g_\nu(z_\lambda, m) = 0 \qquad (\lambda=0, 1, \ldots n)$$

beständen, ohne dass sämmtliche Grössen $c_0, c_1, \ldots c_n$ den Werth Null hätten. Dann würde aber der Ausdruck

$$\sum_{\nu=0}^{n} c_\nu g_\nu(z, n),$$

der eine ganze Function der Veränderlichen z von nicht höherem als dem $n$ten Grade ist, für jeden Werth von z verschwinden, was nach dem Vorhergehenden nicht stattfinden kann.

V. Da

$$\frac{z^\nu\left(a_0^{n-1}f(z)\right)^m}{m!}e^{-z}$$

eine (transcendente) ganze Function von z ist, so hat das Integral auf der Rechten der Gleichung (17) bei gegebenen Werthen von m, ν, z', z'' einen von dem Integrationswege unabhängigen, eindeutig bestimmten Werth, den man, unter τ eine auf das Intervall $(0 \ldots 1)$ beschränkte reelle Veränderliche verstehend, in der Form

$$\int_0^1 \frac{(z''-z')(z'+(z''-z')\tau)^\nu\left(a_0^{n-1}f(z'+(z''-z')\tau)\right)^m}{m!}e^{-z'-(z''-z')\tau}d\tau$$

darstellen kann. Es lässt sich aber, nach Annahme einer beliebig kleinen positiven Grösse δ, eine ganze Zahl M so bestimmen, dass für jeden Werth von m, der grösser als M ist, und für jeden der betrachteten Werthe von τ

$$\left|\frac{(z''-z')(z'+(z''-z')\tau)^\nu\left(a_0^{n-1}f(z'+(z''-z')\tau)\right)^m}{m!}e^{-z'-(z''-z')\tau}\right|<\delta$$

und somit auch, nach einem bekannten Satze, der absolute Betrag des Integrals

$$\int_{z'}^{z''}\frac{z^\nu\left(a_0^{n-1}f(z)\right)^m}{m!}e^{-z}dz$$

kleiner als δ ist. Aus der Gleichung (17) ergiebt sich also, wenn man beide Seiten derselben mit

$$e^{z''+z'}$$

multiplicirt

$$(19) \qquad \operatorname{Lim.}_{m=\infty}\left\{g_\nu(z', m)e^{z''}-g_\nu(z'', m)e^{z'}\right\}=0. \qquad (\nu=0, 1, \ldots n)$$

VI. Bis jetzt sind die Coefficienten der Function $f(z)$ keiner anderen Beschränkung als der oben (unter II.) angegebenen unterworfen worden. Setzt man aber nunmehr noch fest, dass dieselben sämmtlich gegebene ganze Zahlen sein sollen, so werden, nach dem oben (IV.) Bemerkten, für jeden Werth der Zahl m auch die Coefficienten der Functionen $g_\nu(z, m)$ sämmtlich ganze Zahlen. Man kann ferner, wenn $z_0, z_1, \ldots z_n$ die Wurzeln der Gleichung $f(z) = 0$ sind, den Gleichungen (19) gemäss m so gross annehmen, dass jede der Differenzen

$$g_\nu(z_0, m)\, e^{z_\lambda} - g_\nu(z_\lambda, m)\, e^{z_0} \qquad \begin{pmatrix}\lambda = 0, 1, \ldots n\\ \nu = 0, 1, \ldots n\end{pmatrix}$$

ihrem absoluten Betrage nach kleiner ist, als eine beliebig klein angenommene positive Grösse δ. Zugleich hat dann die Determinante

$$|g_\nu(z_\lambda, m)| \quad (\lambda, \nu = 0, 1 \ldots n)$$

einen von Null verschiedenen Werth, indem unter den Grössen $z_0, z_1, \ldots z_n$ keine zwei gleiche sich finden.

Damit ist ein Satz bewiesen, der im Folgenden hauptsächlich zur Begründung der LINDEMANN'schen Theoreme dienen wird und sich so aussprechen lässt:

»Es sei $f(z)$ eine ganze Function $(n+1)$ Grades der Veränderlichen z mit gegebenen ganzzahligen Coefficienten, die so beschaffen sind, dass die Gleichung

$$f(z) = 0$$

$(n+1)$ von einander verschiedene Wurzeln hat, welche mit

$$z_0, z_1, \ldots z_n$$

bezeichnet werden mögen. Alsdann lässt sich, nach Annahme einer beliebig kleinen positiven Grösse δ, auf manigfaltige Weise ein System von $(n+1)$ ganzen Functionen

$$g_0(z), g_1(z), \ldots g_n(z)$$

des Arguments z von nicht höherem als dem $n$ten Grade, deren Coefficienten sämmtlich ganze Zahlen sind, so bestimmen, dass erstens jede der Differenzen

$$g_\nu(z_0)\, e^{z_\lambda} - g_\nu(z_\lambda)\, e^{z_0} \qquad \begin{pmatrix}\lambda = 0, 1, \ldots n\\ \nu = 0, 1, \ldots n\end{pmatrix}$$

ihrem absoluten Betrage nach kleiner als δ ist, und zweitens die Determinante

$$|g_\nu(z_\lambda)| \quad (\lambda, \nu = 0, 1, \ldots n)$$

einen von Null verschiedenen Werth hat.«

§. 2.

Beweis, dass die LUDOLPH'sche Zahl π eine transcendente Zahl ist.

Da $e^{\pi i} = -1$ ist und die Function e^x nur für solche Werthe ihres Arguments, welche (ungrade) Vielfache von πi sind, den Werth -1 annimmt, so kann dem zu beweisenden Satze der folgende substituirt werden:

»Die Grösse e^x+1 hat, wenn x eine algebraische Zahl ist, stets einen von Null verschiedenen Werth.«

Es bedeute x_1 irgend eine gegebene algebraische Zahl. Dieselbe sei Wurzel einer Gleichung $r$ten Grades:

$$x^r + c_1 x^{r-1} + \ldots + c_r = 0,$$

deren Coefficienten sämmtlich rationale Zahlen sind — wobei angenommen werden darf, dass $r > 1$ sei, weil für $r = 1$ $e^{x_1} = e^{-c_1}$ und somit eine positive Grösse ist. Die vorstehende Gleichung hat dann ausser x_1 noch $(r-1)$ andere Wurzeln: werden diese mit $x_2, \ldots x_r$ bezeichnet, so ist es, damit der aufgestellte Satz bestehe, nothwendig und hinreichend, dass die Grösse

$$\prod_{\lambda=1}^{r}(e^{x_\lambda}+1)$$

einen von Null verschiedenen Werth habe. Dies aber lässt sich folgendermaassen zeigen:

Es seien $\xi_1, \xi_2, \ldots \xi_r$ von einander unabhängige Veränderliche, so hat man

$$\prod_{\lambda=1}^{r}(e^{\xi_\lambda}+1) = \sum_{\varepsilon_1, \varepsilon_2 \ldots \varepsilon_r} e^{\varepsilon_1\xi_1 + \varepsilon_2\xi_2 + \ldots + \varepsilon_r\xi_r}, \qquad (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_r = 0, 1)$$

oder, wenn man $2^r = p$ setzt und die p Functionen

$$\varepsilon_1\xi_1 + \varepsilon_2\xi_2 + \ldots + \varepsilon_r\xi_r, \qquad (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_r = 0, 1)$$

in irgend einer Ordnung genommen, mit $\zeta_0, \zeta_1, \ldots \zeta_{p-1}$ bezeichnet,

$$\prod_{\lambda=1}^{r}(e^{\xi_\lambda}+1) = \sum_{\mu=0}^{p-1} e^{\zeta_\mu}.$$

Sind also

$$z_0, z_1, \ldots z_{p-1}$$

die Werthe, welche

$$\zeta_0, \zeta_1, \ldots \zeta_{p-1}$$

dadurch annehmen, dass man

$$\xi_1 = x_1, \; \xi_2 = x_2, \ldots \xi_r = x_r$$

setzt, so ist

$$\prod_{\lambda=1}^{r}(e^{x_\lambda}+1) = \sum_{\mu=0}^{p-1} e^{z_\mu}.$$

Die Anzahl der von einander verschiedenen Werthe, welche in der Reihe $z_0, z_1, \ldots z_{p-1}$ vorhanden sind, sei $n+1$, und es mögen die Ausdrücke $\zeta_0, \zeta_1, \ldots \zeta_{p-1}$ so geordnet sein, dass unter den $(n+1)$ Grössen $z_0, z_1, \ldots z_n$ keine zwei gleiche sich finden, und $z_0 = 0$ ist; wobei zu bemerken, dass $n+1 > 1$ ist, weil in der Reihe $z_0, z_1, \ldots z_{p-1}$ auch die Grössen $x_1, \ldots x_r$ enthalten sind. Dann kann man, unter z eine unbestimmte Grösse verstehend, eine ganze Function $f(z)$ vom Grade $n+1$ herstellen, welche nur ganzzahlige Coefficienten hat und für $z = z_0, z_1, \ldots z_n$ verschwindet. Es kann nämlich das Product

$$\prod_{\mu=0}^{p-1}(z - \zeta_\mu)$$

dargestellt werden als ganze Function der Grössen $z, \xi_1, \xi_2, \ldots \xi_r$ mit ganzzahligen Coefficienten, welche sich nicht ändert, wenn die Grössen $\xi_1, \xi_2, \ldots \xi_r$ in beliebiger Weise permutirt werden, und sich somit, wenn man für $\xi_1, \xi_2, \ldots \xi_r$ die Wurzeln einer Gleichung $r$ten Grades mit lauter rationalen Zahlcoefficienten einsetzt, in eine ganze Function $p$ten Grades von z mit eben solchen Coefficienten verwandelt. Es lässt sich also $\prod_{\mu=0}^{p-1}(z - z_\mu)$ als ganze Function von z mit lauter rationalen Zahlcoefficienten darstellen; dividirt man diese Function dann durch den grössten Theiler, den sie mit ihrer ersten Ableitung gemein hat, so ist der Quotient eine ganze Function $(n+1)$ten Grades von z, aus der man, indem man sie mit einer passenden ganzen Zahl multiplicirt, eine Function

$$f(z) = a_0 z^{n+1} + a_1 z^n + \ldots + a_{n+1}$$

erhält, welche lauter ganzzahlige Coefficienten hat und für $z = z_0, z_1, \ldots z_n$ verschwindet.

Nachdem diese Function $f(z)$ hergestellt worden, kann man aus ihr, nach Annahme einer beliebig kleinen positiven Grösse δ, ein System von $n+1$ ganzen Functionen

$$g_0(z), g_1(z), \ldots g_n(z)$$

von der im Schlusssatz (VI.) des vorigen Paragraphen angegebenen Beschaffenheit ableiten, so dass, wenn man

$$g_\nu(0)e^{z_\lambda} - g_\nu(z_\lambda) = \varepsilon_{\lambda,\nu}\delta \qquad \begin{pmatrix}\lambda = 0, 1, \ldots n \\ \nu = 0, 1, \ldots n\end{pmatrix}$$

setzt, jede der Grössen $\varepsilon_{\lambda,\nu}$ ihrem absoluten Betrage nach kleiner als 1 ist.

Die Grössen $g_\nu(z_\lambda)$ sind sämmtlich algebraische Zahlen; multiplicirt man jede derselben mit a_0^n, so verwandeln sie sich alle in ganze algebraische Zahlen.[1]

---

[1] Nach der von Hrn. KRONECKER eingeführten Terminologie heisst eine Grösse x eine **algebraische Zahl**, wenn sie einer algebraischen Gleichung von der Form

$$x^r + A_1 x^{r-1} + \ldots + A_r = 0,$$

in der die Coefficienten $A_1, \ldots A_r$ sämmtlich **rationale** Zahlen sind, genügt. Sind insbesondere diese Coefficienten sämmtlich ganze Zahlen, so wird x eine **ganze**

Nimmt man nun δ so klein an, dass $|(p-1)\,a_0^n\delta| < 1$ ist, so ergiebt sich aus der vorstehenden Gleichung

$$a_0^n g_\nu(0)\sum_{\mu=0}^{p-1} e^{z_\mu} = \sum_{\mu=0}^{p-1} a_0^n g_\nu(z_\mu) + \varepsilon_\nu, \qquad (\nu=0,1\ldots n)$$

wo jede der Grössen ε_ν dem absoluten Betrage nach kleiner als 1 ist.

Es ist aber

$$\sum_{\mu=0}^{p-1} a_0^n g_\nu(\zeta_\mu)$$

für jeden Werth von ν eine symmetrische ganze Function der Grössen $\xi_1, \xi_2, \ldots \xi_r$ mit ganzzahligen Coefficienten; also ist die ganze algebraische Zahl

$$\sum_{\mu=0}^{p-1} a_0^n g_\nu(z_\mu)$$

zugleich eine rationale Zahl, d. h. sie ist eine ganze Zahl im gewöhnlichen Sinne. Ferner lässt sich zeigen, dass die $n+1$ Zahlen

$$\sum_{\mu=0}^{p-1} a_0^n g_\nu(z_\mu) \qquad (\nu=0,1,\ldots,n)$$

nicht sämmtlich den Werth Null haben. Man hat nämlich

$$\sum_{\mu=0}^{p-1} a_0^n g_\nu(z_\mu) = \sum_{\lambda=0}^{n} N_\lambda g_\nu(z_\lambda),$$

wo $N_0, N_1, \ldots N_n$ sämmtlich positive ganze Zahlen sind; es müsste also, wenn die in Rede stehenden Zahlen alle den Werth Null haben sollten, die Determinante

$$|g_\nu(z_\lambda)| \quad (\lambda, \nu = 0, 1, \ldots, n)$$

gleich Null sein, was nicht der Fall ist.

Es giebt also mindestens einen bestimmten Werth von ν, für welchen

$$\sum_{\mu=0}^{p-1} a_0^n g_\nu(z_\mu)$$

eine von Null verschiedene ganze rationale Zahl, und somit

$$\sum_{\mu=0}^{p-1} a_0^n g_\nu(z_\mu) + \varepsilon_\nu = a_0^n g_\nu(0)\sum_{\mu=0}^{p-1} e^{z_\mu} = a_0^n g_\nu(0)\prod_{\lambda=1}^{r}(e^{x_\lambda}+1)$$

nicht gleich Null ist.

---

algebraische Zahl genannt, und es ist in diesem Falle auch jede ganze rationale Function von x mit lauter ganzzahligen Coefficienten eine ganze algebraische Zahl.

Hiernach sind, da

$$a_0^n f(z) = (a_0 z)^{n+1} + a_1(a_0 z)^n + \ldots + a_n a_0^{n-1}(a_0 z) + a_{n+1}a_0^n$$

ist, $a_0 z_0, a_0 z_1, \ldots a_0 z_n$ sämmtlich ganze algebraische Zahlen, und dasselbe gilt also auch, da der Grad der Function $g_\nu(z)$ nicht grösser als n ist, von jeder der Grössen $a_0^n g_\nu(z_\lambda)$.

Daraus folgt unmittelbar, dass das Product

$$\prod_{\lambda=1}^{r}(e^{x_\lambda}+1)$$

und somit auch jeder einzelne Factor desselben einen von Null verschiedenen Werth hat; w. z. b. w.

Damit ist, dem oben Bemerkten gemäss, dargethan, dass die Zahl πi und daher auch π selbst eine transcendente Zahl ist.

Als ein Corollar zu diesem Satze ergiebt sich, dass die »Quadratur des Kreises« eine unlösbare Aufgabe ist, wenn verlangt wird, dass sie durch eine geometrische Construction, bei der nur algebraische Curven und Flächen zur Anwendung kommen, bewerkstelligt werde. (Vergl. den Schlusssatz des §. 3)

§. 3.

Allgemeinere, auf die Exponentialfunction sich beziehende Sätze.

I. Zunächt ist der folgende Hülfssatz zu beweisen. Es seien gegeben $k+1$ Reihen von je r Grössen:

$$\begin{array}{ll}
(1) & A'_1\ ,\ A'_2\ ,\ \ldots A'_r \\
(2) & A''_1\ ,\ A''_2\ ,\ \ldots A''_r \\
\vdots & \ldots\ldots\ldots\ldots \\
(k) & A^{(k)}_1,\ A^{(k)}_2,\ \ldots A^{(k)}_r \\
(k+1) & x_1\ ,\ x_2\ ,\ \ldots x_r\ ,
\end{array}$$

wobei angenommen werde, dass in jeder der k ersten Reihen wenigstens eine Grösse vorkomme, die einen von Null verschiedenen Werth hat, und in der letzten Reihe keine zwei gleiche Grössen sich finden. Man bilde das Product

$$\sum_{\mathfrak{a}=1}^{r} A'_{\mathfrak{a}} e^{x_{\mathfrak{a}}} \cdot \sum_{\mathfrak{b}=1}^{r} A''_{\mathfrak{b}} e^{x_{\mathfrak{b}}} \ldots \sum_{\mathfrak{k}=1}^{r} A^{(k)}_{\mathfrak{k}} e^{x_{\mathfrak{k}}}$$

und bringe dasselbe, welches mit P bezeichnet werde, auf die Form[1]

$$P = \sum_{\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}} A'_{\mathfrak{a}} A''_{\mathfrak{b}} \ldots A^{(k)}_{\mathfrak{k}} e^{x_{\mathfrak{a}}+x_{\mathfrak{b}}+\cdots x_{\mathfrak{k}}}. \qquad (\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}=1,\ldots r)$$

---

[1] Es ist nothwendig, dass bei dieser Umformung von P das Argument der Exponentialgrösse, durch welche das Product aus den Factoren

$$e^{x_{\mathfrak{a}}},\ e^{x_{\mathfrak{b}}},\ \ldots e^{x_{\mathfrak{k}}}$$

dargestellt wird, gleich $x_{\mathfrak{a}}+x_{\mathfrak{b}}+\cdots+x_{\mathfrak{k}}$ genommen werde; und nicht etwa, was an sich gestattet wäre, gleich dieser Summe plus einem Vielfachen von $2\pi i$. Ohne diese Festsetzung würden die im Folgenden einzuführenden Grössen C_λ nicht gehörig bestimmt sein.

Aus der Gesammtheit der Werthe, welche die aus k Gliedern bestehende Summe

$$x_{\mathfrak{a}} + x_{\mathfrak{b}} + \ldots + x_{\mathfrak{k}}$$

annimmt, wenn man für $\mathfrak{a}, \mathfrak{b}, \ldots \mathfrak{k}$ alle möglichen Systeme von k der Reihe $1, 2, \ldots r$ entnommenen Zahlen setzt, hebe man die von einander verschiedenen, deren Anzahl $n+1$ sein möge, heraus; bezeichnet man dieselben mit $z_0, z_1, \ldots z_n$, so ergiebt sich

$$P = \sum_{\lambda=0}^{n} C_\lambda e^{z_\lambda},$$

wo für jeden bestimmten Werth von λ

$$C_\lambda = \sum_{\mathfrak{a}, \mathfrak{b}, \ldots \mathfrak{k}}' A'_{\mathfrak{a}} A''_{\mathfrak{b}} \ldots A^{(k)}_{\mathfrak{k}}$$

ist, unter der Bedingung, dass die Summation sich über diejenigen Werthsysteme $\mathfrak{a}, \mathfrak{b}, \ldots \mathfrak{k}$, für welche $x_{\mathfrak{a}} + x_{\mathfrak{b}} + \ldots + x_{\mathfrak{k}} = z_\lambda$ ist, erstrecke. Es ist nun zu beweisen, dass unter den so definirten Grössen C_λ mindestens eine sich findet, die nicht gleich Null ist.

Die Grössen $x_1, \ldots x_r$ können sowohl complexe als reelle Werthe haben. Man betrachte eine complexe Grösse $t + t'i$ (wo t, t' reelle Grössen bezeichnen) als positiv, wenn $t > 0$ oder auch $t = 0, t' > 0$, dagegen als negativ, wenn $t < 0$ oder auch $t = 0, t' < 0$ ist; dann darf man, da das Product P seinen Werth nicht ändert, wenn in dem gegebenen Grössensysteme irgend zwei Colonnen unter einander vertauscht werden, voraussetzen, es seien die Grössen $x_1, x_2, \ldots x_r$ so geordnet, dass die Differenzen

$$x_1 - x_2, x_2 - x_3, \ldots x_{r-1} - x_r$$

sämmtlich positive Grössen sind.

Dies vorausgesetzt, nehme man aus jeder der obigen Reihen $(1), (2), \ldots (k)$ die erste Grösse, welche nicht den Werth Null hat, heraus; diese sei A'_α in der ersten Reihe, A''_β in der zweiten, $\ldots, A^{(k)}_\varkappa$ in der $k$ten, so ist

$$A'_\alpha A''_\beta \ldots A^{(k)}_\varkappa e^{x_\alpha + x_\beta + \cdots + x_\varkappa}$$

eines der Glieder, aus denen P zusammengesetzt wird. Irgend ein anderes Glied, dessen Coefficient nicht den Werth Null hat, sei

$$A'_{\mathfrak{a}} A''_{\mathfrak{b}} \ldots A^{(k)}_{\mathfrak{k}} e^{x_{\mathfrak{a}} + x_{\mathfrak{b}} + \cdots + x_{\mathfrak{k}}},$$

so ist

$$\mathfrak{a} \geqq \alpha, \mathfrak{b} \geqq \beta, \ldots \mathfrak{k} \geqq \varkappa,$$

und es giebt unter den Zahlen $\mathfrak{a}, \mathfrak{b}, \ldots \mathfrak{k}$ mindestens eine, welche grösser ist, als die entsprechende der Zahlen $\alpha, \beta, \ldots \varkappa$. Folglich ist

$(x_a+x_b+\ldots+x_t)-(x_\alpha+x_\beta+\ldots+x_\nu)=(x_a-x_\alpha)+(x_b-x_\beta)+\ldots+(x_t-x_\nu)$ eine positive Grösse und somit $(x_a+x_b+\ldots+x_t)$ nicht gleich $(x_\alpha+x_\beta+\ldots+x_\nu)$. Es findet sich also unter den Gliedern

$$A'_a A''_b \ldots A^{(k)}_t e^{x_a+x_b+\ldots+x_t}$$

keines, in welchem das Argument der Exponentialgrösse denselben Werth hätte, wie in dem Gliede

$$A'_\alpha A''_\beta \ldots A^{(k)}_\nu e^{x_\alpha+x_\beta+\ldots+x_\nu};$$

der Coefficient des letzteren ist also eine der Grössen C_λ, unter denen sich hiernach unter allen Umständen eine findet, welche nicht gleich Null ist.[1]

II. Nunmehr seien, wie in §. 2, $x_1, x_2, \ldots x_r$ die Wurzeln einer Gleichung $r$ten Grades

$$x^r+c_1x^{r-1}+\ldots+c_r=0,$$

deren Coefficienten sämmtlich gegebene rationale Zahlen sind, und deren Discriminante einen von Null verschiedenen Werth hat. Ferner seien $N_1, N_2, \ldots N_r$ gegebene ganze Zahlen, unter denen wenigstens eine nicht gleich Null ist. Dann lässt sich beweisen, dass die Summe

$$\sum_{\varrho=1}^{r} N_\varrho e^{x_\varrho}$$

niemals den Werth Null hat.

Man bezeichne die Grössen, welche aus der vorstehenden Summe dadurch hervorgehen, dass man in derselben die Grössen $x_1, \ldots x_r$ auf jede mögliche Weise permutirt, mit

$$X', X'', \ldots X^{(k)},$$

wo $k=r!$ ist, so hat, wenn n irgend eine der Zahlen $1, 2, \ldots k$ bedeutet, $X^{(n)}$ die Form

$$X^{(n)}=\sum_{\varrho=1}^{r} N_\varrho^{(n)} e^{x_\varrho},$$

wo die Reihe der Zahlen $N_1^{(n)}, N_2^{(n)}, \ldots N_r^{(n)}$ aus der Reihe $N_1, N_2, \ldots N_r$ durch eine bestimmte Permutation der Glieder der letzteren hervorgeht. Setzt man dann

$$P=\prod_{n=1}^{k} X^{(n)},$$

[1] Dass man in der Reihe der Grösse $C_0, C_1, \ldots C_n$ eine, die nicht gleich Null ist, ohne Weiteres auf die angegebene Weise ermitteln könne, wenn man die Grössen $x_1, \ldots x_r$ so auf einander folgen lässt, dass die Differenzen $x_1-x_2, x_2-x_3, \ldots x_{n-1}-x_n$ sämmtlich positive Werthe (in dem festgesetzten Sinne) erhalten, ist eine Bemerkung, die ich Hrn. DEDEKIND verdanke.

so ist

$$(1) \qquad P = \sum_{a, b, \ldots t} N'_a N''_b \ldots N_t^{(k)} e^{x_a + x_b + \cdots + x_t}, \qquad (a, b, \ldots t = 1, 2, \ldots r)$$

woraus sich, wenn man die von einander verschiedenen Werthe der Grössen

$$x_a + x_b + \ldots + x_t, \qquad (a, b, \ldots t = 1, \ldots r)$$

wie in (I.), mit $z_0, z_1, \ldots z_n$ bezeichnet,

$$(2) \qquad P = \sum_{\lambda=0}^{n} C_\lambda e^{z_\lambda}$$

ergiebt, wo jetzt die Grössen C_λ sämmtlich ganze Zahlen sind, von denen, dem in (I.) Bewiesenen gemäss, wenigstens eine nicht gleich Null ist.

Mittels der gegebenen Gleichung, deren Wurzeln $x_1, x_2, \ldots x_r$ sind, kann man nun eine Gleichung $(n+1)$ten Grades mit lauter ganzzahligen Coefficienten herstellen, deren Wurzeln die Grössen $z_0, z_1, \ldots z_n$ sind. Bildet man nämlich, unter $z, \xi_1, \xi_2, \ldots \xi_r$ von einander unabhängige Veränderliche verstehend, aus den durch die Formel

$$z - (\xi_a + \xi_b + \ldots + \xi_t) \qquad (a, b, \ldots t = 1, \ldots r)$$

repräsentirten Grössen ein Product, so ist dasselbe eine ganze Function von $z, \xi_1, \ldots \xi_r$ mit lauter ganzzahligen Coefficienten, und zugleich eine symmetrische Function von $\xi_1, \xi_2, \ldots \xi_r$, verwandelt sich also, wenn man $\xi_1 = x_1, \xi_2 = x_2, \ldots \xi_r = x_r$ setzt, in eine ganze Function von z mit lauter rationalen Zahlcoefficienten, welche für $z = z_0, z_1, \ldots z_n$, und zwar nur für diese Werthe von z, verschwindet. Dividirt man dann diese Function durch den grössten Theiler, den sie mit ihrer ersten Ableitung gemein hat, so ist der Quotient eine ganze Function $(n+1)$ten Grades, aus der man, indem man sie mit einer passenden ganzen Zahl multiplicirt, eine ganze Function

$$(3) \qquad f(z) = a_0 z^{n+1} + a_1 z^n + \ldots + a_{n+1}$$

erhält, welche lauter ganzzahlige Coefficienten hat und für $z = z_0, z_1, \ldots z_n$ verschwindet.

Aus dieser Function $f(z)$ kann man nun, nach Annahme einer beliebig kleinen positiven Grösse δ, ein System von $(n+1)$ ganzen Functionen

$$g_0(z), g_1(z), \ldots g_n(z)$$

von der im Schlusssatze (VI.) des §. 1 angegebenen Beschaffenheit ableiten, so dass, wenn

$$(4) \qquad g_\nu(z_0) e^{z_\lambda} - g_\nu(z_\lambda) e^{z_0} = \varepsilon_{\lambda,\nu} \delta \qquad \binom{\lambda = 0, 1, \ldots n}{\nu = 0, 1, \ldots n}$$

gesetzt wird, jede der Grössen $\varepsilon_{\lambda,\nu}$ ihrem absoluten Betrage nach kleiner als 1 ist. Dann ergiebt sich aus dem obigen Ausdrucke der Grösse P

$$(5) \qquad a_0^n g_\nu(z_0) e^{-z_0} P = \sum_{\lambda=0}^{n} C_\lambda a_0^n g_\nu(z_\lambda) + a_0^n e^{-z_0} \delta \sum_{\lambda=0}^{n} \varepsilon_{\lambda,\nu} C_\lambda. \qquad (\nu=0,1\ldots n)$$

Hier ist nun jede der Grössen $a_0^n g_\nu(z_\lambda)$ eine ganze algebraische Zahl; es lässt sich aber zeigen, dass jede der $(n+1)$ Summen

$$\sum_{\lambda=0}^{n} C_\lambda a_0^n g_\nu(z_\lambda)$$

eine ganze rationale Zahl, d. h. also eine ganze Zahl im gewöhnlichen Sinne ist.

Versteht man wieder unter $z, \xi_1, \xi_2, \ldots \xi_r$ von einander unabhängige Veränderliche, so ist das Product

$$\sum_{\mathfrak{a}=1}^{r} N'_{\mathfrak{a}} e^{\xi_{\mathfrak{a}}} \cdot \sum_{\mathfrak{b}=1}^{r} N''_{\mathfrak{b}} e^{\xi_{\mathfrak{b}}} \ldots \sum_{\mathfrak{k}=1}^{r} N_{\mathfrak{k}}^{(k)} e^{\xi_{\mathfrak{k}}}$$

und somit auch die Summe

$$\sum_{\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}} N'_{\mathfrak{a}} N''_{\mathfrak{b}} \ldots N_{\mathfrak{k}}^{(k)} e^{\xi_{\mathfrak{a}} + \xi_{\mathfrak{b}} + \cdots + \xi_{\mathfrak{k}}} \qquad (\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}=1,\ldots r)$$

eine symmetrische Function von $\xi_1, \xi_2, \ldots \xi_r$. Entwickelt man dieselbe in eine Potenzreihe von $\xi_1, \xi_2, \ldots \xi_r$, so ist die Summe der Glieder $\mathfrak{m}$ter Dimension dieser Reihe, nämlich

$$\frac{1}{\mathfrak{m}!} \sum_{\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}} N'_{\mathfrak{a}} N''_{\mathfrak{b}} \ldots N_{\mathfrak{k}}^{(k)} (\xi_{\mathfrak{a}} + \xi_{\mathfrak{b}} + \ldots + \xi_{\mathfrak{k}})^{\mathfrak{m}}, \qquad (\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}=1,\ldots r)$$

und daher auch, wenn $g(z)$ eine beliebige ganze Function von z ist,

$$\sum_{\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}} N'_{\mathfrak{a}} N''_{\mathfrak{b}} \ldots N_{\mathfrak{k}}^{(k)} g(\xi_{\mathfrak{a}} + \xi_{\mathfrak{b}} + \cdots + \xi_{\mathfrak{k}}) \qquad (\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}=1,\ldots r)$$

eine symmetrische ganze rationale Function der Grössen $\xi_1, \xi_2, \ldots \xi_r$. Die letztere erhält also, wenn man (für $\nu = 0, 1, \ldots n$) $g(z) = g_\nu(z)$ annimmt und $\xi_1 = x_1, \xi_2 = x_2, \ldots \xi_r = x_r$ setzt, einen rationalen Werth, der sich durch Multiplication mit a_0^n in eine ganze Zahl verwandelt. Es ist aber nach dem Obigen, für jeden bestimmten Werth von λ

$$(6) \qquad C_\lambda = \sum_{\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}}' N'_{\mathfrak{a}} N''_{\mathfrak{b}} \ldots N_{\mathfrak{k}}^{(k)},$$

wenn die Summation über diejenigen Zahlensysteme, für welche $x_{\mathfrak{a}} + x_{\mathfrak{b}} + \ldots + x_{\mathfrak{k}} = z_\lambda$ ist, erstreckt wird, und daher

$$(7) \qquad \sum_{\lambda=0}^{n} C_\lambda a_0^n g_\nu(z_\lambda) = \sum_{\mathfrak{a},\mathfrak{b},\ldots\mathfrak{k}} N'_{\mathfrak{a}} N''_{\mathfrak{b}} \ldots N_{\mathfrak{k}}^{(k)} a_0^n g_\nu(x_{\mathfrak{a}} + x_{\mathfrak{b}} + \ldots + x_{\mathfrak{k}}),$$

also auch $\sum_{\lambda=0}^{n} C_\lambda a_0^n g_\nu(z_\lambda)$ eine ganze rationale Zahl.

Es lässt sich aber auch zeigen, dass diese Zahl wenigstens für einen Werth von ν einen von Null verschiedenen Werth hat. Da nämlich die Grössen C_λ (nach I.) nicht sämmtlich gleich Null sind, so müsste, wenn alle $(n+1)$ Zahlen

$$\sum_{\lambda=0}^{n} C_\lambda a_0^n g_\lambda(z_\nu) \qquad (\nu=0,1,\ldots n)$$

den Werth Null hätten, die Determinante

$$|g_\nu(z_\lambda)| \quad (\lambda, \nu = 0, 1, \ldots n)$$

gleich Null sein, was nicht der Fall ist.

Nimmt man also die Grösse δ so klein an, dass jede der Grössen

$$a_0^n e^{-z_0} \delta \sum_{\lambda=0}^{n} \varepsilon_{\lambda,\nu} C_\lambda \qquad (\nu=0,1,\ldots n)$$

dem absoluten Betrage nach kleiner als 1 ist, so findet sich unter den Grössen auf der Rechten der Gleichungen (5) wenigstens eine, die nicht gleich Null ist; woraus sich unmittelbar ergiebt, dass das Product P. und somit auch die Grösse

$$\sum_{t=1}^{r} N_t e^{x_t},$$

welche ein Factor dieses Productes ist, einen von Null verschiedenen Werth hat; was zu beweisen war.

Selbstverständlich gilt dieser Satz auch, wenn unter $N_1, N_2, \ldots N_r$ rationale Zahlen verstanden werden.

Nimmt man für $x_1, x_2, \ldots x_r$ irgend r von einander verschiedene ganze Zahlen an, so ergiebt sich als ein besonderer Fall des vorstehenden Theorems der von Hermite bewiesene Satz, dass die Zahl e keine algebraische Zahl ist.

Eine naheliegende Verallgemeinerung des Theorems ergiebt sich folgendermaassen:

Sind $x_1, x_2, \ldots x_r$ irgend r gegebene, von einander verschiedene algebraische Zahlen, so lässt sich stets eine algebraische Gleichung — im Allgemeinen von höherem als dem $r$ten Grade — mit lauter rationalen Zahlcoefficienten und nicht verschwindender Determinante herstellen, unter deren Wurzeln die gegebenen Grössen $x_1, x_2, \ldots x_r$ sich finden. Ist der Grad dieser Gleichung gleich r, so sind $x_1, x_2, \ldots x_r$ solche r Grössen, für welche das bewiesene Theorem unmittelbar gilt. Hat die Gleichung aber ausser $x_1, x_2, \ldots x_r$ noch l andere Wurzeln:

$$x_{r+1}, \ldots x_{r+l},$$

und werden unter $N_1, N_2, \ldots N_{r+l}$ rationale Zahlen verstanden, so ist

$$\sum_{t=1}^{r+l} N_t e^{x_t}$$

nur in dem Falle gleich Null, wo $N_1, N_2, \ldots N_{r+l}$ sämmtlich den

Werth Null haben. Nimmt man also für jeden Werth von ρ, der grösser als r ist, $N_\rho = 0$ an, so ergiebt sich:

Werden unter $x_1, x_2, \ldots x_r$ irgend r von einander verschiedene algebraische Zahlen, unter $N_1, N_2, \ldots N_r$ aber beliebige rationale Zahlen verstanden, so kann die Gleichung

$$\sum_{\rho=1}^{r} N_\rho e^{x_\rho} = 0$$

nur dadurch befriedigt werden, dass man jeder der Zahlen N_ρ den Werth Null giebt.

III. Jetzt seien

$$X_1, X_2, \ldots X_r$$
$$x_1, x_2, \ldots x_r$$

zwei Systeme von je r gegebenen algebraischen Zahlen, wobei angenommen werde, dass die Grössen $X_1, X_2, \ldots X_r$ nicht sämmtlich gleich Null seien, und unter den $x_1, x_2, \ldots x_r$ keine zwei gleiche sich finden.

Die Grössen $X_1, X_2, \ldots X_r$ lassen sich durch eine Grösse ξ, welche eine der Wurzeln einer bestimmten irreductibelen algebraischen Gleichung mit lauter rationalen Zahlcoefficienten ist, in der Form

$$X_1 = G_1(\xi), \; X_2 = G_2(\xi), \ldots X_r = G_r(\xi)$$

ausdrücken, wo $G_1(\xi), G_2(\xi), \ldots G_r(\xi)$ ganze Funtionen von ξ, deren Coefficienten sämmtlich rationale Zahlen sind, bedeuten. Bezeichnet man mit ξ' irgend eine andere Wurzel der genannten Gleichung, so sind die Grössen

$$G_1(\xi'), G_2(\xi'), \ldots G_r(\xi')$$

nicht alle gleich Null, weil der bekannten Eigenschaft einer irreductibelen Gleichung gemäss $G_\rho(\xi')$ nicht gleich Null sein kann, wenn nicht (für denselben Werth von ρ) auch $G_\rho(\xi) = 0$ ist. Nun seien $\xi, \xi', \ldots \xi^{(k-1)}$ sämmtliche Wurzeln der in Rede stehenden Gleichung, so bringe man das Product

$$P = \sum_{a=1}^{r} G_a(\xi) e^{x_a} \cdot \sum_{b=1}^{r} G_b(\xi') e^{x_b} \ldots \sum_{t=1}^{r} G_t^{(k-1)}(\xi^{(k-1)}) e^{x_t}$$

auf die in (I) beschriebene Weise, indem man für die dort mit

$$A'_\rho, A''_\rho, \ldots A^{(k)}_\rho \qquad (\rho = 1, 2, \ldots n)$$

bezeichneten unbestimmten Grössen beziehlich

$$G_\rho(\xi), G_\rho(\xi'), \ldots G_\rho(\xi^{(k-1)})$$

substituirt, auf die Form

$$\sum_{\lambda=0}^{n} C_\lambda e^{z_\lambda},$$

wo die Grössen $z_0, z_1, \ldots z_n$ also von einander verschiedene algebraische Zahlen sind, die $C_0, C_1, \ldots C_n$ aber zunächst als symmetrische ganze Functionen von $\xi, \xi', \ldots \xi^{(k_r-1)}$ mit lauter rationalen Zahlcoefficienten und sodann sämmtlich als **rationale** Zahlen dargestellt werden können. Da nun $C_0, C_1, \ldots C_n$ nicht sämmtlich gleich Null sind, so hat nach dem Schlusssatze von (II.) die Grösse

$$\sum_{\lambda=0}^{n} C_\lambda e^{z_\lambda}$$

unter den in Betreff der Grössen $X_1, X_2, \ldots X_r, x_1, x_2, \ldots x_r$ gemachten Voraussetzungen einen von Null verschiedenen Werth. Dasselbe gilt also auch von dem Producte P und somit auch von dem Ausdrucke

$$\sum_{\varrho=1}^{r} X_\varrho e^{x_\varrho} = \sum_{\varrho=1}^{r} G_\varrho(\xi) e^{x_\varrho},$$

der ein Factor des Productes ist.

Damit ist bewiesen:

»**Werden unter $x_1, x_2, \ldots x_r$ irgend r von einander verschiedene, unter $X_1, X_2, \ldots X_r$ aber beliebige algebraische Zahlen verstanden, so kann die Gleichung**

$$\sum_{\varrho=1}^{r} X_\varrho e^{x_\varrho} = 0$$

nur in dem Falle, wo $X_1, X_2, \ldots X_r$ sämmtlich den Werth Null haben, bestehen.«

In diesem von Lindemann ohne ausgeführten Beweis aufgestellten allgemeinen Satze finden die von Hermite begonnenen Untersuchungen über die Exponentialfunction ihren Abschluss.

IV. Schliesslich mögen noch einige, aus dem vorstehenden Theorem unmittelbar sich ergebende specielle Sätze angeführt werden.

Nimmt man $r = 2$, $X_1 = -1$, $x_2 = 0$ an und setzt x für x_1, X für X_2, so ergiebt sich, dass die Gleichung $e^x = X$ nicht bestehen kann, wenn x, X beide algebraische Zahlen sind und zugleich x einen von Null verschiedenen Werth hat. Daraus folgt:

»Die Exponentialgrösse e^x ist stets eine **transcendente** Zahl, wenn x eine von Null verschiedene **algebraische** Zahl ist.«

»Der natürliche Logarithmus einer algebraischen Zahl X ist immer eine transcendente Zahl, wenn X nicht den Werth 1 hat.«

Diese beiden, von Lindemann besonders hervorgehobenen Sätze scheinen mir zu den schönsten Sätzen der Arithmetik zu gehören.

Nimmt man ferner

$$r = 3, X_1 = i, X_2 = -i, x_2 = -x_1, x_3 = 0$$

an und setzt $\frac{xi}{2}$ für x_1, X für X_3, so ergiebt sich, dass die Gleichung

$$2 \sin \frac{x}{2} = X$$

nicht bestehen kann, wenn x, X beide algebraische Zahlen sind und x einen von Null verschiedenen Werth hat. Daraus folgt:

Ein Kreisbogen, dessen Sehne, durch den Halbmesser des Kreises gemessen, eine algebraisch ausdrückbare Länge hat, kann nicht durch eine geometrische Construction, bei der nur algebraische Curven und Flächen zur Anwendung kommen, rectificirt werden; eben so wenig ist der zu einem solchen Bogen gehörige Kreissector durch eine derartige Construction quadrirbar.

Hat nämlich in einem Kreise, dessen Halbmesser als Längeneinheit angenommen wird, ein Bogen die Länge x, seine Sehne also die Länge $2 \sin \frac{x}{2}$ und der zugehörige Kreissector den Inhalt $\frac{1}{2}x$, so würde, wenn durch eine Construction der angegebenen Art der Bogen rectificirbar oder der Sector quadrirbar wäre, daraus eine algebraische Gleichung zwischen x und $2 \sin \frac{x}{2}$ sich ergeben. Eine solche Gleichung existirt aber nicht, wenn $2 \sin \frac{x}{2}$, wie angenommen, eine algebraische Zahl ist.

Ueber die Transcendenz der Zahlen e und π.*)

Von

David Hilbert in Königsberg i. Pr.

Man nehme an, die Zahl e genüge der Gleichung n^{ten} Grades

$$a + a_1 e + a_2 e^2 + \cdots + a_n e^n = 0,$$

deren Coefficienten $a, a_1, \ldots, a_n$ ganze rationale Zahlen sind. Wird die linke Seite dieser Gleichung mit dem Integral

$$\int_0^\infty = \int_0^\infty z^\varrho [(z-1)\,(z-2)\cdots(z-n)]^{\varrho+1}\, e^{-z} dz$$

multiplicirt, wo ϱ eine ganze positive Zahl bedeutet, so entsteht der Ausdruck

$$a\int_0^\infty + a_1 e\int_0^\infty + a_2 e^2\int_0^\infty + \cdots + a_n e^n \int_0^\infty$$

und dieser Ausdruck zerlegt sich in die Summe der beiden folgenden Ausdrücke:

$$\mathsf{P}_1 = a\int_0^\infty + a_1 e\int_1^\infty + a_2 e^2\int_2^\infty + \cdots + a_n e^n \int_n^\infty,$$

$$\mathsf{P}_2 = \qquad a_1 e\int_0^1 + a_2 e^2\int_0^2 + \cdots + a_n e^n \int_0^n.$$

Die Formel

$$\int_0^\infty z^\varrho e^{-z} dz = \varrho!$$

zeigt, dass das Integral $\int_0^\infty$ eine ganze rationale durch $\varrho!$ theilbare Zahl ist und ebenso leicht folgt, wenn man bezüglich die Substitutionen $z = z' + 1,\ z = z' + 2, \ldots, z = z' + n$ anwendet, dass

$$e\int_1^\infty,\ e^2\int_2^\infty, \ldots, e^n\int_n^\infty$$

ganze rationale durch $(\varrho+1)!$ theilbare Zahlen sind. Daher ist auch

*) Abdruck aus Nr. 2 der Göttinger Nachrichten v. J. 1893.

P_1 eine durch $\varrho!$ theilbare ganze Zahl und zwar gilt, wie man sieht, nach dem Modul $\varrho+1$ die Congruenz

$$(1) \qquad \frac{P_1}{\varrho!} \equiv \pm a(n!)^{\varrho+1}. \qquad (\varrho+1)$$

Andererseits ist, wenn mit K bezüglich k die absolut grössten Werthe bezeichnet werden, welche die Functionen

$$z(z-1)(z-2)\ldots(z-n)$$

bezüglich

$$(z-1)(z-2)\ldots(z-n)e^{-z}$$

in dem Intervalle $z=0$ bis $z=n$ annehmen:

$$\left|\int_0^1\right| < kK^\varrho, \quad \left|\int_0^2\right| < 2kK^\varrho, \cdots \left|\int_0^n\right| < nkK^\varrho$$

und hieraus folgt, wenn zur Abkürzung

$$\varkappa = \{|a_1 e| + 2|a_2 e^2| + \cdots + n|a_n e^n|\}k$$

gesetzt wird, die Ungleichung

$$(2) \qquad |P_2| < \varkappa K^\varrho.$$

Nun bestimme man eine ganze positive Zahl ϱ, welche *erstens* durch die ganze Zahl $a \,.\, n!$ theilbar ist und für welche *zweitens* $\varkappa \frac{K^\varrho}{\varrho!} < 1$ wird. Es ist dann $\frac{P_1}{\varrho!}$ infolge der Congruenz (1) eine nicht durch $\varrho+1$ theilbare und daher nothwendig von 0 verschiedene ganze Zahl und da überdies $\frac{P_2}{\varrho!}$ infolge der Ungleichung (2) absolut genommen kleiner als 1 wird, so ist die Gleichung

$$\frac{P_1}{\varrho!} + \frac{P_2}{\varrho!} = 0$$

unmöglich.

Man nehme an, es sei π eine algebraische Zahl und zwar genüge die Zahl $\alpha_1 = i\pi$ einer Gleichung n^{ten} Grades mit ganzzahligen Coefficienten. Bezeichnen wir dann mit $\alpha_2, \ldots, \alpha_n$ die übrigen Wurzeln dieser Gleichung, so muss, da $1+e^{i\pi}$ den Werth 0 hat, auch der Ausdruck

$$(1+e^{\alpha_1})(1+e^{\alpha_2})\cdots(1+e^{\alpha_n}) = 1 + e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_N}$$

den Werth 0 haben und hierin sind, wie man leicht sieht, die N Exponenten $\beta_1, \ldots, \beta_N$ die Wurzeln einer Gleichung N^{ten} Grades mit ganzzahligen Coefficienten. Sind überdies etwa die M Exponenten $\beta_1, \ldots, \beta_M$ von 0 verschieden, während die übrigen verschwinden, so sind diese M Exponenten $\beta_1, \ldots, \beta_M$ die Wurzeln einer Gleichung M^{ten} Grades von der Gestalt

$$f(z) = bz^M + b_1 z^{M-1} + \cdots + b_M = 0,$$

deren Coefficienten ebenfalls ganze rationale Zahlen sind und in welcher insbesondere der letzte Coefficient b_M von Null verschieden ist. Der obige Ausdruck erhält dann die Gestalt

$$a + e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_M},$$

wo a eine ganze positive Zahl ist.

Man multiplicire diesen Ausdruck mit dem Integral

$$\int_0^\infty = \int_0^\infty z^\varrho [g(z)]^{\varrho+1} e^{-z} dz,$$

wo ϱ wiederum eine ganze positive Zahl bedeutet und wo zur Abkürzung $g(z) = b^M f(z)$ gesetzt ist; dann ergiebt sich

$$a\int_0^\infty + e^{\beta_1}\int_0^\infty + e^{\beta_2}\int_0^\infty + \cdots + e^{\beta_M}\int_0^\infty$$

und dieser Ausdruck zerlegt sich in die Summe der beiden folgenden Ausdrücke:

$$\mathsf{P}_1 = a\int_0^\infty + e^{\beta_1}\int_{\beta_1}^\infty + e^{\beta_2}\int_{\beta_2}^\infty + \cdots + e^{\beta_M}\int_{\beta_M}^\infty,$$

$$\mathsf{P}_2 = \qquad e^{\beta_1}\int_0^{\beta_1} + e^{\beta_2}\int_0^{\beta_2} + \cdots + e^{\beta_M}\int_0^{\beta_M},$$

wo allgemein das Integral $\int_{\beta_i}^\infty$ in der complexen z-Ebene vom Punkte $z = \beta_i$ längs einer zur Axe der reellen Zahlen parallelen Geraden bis zu $z = +\infty$ hin und das Integral $\int_0^{\beta_i}$ vom Punkte $z = 0$ längst der geraden Verbindungslinie bis zum Punkte $z = \beta_i$ hin zu erstrecken ist.

Das Integral $\int_0^\infty$ ist wieder gleich einer ganzen rationalen durch $\varrho!$ theilbaren Zahl und zwar gilt, wie man sieht, nach dem Modul $\varrho + 1$ die Congruenz

$$\frac{1}{\varrho!}\int_0^\infty \equiv b^{\varrho M+M} b_M^{\varrho+1}. \qquad (\varrho+1)$$

Mittelst der Substitution $z = z' + \beta_i$ und wegen $g(\beta_i) = 0$ ergiebt sich ferner

$$e^{\beta_i}\int_{\beta_i}^\infty = \int_0^\infty (z'+\beta_i)^\varrho [g(z'+\beta_i)]^{\varrho+1} e^{-z'} dz' = (\varrho+1)!\; G(\beta_i),$$

wo $G(\beta_i)$ eine ganze ganzzahlige Function von β_i bedeutet, deren Grad in β_i unterhalb der Zahl $\varrho M + M$ bleibt und deren Coefficienten sämmtlich durch $b^{\varrho M+M}$ theilbar sind. Da $\beta_1, \ldots, \beta_M$ die Wurzeln der ganzzahligen Gleichung $f(z) = 0$ sind und mithin durch Multiplication mit dem ersten Coefficienten b zu *ganzen* algebraischen Zahlen werden, so ist

$$G(\beta_1) + G(\beta_2) + \cdots + G(\beta_M)$$

nothwendig eine *ganze rationale* Zahl. Hieraus folgt, dass der Ausdruck P_1 gleich einer ganzen rationalen durch $\varrho!$ theilbaren Zahl wird und zwar gilt nach dem Modul $\varrho + 1$ die Congruenz

$$\text{(3)} \qquad \frac{\mathsf{P}_1}{\varrho!} \equiv a b^{\varrho M + M} b_M^{\varrho+1}. \qquad (\varrho + 1)$$

Andererseits ist, wenn mit K bezüglich k die grössten absoluten Beträge bezeichnet werden, welche die Functionen $zg(z)$ bezüglich $g(z)e^{-z}$ auf den geradlinigen Integrationsstrecken zwischen $z = 0$ bis $z = \beta_i$ annehmen:

$$\left| \int_0^{\beta_i} \right| < |\beta_i| k K^\varrho \qquad (i = 1, 2, \ldots, M)$$

und hieraus folgt, wenn zur Abkürzung

$$\varkappa = \{ |\beta_1 e^{\beta_1}| + |\beta_2 e^{\beta_2}| + \cdots + |\beta_M e^{\beta_M}| \} k$$

gesetzt wird, die Ungleichung

$$\text{(4)} \qquad |\mathsf{P}_2| < \varkappa K^\varrho.$$

Nun bestimme man eine ganze positive Zahl ϱ, welche *erstens* durch $a b b_M$ theilbar ist und für welche *zweitens* $\varkappa \frac{K^\varrho}{\varrho!} < 1$ wird. Es ist dann $\frac{\mathsf{P}_1}{\varrho!}$ in Folge der Congruenz (3) eine nicht durch $\varrho + 1$ theilbare und daher nothwendig von 0 verschiedene ganze Zahl und da überdies $\frac{\mathsf{P}_2}{\varrho!}$ in Folge der Ungleichung (4) absolut genommen kleiner als 1 wird, so ist die Gleichung

$$\frac{\mathsf{P}_1}{\varrho!} + \frac{\mathsf{P}_2}{\varrho!} = 0$$

unmöglich.

Es ist leicht zu erkennen, wie auf dem eingeschlagenen Wege ebenso einfach auch der allgemeinste Lindemann'sche Satz über die Exponentialfunction sich beweisen lässt.

Königsberg i. Pr., den 5. Januar 1893.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

QUADRATURE OF THE CIRCLE.

By EDWARD J. GOODWIN, Solitude, Indiana.

Published by the request of the author.

A circular area is equal to the square on a line equal to the quadrant of the circumference; and the area of a square is equal to the area of the circle whose circumference is equal to the perimeter of the square.

To quadrate the circle is to find the side of a square whose perimeter equals that of the given circle; rectification of the circle requires to find a right line equal to the circumference of the given circle. The square on a line equal to the arc of 90° fulfills both of the said requirements.

It is impossible to quadrate the circle by taking the diameter as the linear unit, because the square root of the product of the diameter by the quadrant of the circumference produces the side of a square which equals 9 when the quadrant equals 8.

It is not mathematically consistent that it should take the side of a square whose perimeter equals that of a greater circle to measure the space contained within the limits of a less circle.

Were this true, it would require a piece of tire iron 18 feet to bind a wagon wheel 16 feet in circumference.

This new measure of the circle has happily brought to light the ratio of the chord and arc of 90°, which is as 7:8; and also the ratio of the diagonal and one side of a square, which is as 10:7. These two ratios show the numerical relation of diameter to circumference to be as $\frac{5}{4}:4$.

Authorities will please note that while the finite ratio ($\frac{5}{4}:4$) represents the area of the circle to be more than the orthodox ratio, yet the ratio (3.1416) represents the area of a circle whose circumference equals 4 two % greater than the finite ratio ($\frac{5}{4}:4$), as will be seen by comparing the terms of their respective proportions, stated as follows: 1:3.20::1.25:4, 1:3.1416::1.2732:4.

It will be observed that the product of the extremes is equal to the product of the means in the first statement, while they fail to agree in the second proportion. Furthermore, the square on a line equal to the arc of 90° shows very clearly that the ratio of the circle is the same in principle as that of the square. For example, if we multiply the perimeter of a square (the sum of its sides) by $\frac{1}{4}$ of one side the product equals the sum of two sides by $\frac{1}{2}$ of one side, which equals the square on one side.

Again, the number required to express the units of length in $\frac{1}{4}$ of a right line, is the square root of the number representing the squares of the linear unit bounded by it in the form of a square whose ratio is as 1:4.

These properties of the ratio of the square apply to the circle without an exception, as is further sustained by the following formula to express the numerical measure of both *circle* and *square.*

Let C represent the circumference of a circle whose quadrant is *unity*, Q $\frac{1}{2}$ the quadrant, and CQ^2 will apply to the numerical measure of a circle and a square.

We are now able to get the true and finite dimensions of a circle by the exact ratio $\frac{5}{4}:4$, and have simply to divide the circumference by 4 and square the quotient to compute the area.

HOUSE BILL NO. 246, INDIANA STATE LEGISLATURE, 1897

WILL E. EDINGTON, DePauw University

This paper has grown out of a number of requests for information over a number of years, by students and others, concerning some supposed action taken by the Indiana State Legislature with regard to fixing the value of *pi*, that is, the result of dividing the length of the circumference of a circle by the length of its diameter, at a certain value that was different from the true value. Of course the interest in and wonder at such an action lies in the presumption of a group of supposedly fairly well educated men to attempt to legislate upon something not in the realm of legislation.

The only reference to this action known to me until recently was an article by Professor C. A. Waldo in the Proceedings of the Academy for 1916. Professor Waldo was a member of the mathematics department at Purdue at the time the action was taken, but he was professor of mathematics at Washington University at St. Louis when he wrote the article. His article is merely a running account based on memory, and unfortunately there are several errors. The date of the action is given as 1899, and the description of the progress of the bill is inaccurate and somewhat misleading, as might be expected in a memory account written twenty years later, probably without any special check being made as to the facts. The error in the date caused me to spend many futile hours searching through the House and Senate Journals for 1899 for some record of the action.

However, during the past summer there appeared in the Contributors' Club of the July, 1935, number of The Atlantic Monthly an article by Professor Thomas F. Holgate, of Northwestern University, entitled "Rules for Making Pi Digestible", in which a statement of House Bill No. 246 was given, together with the correct date 1897. Armed with this information I have attempted to ascertain the facts as to the inception of the bill, its author, the history of the bill in the Legislature, and the atmosphere surrounding it during the successive steps of its progress through the legislative mill. The facts which I present here have been secured from the bill itself, on file in the Indiana State Library, the Journals of the House and Senate for 1897, and the files of the three Indianapolis papers for January and February, 1897. My purpose in presenting this paper is to have on record in an accessible place the facts concerning this most interesting piece of attempted legislation. The drawing of morals I leave to others.

The author of the bill was Edwin J. Goodwin, M. D., of Solitude, Posey County. The bill was introduced in the House by Mr. Taylor I. Record, Representative from Posey County, on January 18, 1897. In the Indiana Historical Library may be seen the handwritten copy together with a typewritten copy of the bill, and a handwritten record of actions taken by the Legislature. Some one has written on the page containing the record of the actions the following: "Dr. E. J. Goodman, No. 72 Ohio Street—Author". However, this is in error, for the author was Dr. E. J. Goodwin. Following is a copy of the bill:

HOUSE BILL NO. 246

A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature of 1897.

Section 1. Be it enacted by the General Assembly of the State of Indiana: It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. The diameter employed as the linear unit according to the present rule in computing the circle's area is entirely wrong, as it represents the circle's area one and one-fifth times the area of a square whose perimeter is equal to the circumference of the circle. This is because one-fifth of the diameter fails to be represented four times in the circle's circumference. For example: if we multiply the perimeter of a square by one-fourth of any line one-fifth greater than one side, we can in like manner make the square's area to appear one fifth greater than the fact, as is done by taking the diameter for the linear unit instead of the quadrant of the circle's circumference.

Section 2. It is impossible to compute the area of a circle on the diameter as the linear unit without tresspassing upon the area outside of the circle to the extent of including one-fifth more area than is contained within the circle's circumference, because the square on the diameter produces the side of a square which equals nine when the arc of ninety degrees equals eight. By taking the quadrant of the circle's circumference for the linear unit, we fulfill the requirements of both quadrature and rectification of the circle's circumference. Furthermore, it has revealed the ratio of the chord and arc of ninety degrees, which is as seven to eight, and also the ratio of the diagonal and one side of a square which is as ten to seven, disclosing the fourth important fact, that the ratio of the diameter and circumference is as five-fourths to four; and because of these facts and the futher fact that the rule in present use fails to work both ways mathematically, it should be discarded as wholly wanting and misleading in its practical applications.

Section 3. In further proof of the value of the author's proposed contribution to education, and offered as a gift to the State of Indiana, is the fact of his solutions of the trisection of the angle, duplication of the cube and quadrature of the circle having been already accepted as contributions to science by the American Mathematical Monthly, the leading exponent of mathematical thought in this country. And be it remembered that these noted problems had been long since given up by scientific bodies as unsolvable mysteries and above man's ability to comprehend.

Following the introduction of the bill it was referred to the House Committee on Canals. Just why it should be referred to this committee, frequently called the Committee on Swamp Lands, is difficult to understand. The following items appeared in the newspapers on January 19, 1897, under the heading of New House Bills:

H.B. 246. By Mr. Ricard: Bill telling how to "square a circle". Swamp Lands.[1]
Mr. Recard, H.B. 246—Bill telling how to "square a circle". Swamp Lands.[2]

On January 19, 1897, Representative M. B. Butler, of Steuben County, chairman of the Committee on Canals, submitted the following report:

Your Committee on Canals, to which was referred House Bill No. 246, entitled a bill for an act entitled an act for the introduction of a mathematical truth, etc., has had the same under consideration and begs leave to report the same back to the House with the recommendation that said bill be referred to the Committee on Education.[3]

The bill was accordingly referred to the Committee on Education.

---

[1]The Indianapolis Journal, January 19, 1897, p. 3, col. 4.
[2]The Indianapolis Sentinel, January 19, 1897, p. 2, col. 3.
[3]House Journal, 1897, p. 213.

On January 20, 1897, the following appeared in The Indianapolis Sentinel:

TO SQUARE THE CIRCLE

Claims Made That This Old Problem Has Been Solved.

The bill telling how to square a circle, introduced in the House by Mr. Record, is not intended to be a hoax. Mr. Record knows nothing of the bill with the exception that he introduced it by request of Dr. Edwin Goodwin of Posey County, who is the author of the demonstration. The latter and State Superintendent of Public Instruction Geeting believe that it is the long-sought solution of the problem, and they are seeking to have it adopted by the legislature. Dr. Goodwin, the author, is a mathematician of note. He has it copyrighted and his proposition is that if the legislature will indorse the solution he will allow the state to use the demonstration in its textbooks free of charge. The author is lobbying for the bill.[4]

The records concerning the bill in the Committee on Education, as given in the House Journal, are somewhat confused. In the Calendar of House Bills Introduced and Action Thereon, the record reads

Referred back February 2 and
Recommitted to the same committee
Reported back February 4.[5]

In the House Journal proper, however, one finds under the date of February 2, 1897, that Representative S. E. Nicholson, of Howard County, chairman of the Committee on Education, reported to the House.

Your Committee on Education, to which was referred House Bill No. 246, entitled a bill for an act entitled an act introducing a new mathematical truth, has had same under consideration, and begs leave to report the same back to the House with the recommendation that said bill do pass.[6]

The report was concurred in, and on February 5, 1897, it was brought up for the second reading, following which it was considered engrossed. Then "Mr. Nicholson moved that the constitutional rule requiring bills to be read on three days be suspended, that the bill may be read a third time now".[7] The constitutional rule was suspended by a vote of 72 to 0 and the bill was then read a third time. It was passed by a vote of 67 to 0, and the Clerk of the House was directed to inform the Senate of the passage of the bill.

After the bill had passed the House the Indianapolis papers reported on it as follows:

NEW MATHEMATICAL TRUTH

Mr. Record's bill "introducing a new mathematical truth", was passed under a suspension of the constitutional rule.[8]

DR. GOODWIN'S THEAOREM

Resolution Adopted by the House of Representatives.

Following is the text of the resolution adopted by the House relative to the mathematical theorem of E. J. Goodwin, M.D., Solitude, Posey County. Whereas, it has been found (The remainder of Section 1 and all of Section 2 is printed).[9]

---

[4] The Indianapolis Sentinel, January 20, 1897, p. 5, col. 4.
[5] House Journal, 1897, Appendix, p. 183.
[6] House Journal, 1897, p. 489.
[7] House Journal, 1897, p. 588.
[8] The Indianapolis News, February 5, 1897, p. 2, col. 3.
[9] The Indianapolis News, February 6, 1897, p. 11, col. 4.

NO OPPOSITION TO IT

The bill introduced by Mr. Record telling how to "square the circle" was unanimously passed by the house yesterday afternoon. It indorses a mathematical demonstration of which Dr. Edwin Goodwin of Posey County is author.[10]

MATHEMATICAL BILL PASSED

Record's bill containing the discovery of Dr. Goodwin, of Posey County, for computing the area of a circle, was handed down upon second reading. Mr. Nicholson explained that Dr. Goodwin had a copyright on his discovery and had offered this bill in order that it might be free to the schools of Indiana. The bill was taken up and passed under suspension of rules. This is the strangest bill that has ever passed an Indiana Assembly. It reads as follows (The whole bill as given at the beginning of this paper is then printed).[11]

The contents of the bill thus became known in this and other states and, of course, became the target for ridicule. However, let us follow the progress of the bill.

Engrossed House Bill No. 246 was referred to the Senate on February 10, 1897, and was read for the first time on February 11 and referred to the Committee on Temperance. One wonders whether this was done intentionally, for certainly the bill could have been referred to no committee more appropriately named. On February 12 Senator Harry S. New, of Marion County, Chairman of the Committee on Temperance made the following report to the Senate:

Your Committee on Temperance, to which was referred House Bill No. 246, introduced by Mr. Record, has had the same under consideration, and begs leave to report the same back to the Senate with the recommendation that said bill do pass.[12]

On the afternoon of February 12 "Senator Bozeman called up House Bill No. 246. The bill was read a second time by title. Senator Hogate moved to amend the bill by striking out the enacting clause. The motion was lost. Senator Hubbell moved to postpone the further consideration of this bill indefinitely. Which motion prevailed."[13]

This cold recital of Senate action gives no clue as to how these actions were taken but the reports in the papers do, as follows:

House Bill 246, providing for the official adoption of the demonstration for squaring the circle, was killed.[14]

THE MATHEMATICAL BILL

Fun Making in the Senate Yesterday Afternoon—* * *

Representative Record's mathematical bill legalizing a formula for squaring the circle was brought up and made fun of. The Senators made bad puns about it, ridiculed it and laughed over it. The fun lasted half an hour. Senator Hubbell said that it was not meet for the Senate, which was costing the State $250 a day, to waste its time in such frivolity. He said that in reading the leading newspapers of Chicago and the East, he found that the Indiana State Legislature had laid itself open to ridicule by the action already taken on the bill. He thought consideration of such a proposition was not dignified or worthy of the Senate. He moved the indefinite postponement of the bill, and the motion carried.[15]

---

[10]The Indianapolis Sentinel, February 6, 1897, p. 2, col. 4.
[11]The Indianapolis Journal, February 6, 1897, p. 8, cols. 2, 3.
[12]Senate Journal, 1897, p. 649.
[13]Senate Journal, 1897, p. 677.
[14]The Indianapolis Sentinel, February 13, 1897, p. 2, col. 4.
[15]The Indianapolis News, February 13, 1897, p. 11, col. 3.

WORK OF THE SENATE

A lot of fun was had with Mr. Record's mathematical bill. It came from the House a few days ago and went to the committee on temperance. It was called up by Senator Bozeman. It was indefinitely postponed, as not being a subject for legislation. Senator Hubbell characterized the bill as utter folly. The Senate might as well try to legislate water to run up hill as to establish mathematical truth by law. Leading papers all over the country, he said, were ridiculing the Indiana Legislature. It was outrageous that the State of Indiana should pay $250 a day to have time wasted on such frivolous matters.

STRUCK A POPULAR CHORD

Senator Drummond did not want the rule suspended until he had some information as to the purpose of the bill.

"It may be I am densely ignorant on this question of Mathematics," he said. "Consent! Consent!" said Senator Ellison. There was loud laughter at this sally.

Although the bill was not acted on favorably no one who spoke against it intimated that there was anything wrong with the theories it advances. All of the senators who spoke on the bill admitted that they were ignorant of the merits of the proposition. It was simply regarded as not being a subject for legislation.[16]

Naturally the question arises as to what happened to bring about the defeat of this bill after it had passed the House, apparently had the backing of the State Superintendent of Education, was actively being lobbied for, and had passed the first reading in the Senate without comment. To answer this question one must read Professor Waldo's account.

As the session of the Legislature was drawing toward its close it chanced to be the duty of the writer to visit the State Capitol and make sure that the Academy appropriation was cared for. When admitted to the floor of the House, imagine his surprize when he discovered that he was in the midst of a debate upon a piece of mathematical legislation. An ex-teacher from the eastern part of the state was saying: "The case is perfectly simple. If we pass this bill which establishes a new and correct value for π, the author offers to our state without cost the use of his discovery and its free publication in our school text books, while everyone else must pay him a royalty." The roll was then called and the bill passed its third and final reading in the lower house. A member then showed the writer a copy of the bill just passed and asked him if he would like an introduction to the learned doctor, its author. He declined the courtesy with thanks remarking that he was acquainted with as many crazy people as he cared to know.

That evening the senators were properly coached and shortly thereafter as it came to its final reading in the upper house they threw out with much merriment the epoch making discovery of the Wise Man from the Pocket.[17]

[16]The Indianapolis Journal, February 13, 1897, p. 3, cols. 4, 5.
[17]Proceedings of the Indiana Academy of Science 1916:445-446. 1917.

Old Intelligencer

The Legal Values of Pi
by David Singmaster

I have long been interested in the notorious attempt to legislate a legal value of π. I have read several articles [2,5,6,7,8,10] on the history of this attempt and it has been mentioned in the popular press recently [1,9]. From these it is clear that only Greenblatt [5] and Hallerberg [7] have tried to understand the obscure content of the proposed bill. Greenblatt found four different values of π given in it! Hallerberg has tried to understand how some of the values came about.

I have found six different values given in the Bill and three other values in an earlier article by the same author [3] and in contemporary interviews with him [5, pp. 388–389].

In this note, I will briefly sketch the history of the proposal, drawing heavily on the definitive article of Hallerberg [6] and then I will quote the versions of the proposals and then derive the values of π described.

Historical Sketch

Edward Johnston Goodwin (1828?–1902) was a doctor in Solitude, Indiana. Hallerberg [7, p. 139] has recently discovered three versions of a monograph by Goodwin. The 1982 version, entitled "Universal Inequality is the Law of All Creation", includes the following comments:

> During the first week in March, 1888, the author was supernaturally taught the exact measure of the circle . . . no authority in the science of numbers can tell how the ratio was discovered. . . .

(I think the evidence will lead us to agree with the latter assertion!)

He copyrighted his results in the U.S., England, Germany, Belgium, France, Austria and Spain. He attempted to present his solution at the educational exhibit of the 1893 Columbian Exposition in Chicago. When the organisers learned what he planned to present, his permit was revoked and he was told to present his solution to the mathematical journals.

The *American Mathematical Monthly*, then in its first year, was privately published and often inserted whatever material was at hand. In 1894, Goodwin's article "Quadrature of the Circle" appeared in the Queries and Information section of the seventh issue of the first volume of the *Monthly*, with the note "Published at the request of the author". The first paragraph is followed by "(copyrighted by the author, 1889. All rights reserved.)" As we will see, the proposed Bill is almost, but not quite, identical to this article.

Dr. Goodwin persuaded his local representative, Taylor I. Record, to introduce a Bill, written by Goodwin. House Bill No. 246 was introduced in the Indiana House of Representatives on 18 January 1897. The Bill was referred to the Committee on Canals, generally known as the Committee on Swamp Lands, 'midst general cheerfulness . . . where, in the swamps, the bill would find a deserved grave' [6, p. 385, where it is quoted from *Der Tägliche Telegraph*, A German language paper in Indianapolis]. Surprisingly (or perhaps not), the committee returned the Bill for reference to the Committee on Education. On 2 February, the Committee on Education reported the Bill back "with the recommendation that said bill do pass". The House duly passed it then and again at a second reading on 5 February, when the rules were suspended to permit the third reading as well. The Bill was then forwarded to the Senate, where it was referred to the Committee on Temperance on 11 February.

The first newspaper report, on 20th January, commented that: "The bill . . . is not intended to be a hoax. Dr. Goodwin . . . and State Superintendent of Public Instruction Geeting, believe that it is the long-sought solution . . . Dr. Goodwin, the author, is a mathematician of note. He has it copyrighted and his proposition is that if the legislative will indorse the solution he will allow the state to use [it] free of charge" [2, p. 208, quoted from *The Indianapolis Sentinel*].

By 6 February, the newspapers began to realise what was happening. The *Indianapolis Journal* commented: "This is the strangest bill that has ever passed an Indiana assembly" and printed the entire Bill, [2, p. 209]. The Bill became a source of much merriment in the press, as far away as Chicago and, later, New York.

Professor C. A. Waldo, a mathematician at Purdue University, came to the state capitol on Academy business at this time. He came to the House session on 5 February and was surprised to discover a debate on mathematical legislation. He recalls:

> An ex-teacher from the eastern part of the state was saying; "The case is perfectly simple. If we pass this bill which establishes a new and correct value for π, the author offers to our state without cost the use of his discovery and its free publication in our school text books, while everyone else must pay him a royalty." . . . A member then showed the writer a copy of the bill just passed and asked him if he would like an introduction to the learned doctor, its author. He declined the courtesy with thanks remarking that he was acquainted with as many crazy people as he cared to know.
>
> That evening the senators were properly coached and shortly thereafter . . . they threw out with much merriment the epoch making discovery. . . [10].

The Committee on Temperance reported the Bill back to the Senate on 12 February "with the recommendation that said bill do pass". In the afternoon, it was brought before the Senate and "the Senators made bad puns about it," [2, p. 209, from the *Indianapolis News*]. "Senator Hubbell characterized the bill as utter folly. The Senate might as well try to legislate water to run up hill. . . . Although the bill was not acted on favorably no one who spoke against it intimated that there was anything wrong with the theories it advances. All of the Senators who spoke on the bill admitted that they were ignorant of the merits of the proposition. It was simply regarded as not being a subject for legislation" [2, p. 210, from the *Indianapolis Journal*]. The Bill was postponed indefinitely and so never became law.

The American Mathematical Monthly Article

Below, I reproduce the text of Goodwin's article [3] in the *American Mathematical Monthly*, July 1894. I have inserted the marginal numbers for reference to the starred assertions.

1. * A circular area is equal to the square on a line equal
1.' to the quadrant of the circumference; * and the area of a square is equal to the area of the circle whose circumference is equal to the perimeter of the square.

(Copyrighted by the author, 1889. All rights reserved.)

To quadrate the circle is to find the side of a square whose perimeter equals that of the given circle; rectification of the circle requires to find a right line equal to the circumference of the given circle. The square on a line equal to the arc of 90° fulfills both of the said requirements.

2. * It is impossible to quadrate the circle by taking the diameter as the linear unit, because the square root of the product of the diameter by the quadrant of the circumference produces the side of a square which equals 9 when the quadrant equals 8.

It is not mathematically consistent that it should take the side of a square whose perimeter equals that of a greater circle to measure the space contained within the limits of a less circle.

3. * Were this true, it would require a piece of tire iron 18 feet to bind a wagon wheel 16 feet in circumference.

4. * This new measure of the circle has happily brought to light the ratio of the chord and arc of 90°, which is
4a. as 7:8, * and also the ratio of the diagonal and one side
5. of a square, which is as 10:7. * These two ratios show the numerical relation of diameter to circumference to be $\frac{5}{4}$:4.

6. * Authorities will please note that while the finite ratio $\frac{5}{4}$:4) represents the area of the circle to be more than the orthodox ratio, yet the ratio (3.1416) represents the area of a circle whose circumference equals 4 two % greater than the finite ratio $\frac{5}{4}$:4), as will be seen by comparing the terms of their respective proportions, stated as follows: 1:3.20::1.25:4, 1:3.1416::1.2732:4.

It will be observed that the product of the extremes is equal to the product of the means in the first statement, while they fail to agree in the second proportion. Furthermore, the square on a line equal to the arc of 90° shows very clearly that the ratio of the circle is the
7. same in principle as that of the square. * For example, if we multiply the perimeter of a square (the sum of its sides) by $\frac{1}{4}$ of one side the product equals the sum of two sides by $\frac{1}{2}$ of one side, which equals the square on one side.

Again, the number required to express the units of length in $\frac{1}{4}$ of a right line, is the square root of the number representing the squares of the linear unit bounded by it in the form of a square whose ratio is as 1:4.

These properties of the ratio of the square apply to the circle without an exception, as is further sustained by the following formula to express the numerical measure of both *circle* and *square*.

8. * Let C represent the circumference of a circle whose quadrant is *unity*, Q $\frac{1}{2}$ the quadrant, and CQ^2 will apply to the numerical measure of a circle and a square.

9. * We are now able to get the true and finite dimensions of a circle by the exact ratio $\frac{5}{4}$:4, and have simply to divide the circumference by 4 and square the quotient to compute the area.

Analysis of the Article. Let us consider a circle of radius R, diameter D, circumference C, area A, and a square of side S. (Numerical values are rounded.)

1. This says $A = (C/4)^2$.
That is, $\pi R^2 = (\pi R/2)^2$, whence $\pi = 4$.

1.' This says $S^2 = A$ if $C = 4S$.
Thus $S^2 = \pi R^2$ when $2\pi R = 4S$ i.e., $S = \pi R/2$. This is just a rephrasing of (1), so $\pi = 4$.

2. This says $(D \cdot C/4)^{1/2} = 9$ when $C/4 = 8$. That is $2R \cdot \pi R/2 = 81$ when $\pi R/2 = 8$. This yield $\pi = 256/81 = 3.160494$. Hallerberg [7, p. 138] says that the phrase 'the diameter as the liner unit' refers to the presence of D in the formula $A = D \cdot C/4$.

3. Hallerberg [7, p. 138] advances an explanation for this curious assertion, based on Goodwin's assumption that equal perimeters give equal areas. If $C = 16$, then A is correctly $= 20.371833$. A square of the same area would have $S = 4.513517$. Then taking $S = C/4$ gives $C = 18.054067$!

4. This says $\sqrt{2}R : C/4 = 7 : 8$, whence $\pi = 16\sqrt{2}/7 = 3.232488$

4a. However, this asserts that $\sqrt{2} = 10/7 = 1.428571$. Using this in (4), we have $\pi = 160/49 = 3.265306$.

5. Goodwin asserts that (4a) yields $D : C = \frac{5}{4}$:4, whence $\underline{\pi = 16/5 = 3.2}$, though I could see how this followed. However Hallerberg [7, p. 136] has clarified this by observing that Goodwin has used (4) in the form $D/\sqrt{2} : C/4 = 7 : 8$, which does yield the asserted result.

6. If $C = 4$, then $R = 2/\pi$ and $A = 4/\pi$. When $\pi \doteq 16/5$, then A = 5/4 = 1.25, but when $\pi = 3.1416$, then $A = 1.2732$, which is 2% larger. This doesn't yield any new value of π.

7. This says $4S \cdot S/4 = 2S \cdot S/2 = S^2$!

8. This says that $C = 4$ (hence $Q = 1$) implies $A = CQ^2$ (= 1). Since $C = 4$ gives $R = 2/\pi$ and $A = 4/\pi$, this yields $\underline{\pi = 4}$ again.

9. This is a repetition of (1), giving $\underline{\pi = 4}$ again.

House Bill No. 246

Below is the text of the bill. I have added marginal numbers as before. Numbers ≤ 10 refer to analyses given already.

A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the state of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature of 1897.

10. SECTION 1. * Be it enacted by the General Assembly
of the state of Indiana, That it has been found that a
circular area is to the square on a line equal to the quad-
rant of the circumference as the area an equilateral rec-
11. tangle is to the square on one side. * The diameter em-
ployed as a linear unit according to the present rule in
computing the circle's area is entirely wrong, as it rep-
resents the circle's area one and one fifth times the area
of a square whose perimeter is equal to the circumfer-
12. ence of the circle. * This is because one fifth of the di-
ameter fails to be represented four times in the circle's
circumference. For example, if we multiply the perim-
eter of a square by one fourth of any line one-fifth
greater than one side, we can in like manner make the
square's area to appear one-fifth greater than the fact,
as is done by taking the diameter for the linear unit
instead of the quadrant of the circle's circumference.

SECTION 2. It is impossible to compute the area of a
circle on the diameter as the linear unit without tres-
passing upon the area outside of the circle to the extent
of including one-fifth more area than is contained
13. within the circle's circumference, * because the square
on the diameter produces the side of a square which
equals nine when the arc of ninety degrees equals eight.
By taking the quadrant of the circle's circumference for
the linear unit, we fulfill the requirements of both quad-
rature and rectification of the circle's circumference.
4. * Furthermore, it has revealed the ratio of the chord and
4a. arc of ninety degrees, which is as seven to eight, * and
also the ratio of the diagonal and one side of a square
5. which is as ten to seven, * disclosing the fourth impor-
tant fact, that the ratio of the diameter and circumfer-
ence is as five-fourths to four; and because of these facts
and the further fact that the rule in present use fails to
work both ways mathematically, it should be discarded
as wholly wanting and misleading in its practical ap-
plications.

SECTION 3. In further proof of the value of the author's proposed contribution to education, and offered as a gift to the State of Indiana, is the fact of his solutions of the trisection of the angle, duplication of the cube and quadrature of the circle having been already accepted as contributions to science by the *American Mathematical Monthly*, the leading exponent of mathematical thought in this country.

And be it remembered that these noted problems had been long since given up by scientific bodies as unsolvable mysteries and above man's ability to comprehend.

Analysis of the Bill. **10.** This says $A : (C/4)^2 = 1$, which a rephrasing of (1), giving $\underline{\pi = 4.}$

11. Here he says the 'present rule' is $A = \frac{6}{5}(C/4)^2$, whence $\underline{\pi = 10/3 = 3.333333.}$ I think we can agree that this is 'entirely wrong'.

12. I think this may mean that $4 \cdot \frac{4}{5}D = C$, whence $\underline{\pi = 16/5 = 3.2}$, as in (5).

13. I think this says $D^2 = 9^2$ when $C/4 = 8$. This yields $\underline{\pi = 32/9 = 3.555558.}$ However the ratio 9 to 8 sounds like (2) above is intended, but I do not see the relation.

Some Further Values

Hállerberg's article [6] quotes a number of interviews with Goodwin. An article in the *Indianapolis Journal* [6, p. 382] states $\pi = 3.2$. A later article in the same paper [6, pp. 388–389] quotes Goodwin:

14. * Tell anyone who questions my demonstration to de-
scribe a circle whose radius is 5/8 (the diameter thus
being $1\frac{1}{4}$). The circumference is 3.2 according to my
ratio. Then circumscribe the circle with an equilateral
rectangle and compute the area of the rectangle first,
which you can do if you know anything about ge-
ometry. Then allow one square inch of area for the
circle and see if the difference between the area of the
circle and the area of the circumscribed rectangle
proves that the area of the circle is more than one-fifth
11a. * greater than the area of a square with an equal pe-
rimeter.

14. This seems to say $C = 3.2$ when $D = 5/4$, so $\underline{\pi = 64/25 = 2.56}$

11a. This is essentially the same as (11). Here he seems to be saying this is incorrect and that $A > \frac{6}{5}(C/4)^2$ so $\pi < 10/3$.

Another newspaper reported the rules:

$C = 3.2D$ (which is (5)); $A = (C/4)^2$ (which is (1)); and the volume V of a sphere is $V = (C/4)^3$, which yields a new value of $\underline{\pi = \sqrt{32/3} = 3.265986.}$ Hallerberg [7, p. 138] points out that the formula for V is a superb piece of reasoning by analogy.

Goodwin's Trisection and Duplication

Like all circle squarers, Goodwin also trisected the angle and duplicated the cube. This also appeared in the *American Mathematical Monthly* [4], as follows.

(A) The trisection of an angle:

The trisection of a right line taken as the chord of any arc of a circle trisects the angle of the arc;
(B) Duplication of the Cube:

Doubling the dimensions of a cube octuples its contents, and doubling its contents increases its dimensions twenty-five plus one per cent.

By request of the author,
Edw. J. Goodwin,
Solitude, Indiana.

This makes $2^{1/3} = 1.26$ instead of 1.259921.

Conclusions

Originally, my only conclusion was that ignorance is consistently inconsistent. The inconsistency in Goodwin's writing is so great that I wonder if there is really any point (other than amusement) in trying to analyse it. As already noted, Hallerberg [6, p. 394; 7, p. 138] points out that the beginning of Goodwins article shows that he believed that a square and a circle of the same perimeter had the same area and that consequently he would interchange reference to the two shapes. This means that a reference to A may mean S^2 and a reference to C may mean $4S$. Indeed some of the analyses have π occurring several times and these may denote different things! As seen in (5), it is even possible to obtain different results from the same assertions. I am sure that some of the other analyses may have alternative values for π. Hallerberg says that Goodwin's erroneous belief has led many to believe that (1) yields $\pi = 4$, but that this is not what Goodwin intended. However he does not manage to make (1) any clearer to me and I have not managed either. There are several further sections of the article and the Bill which still defy any analysis.

Goodwin appears to give 9 different values of π, namely: 4, 3.555556, 3.333333, 3.265986, 3.265306, 3.232488, 3.2, 3.160494, 2.56. These average to 3.28907, though I haven't accounted for the relative frequency of these values in his assertions. Most circle squarers get just one value. Goodwin is unique in getting so many.

Afterword

Since I wrote the above article, Arnold Arnold has been featured in the media. This suggests a rather more serious conclusion. It is distressingly easy for a crank to convince people that he has a solution. Irrational forces are always present, ready to tear down the ivory towers. If we believe in reason, then our town survival and a decent regard for posterity require us to fight against unreason.

Added in proof.
A few days after sending back the revised version, I came across the following passage in Alexander MacFarlane's *Lectures on Ten British Mathematicians of the Nineteenth Century* (Wiley, New York and Chapman & Hall, London, 1916), on page 31 where he talks about De Morgan's "Budget of Paradoxes".

> I remember that just before the American Association met at Indianapolis in 1890, the local newspapers heralded a great discovery which was to be laid before the assembled savants—a young man living somewhere in the country had squared the circle. While the meeting was in progress I observed a young man going about with a roll of paper in his hand. He spoke to me and complained that the paper containing his discovery had not been received. I asked him whether his object in presenting the paper was not to get it read, printed and published so that everyone might inform himself of the result; to all of which he assented readily. But, said I, many men have worked at this question, and their results have been tested fully, and they are printed for the benefit of anyone who can read; have you informed yourself of their results? To this there was no assent, but the sickly smile of the false paradoxer.

Although Goodwin would have been about 62 in 1890, the coincidence of date and location lead me to believe that this 'young man' must have been Goodwin. MacFarlane's lecture was given in 1901 (they were collected and printed after his death), so perhaps his memory had played him false on the age of the paradoxer. But his lectures do not show many errors, so perhaps we have another Indiana circle squarer contemporary with Goodwin—A coincidence, a coincidence, a most unlikely coincidence!

References

1. Anon. (1983) Reciting to the point of boredom. *The Guardian* (3 Nov.), 1
2. W. E. Edington (1935) House Bill No. 246, Indiana State Legislature, 1897. *Proc. Indiana Acad. Sci., 45*, 206–210
3. E. J. Goodwin (1894) Quadrature of the circle. *Amer. Math. Month., 1*, 246–247
4. E. J. Goodwin (1895) (A) The trisection of an angle; (B) Duplication of the cube. *Amer. Math. Month., 2*, 337
5. M. H. Greenblatt (1965) The "legal" value of π, and some related mathematical anomalies. *American Scientist* (Dec.), 427A–434A
6. A. E. Hallerberg (1975) House Bill No. 246 revisited. *Proc. Indiana Acad. Sci., 84*, 374–399
7. A. E. Hallerberg (1977) Indiana's squared circle. *Math. Mag., 50*, 136–140
8. T. F. Holgate (1935) Rules for making pi digestible. *The Atlantic Monthly* (July), 118–119
9. R. Millington (1982) Letter: The House divided and got a zero answer. *The Guardian* (8 Nov.)
10. C. A. Waldo (1916) What might have been. *Proc. Indiana Acad. Sci., 26*, 445–446

Polytechnic of the South Bank
London, SE1 0AA
England

5

SQUARING THE CIRCLE

(*Journal of the Indian Mathematical Society*, v, 1913, 132)

Let PQR be a circle with centre O, of which a diameter is PR. Bisect PO at H and let T be the point of trisection of OR nearer R. Draw TQ perpendicular to PR and place the chord $RS = TQ$.

Join PS, and draw OM and TN parallel to RS. Place a chord $PK = PM$, and draw the tangent $PL = MN$. Join RL, RK and KL. Cut off $RC = RH$. Draw CD parallel to KL, meeting RL at D.

Then the square on RD will be equal to the circle PQR approximately.

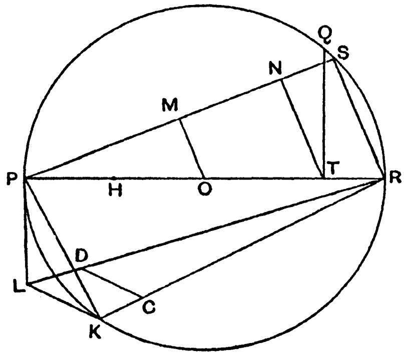
For $$RS^2 = \tfrac{5}{36} d^2,$$
where d is the diameter of the circle.

Therefore $$PS^2 = \tfrac{31}{36} d^2.$$

But PL and PK are equal to MN and PM respectively.

Therefore $$PK^2 = \tfrac{31}{144} d^2, \text{ and } PL^2 = \tfrac{31}{324} d^2.$$

Hence $$RK^2 = PR^2 - PK^2 = \tfrac{113}{144} d^2,$$
and $$RL^2 = PR^2 + PL^2 = \tfrac{355}{324} d^2.$$



But $$\frac{RK}{RL} = \frac{RC}{RD} = \frac{3}{2}\sqrt{\frac{113}{355}},$$
and $$RC = \tfrac{3}{4} d.$$

Therefore $$RD = \frac{d}{2}\sqrt{\frac{355}{113}} = r\sqrt{\pi}, \text{ very nearly.}$$

Note.—If the area of the circle be 140,000 square miles, then RD is greater than the true length by about an inch.

6

MODULAR EQUATIONS AND APPROXIMATIONS TO π

(*Quarterly Journal of Mathematics*, XLV, 1914, 350—372)

1. If we suppose that

$$(1+e^{-\pi\sqrt{n}})(1+e^{-3\pi\sqrt{n}})(1+e^{-5\pi\sqrt{n}})\ldots = 2^{\frac{1}{4}}e^{-\pi\sqrt{n}/24}G_n \quad\ldots\ldots\ldots(1)$$

and

$$(1-e^{-\pi\sqrt{n}})(1-e^{-3\pi\sqrt{n}})(1-e^{-5\pi\sqrt{n}})\ldots = 2^{\frac{1}{4}}e^{-\pi\sqrt{n}/24}g_n, \quad\ldots\ldots\ldots(2)$$

then G_n and g_n can always be expressed as roots of algebraical equations when n is any rational number. For we know that

$$(1+q)(1+q^3)(1+q^5)\ldots = 2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{-\frac{1}{12}} \quad\ldots\ldots\ldots\ldots\ldots(3)$$

and

$$(1-q)(1-q^3)(1-q^5)\ldots = 2^{\frac{1}{6}}q^{\frac{1}{24}}k^{-\frac{1}{12}}k'^{\frac{1}{6}}. \quad\ldots\ldots\ldots\ldots\ldots(4)$$

Now the relation between the moduli k and l, which makes

$$n\frac{K'}{K} = \frac{L'}{L},$$

where $n = r/s$, r and s being positive integers, is expressed by the modular equation of the rsth degree. If we suppose that $k = l'$, $k' = l$, so that $K = L'$, $K' = L$, then

$$q = e^{-\pi L'/L} = e^{-\pi\sqrt{n}},$$

and the corresponding value of k may be found by the solution of an algebraical equation.

From (1), (2), (3), and (4) it may easily be deduced that

$$g_{4n} = 2^{\frac{1}{4}}g_n G_n, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$G_n = G_{1/n}, \quad 1/g_n = g_{4/n}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$(g_n G_n)^8(G_n^8 - g_n^8) = \tfrac{1}{4}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

I shall consider only integral values of n. It follows from (7) that we need consider only one of G_n or g_n for any given value of n; and from (5) that we may suppose n not divisible by 4. It is most convenient to consider g_n when n is even, and G_n when n is odd.

2. Suppose then that n is odd. The values of G_n and g_{2n} are got from the same modular equation. For example, let us take the modular equation of the 5th degree, viz.

$$\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 = 2\left(u^2v^2 - \frac{1}{u^2v^2}\right), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where

$$2^{\frac{1}{4}}q^{\frac{1}{24}}u = (1+q)(1+q^3)(1+q^5)\ldots$$

and

$$2^{\frac{1}{4}}q^{\frac{5}{24}}v = (1+q^5)(1+q^{15})(1+q^{25})\ldots.$$

By changing q to $-q$ the above equation may also be written as

$$\left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3 = 2\left(u^2v^2 + \frac{1}{u^2v^2}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

where $$2^{\frac{1}{4}} q^{\frac{1}{24}} u = (1-q)(1-q^3)(1-q^5)\ldots$$

and $$2^{\frac{1}{4}} q^{\frac{5}{24}} v = (1-q^5)(1-q^{15})(1-q^{25})\ldots.$$

If we put $q = e^{-\pi/\sqrt{5}}$ in (8), so that $u = G_{\frac{1}{5}}$ and $v = G_5$, and hence $u = v$, we see that

$$v^4 - v^{-4} = 1.$$

Hence $$v^4 = \frac{1+\sqrt{5}}{2}, \quad G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{4}}.$$

Similarly, by putting $q = e^{-\pi\sqrt{\frac{2}{5}}}$, so that $u = g_{\frac{2}{5}}$ and $v = g_{10}$, and hence $u = 1/v$, we see that

$$v^6 - v^{-6} = 4.$$

Hence $$v^2 = \frac{1+\sqrt{5}}{2}, \quad g_{10} = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)}.$$

Similarly it can be shewn that

$$G_9 = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{3}}, \quad g_{18} = (\sqrt{2}+\sqrt{3})^{\frac{1}{3}},$$

$$G_{17} = \sqrt{\left(\frac{5+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-3}{8}\right)},$$

$$g_{34} = \sqrt{\left(\frac{7+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-1}{8}\right)},$$

and so on.

3. In order to obtain approximations for π we take logarithms of (1) and (2). Thus

$$\left.\begin{aligned}\pi &= \frac{24}{\sqrt{n}}\log\left(2^{\frac{1}{4}} G_n\right)\\ \pi &= \frac{24}{\sqrt{n}}\log\left(2^{\frac{1}{4}} g_n\right)\end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

approximately, the error being nearly $\frac{24}{\sqrt{n}} e^{-\pi\sqrt{n}}$ in both cases. These equations may also be written as

$$e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}} G_n, \quad e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}} g_n. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

In those cases in which G_n^{12} and g_n^{12} are simple quadratic surds we may use the forms

$$(G_n^{12} + G_n^{-12})^{\frac{1}{12}}, \quad (g_n^{12} + g_n^{-12})^{\frac{1}{12}},$$

instead of G_n and g_n, for we have

$$g_n^{12} = \tfrac{1}{8}e^{\frac{1}{2}\pi\sqrt{n}} - \tfrac{3}{2}e^{-\frac{1}{2}\pi\sqrt{n}},$$

approximately, and so

$$g_n^{12} + g_n^{-12} = \tfrac{1}{8}e^{\frac{1}{2}\pi\sqrt{n}} + \tfrac{13}{2}e^{-\frac{1}{2}\pi\sqrt{n}},$$

approximately, so that

$$\pi = \frac{2}{\sqrt{n}} \log \{8 (g_n^{12} + g_n^{-12})\}, \quad \text{.....................(12)}$$

the error being about $\dfrac{104}{\sqrt{n}} e^{-\pi\sqrt{n}}$, which is of the same order as the error in the formulæ (10). The formula (12) often leads to simpler results. Thus the second of formulae (10) gives

$$e^{\pi\sqrt{18}/24} = 2^{\frac{1}{4}} g_{18}$$

or

$$e^{\frac{1}{4}\pi\sqrt{18}} = 10\sqrt{2} + 8\sqrt{3}.$$

But if we use the formula (12), or

$$e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}} (g_n^{12} + g_n^{-12})^{\frac{1}{12}},$$

we get a simpler form, viz.

$$e^{\frac{1}{8}\pi\sqrt{18}} = 2\sqrt{7}.$$

4. The values of g_{2n} and G_n are obtained from the same equation. The approximation by means of g_{2n} is preferable to that by G_n for the following reasons.

(a) It is more accurate. Thus the error when we use G_{65} contains a factor $e^{-\pi\sqrt{65}}$, whereas that when we use g_{130} contains a factor $e^{-\pi\sqrt{130}}$.

(b) For many values of n, g_{2n} is simpler in form than G_n; thus

$$g_{130} = \sqrt{\left\{(2+\sqrt{5})\left(\frac{3+\sqrt{13}}{2}\right)\right\}},$$

while

$$G_{65} = \left\{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{13}}{2}\right)\right\}^{\frac{1}{4}} \sqrt{\left\{\sqrt{\left(\frac{9+\sqrt{65}}{8}\right)} + \sqrt{\left(\frac{1+\sqrt{65}}{8}\right)}\right\}}.$$

(c) For many values of n, g_{2n} involves quadratic surds only, even when G_n is a root of an equation of higher order. Thus G_{23}, G_{29}, G_{31} are roots of cubic equations, G_{47}, G_{79} are those of quintic equations, and G_{71} is that of a septic equation, while g_{46}, g_{58}, g_{62}, g_{94}, g_{142}, and g_{158} are all expressible by quadratic surds.

5. Since G_n and g_n can be expressed as roots of algebraical equations with rational coefficients, the same is true of G_n^{24} or g_n^{24}. So let us suppose that

$$1 = a g_n^{-24} - b g_n^{-48} + \dots,$$

or

$$g_n^{24} = a - b g_n^{-24} + \dots.$$

But we know that

$$64 e^{-\pi\sqrt{n}} g_n^{24} = 1 - 24 e^{-\pi\sqrt{n}} + 276 e^{-2\pi\sqrt{n}} - \dots,$$

$$64 g_n^{24} = e^{\pi\sqrt{n}} - 24 + 276 e^{-\pi\sqrt{n}} - \dots,$$

$$64a - 64b g_n^{-24} + \dots = e^{\pi\sqrt{n}} - 24 + 276 e^{-\pi\sqrt{n}} - \dots,$$

$$64a - 4096 b e^{-\pi\sqrt{n}} + \dots = e^{\pi\sqrt{n}} - 24 + 276 e^{-\pi\sqrt{n}} - \dots,$$

that is

$$e^{\pi\sqrt{n}} = (64a + 24) - (4096b + 276) e^{-\pi\sqrt{n}} + \dots. \quad \text{.........(13)}$$

Similarly, if $$1 = aG_n^{-24} - bG_n^{-48} + \ldots,$$
then $$e^{\pi\sqrt{n}} = (64a - 24) - (4096b + 276)\, e^{-\pi\sqrt{n}} + \ldots. \qquad \ldots\ldots\ldots(14)$$

From (13) and (14) we can find whether $e^{\pi\sqrt{n}}$ is very nearly an integer for given values of n, and ascertain also the number of 9's or 0's in the decimal part. But if G_n and g_n be simple quadratic surds we may work independently as follows. We have, for example,
$$g_{22} = \sqrt{(1 + \sqrt{2})}.$$
Hence
$$64 g_{22}^{24} = e^{\pi\sqrt{22}} - 24 + 276 e^{-\pi\sqrt{22}} - \ldots,$$
$$64 g_{22}^{-24} = 4096 e^{-\pi\sqrt{22}} + \ldots,$$
so that
$$64\,(g_{22}^{24} + g_{22}^{-24}) = e^{\pi\sqrt{22}} - 24 + 4372 e^{-\pi\sqrt{22}} + \ldots = 64\,\{(1 + \sqrt{2})^{12} + (1 - \sqrt{2})^{12}\}.$$
Hence $$e^{\pi\sqrt{22}} = 2508951{\cdot}9982\ldots.$$
Again $$G_{37} = (6 + \sqrt{37})^{\frac{1}{4}},$$
$$64 G_{37}^{24} = e^{\pi\sqrt{37}} + 24 + 276 e^{-\pi\sqrt{37}} + \ldots,$$
$$64 G_{37}^{-24} = 4096 e^{-\pi\sqrt{37}} - \ldots,$$
so that
$$64\,(G_{37}^{24} + G_{37}^{-24}) = e^{\pi\sqrt{37}} + 24 + 4372 e^{-\pi\sqrt{37}} - \ldots = 64\,\{(6 + \sqrt{37})^{6} + (6 - \sqrt{37})^{6}\}.$$
Hence $$e^{\pi\sqrt{37}} = 199148647{\cdot}999978\ldots.$$
Similarly, from $$g_{58} = \sqrt{\left(\frac{5 + \sqrt{29}}{2}\right)},$$
we obtain
$$64\,(g_{58}^{24} + g_{58}^{-24}) = e^{\pi\sqrt{58}} - 24 + 4372 e^{-\pi\sqrt{58}} + \ldots$$
$$= 64\left\{\left(\frac{5 + \sqrt{29}}{2}\right)^{12} + \left(\frac{5 - \sqrt{29}}{2}\right)^{12}\right\}.$$
Hence $$e^{\pi\sqrt{58}} = 24591257751{\cdot}99999982\ldots.$$

6. I have calculated the values of G_n and g_n for a large number of values of n. Many of these results are equivalent to results given by Weber; for example,
$$G_{13}^{4} = \frac{3 + \sqrt{13}}{2}, \qquad G_{25} = \frac{1 + \sqrt{5}}{2},$$
$$g_{30}^{6} = (2 + \sqrt{5})(3 + \sqrt{10}), \qquad G_{37}^{4} = 6 + \sqrt{37},$$
$$G_{49} = \frac{7^{\frac{1}{4}} + \sqrt{(4 + \sqrt{7})}}{2}, \qquad g_{58}^{2} = \frac{5 + \sqrt{29}}{2},$$
$$g_{70}^{2} = \frac{(3 + \sqrt{5})(1 + \sqrt{2})}{2},$$
$$G_{73} = \sqrt{\left(\frac{9 + \sqrt{73}}{8}\right)} + \sqrt{\left(\frac{1 + \sqrt{73}}{8}\right)},$$
$$G_{85} = \left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{9 + \sqrt{85}}{2}\right)^{\frac{1}{4}},$$

$$G_{97} = \sqrt{\left(\frac{13+\sqrt{97}}{8}\right)} + \sqrt{\left(\frac{5+\sqrt{97}}{8}\right)},$$

$$g_{190}^2 = (2+\sqrt{5})(3+\sqrt{10}),$$

$$G_{385}^2 = \tfrac{1}{8}(3+\sqrt{11})(\sqrt{5}+\sqrt{7})(\sqrt{7}+\sqrt{11})(3+\sqrt{5}),$$

and so on. I have also many results not given by Weber. I give a complete table of new results. In Weber's notation, $G_n = 2^{-\frac{1}{4}} f\{\sqrt{(-n)}\}$ and $g_n = 2^{-\frac{1}{4}} f_1\{\sqrt{(-n)}\}$.

TABLE I.

$$g_{62} + \frac{1}{g_{62}} = \tfrac{1}{2}\{\sqrt{(1+\sqrt{2})} + \sqrt{(9+5\sqrt{2})}\},$$

$$G_{65}^2 = \sqrt{\left\{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{13}}{2}\right)\right\}}\left\{\sqrt{\left(\frac{1+\sqrt{65}}{8}\right)} + \sqrt{\left(\frac{9+\sqrt{65}}{8}\right)}\right\},$$

$$g_{66}^2 = \sqrt{(\sqrt{2}+\sqrt{3})}(7\sqrt{2}+3\sqrt{11})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{7+\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{\sqrt{33}-1}{8}\right)}\right\},$$

$$G_{69}^2 = (3\sqrt{3}+\sqrt{23})^{\frac{1}{4}}\left(\frac{5+\sqrt{23}}{4}\right)^{\frac{1}{4}}\left\{\sqrt{\left(\frac{6+3\sqrt{3}}{4}\right)} + \sqrt{\left(\frac{2+3\sqrt{3}}{4}\right)}\right\},$$

$$G_{77}^2 = \{\tfrac{1}{2}(\sqrt{7}+\sqrt{11})(8+3\sqrt{7})\}^{\frac{1}{4}}\left\{\sqrt{\left(\frac{6+\sqrt{11}}{4}\right)} + \sqrt{\left(\frac{2+\sqrt{11}}{4}\right)}\right\},$$

$$G_{81}^3 = \frac{(2\sqrt{3}+2)^{\frac{1}{3}}+1}{(2\sqrt{3}-2)^{\frac{1}{3}}-1},$$

$$g_{90} = \{(2+\sqrt{5})(\sqrt{5}+\sqrt{6})\}^{\frac{1}{6}}\left\{\sqrt{\left(\frac{3+\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{\sqrt{6}-1}{4}\right)}\right\},$$

$$g_{94} + \frac{1}{g_{94}} = \tfrac{1}{2}\{\sqrt{(7+\sqrt{2})} + \sqrt{(7+5\sqrt{2})}\},$$

$$g_{98} + \frac{1}{g_{98}} = \tfrac{1}{2}\{\sqrt{2} + \sqrt{(14+4\sqrt{14})}\},$$

$$g_{114}^2 = \sqrt{(\sqrt{2}+\sqrt{3})}(3\sqrt{2}+\sqrt{19})^{\frac{1}{4}}\left\{\sqrt{\left(\frac{23+3\sqrt{57}}{8}\right)} + \sqrt{\left(\frac{15+3\sqrt{57}}{8}\right)}\right\},$$

$$G_{117} = \tfrac{1}{2}\left(\frac{3+\sqrt{13}}{2}\right)^{\frac{1}{4}}(2\sqrt{3}+\sqrt{13})^{\frac{1}{6}}\{3^{\frac{1}{4}} + \sqrt{(4+\sqrt{3})}\},$$

$$\left.\begin{aligned} G_{121} + \frac{1}{G_{121}} &= \left(\frac{11}{2}\right)^{\frac{1}{6}}\left\{\left(3+\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}} + \left(3-\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}\right\} \\ \frac{1}{G_{121}} &= \frac{1}{3\sqrt{2}}[(11-3\sqrt{11})^{\frac{1}{3}}\{(3\sqrt{11}+3\sqrt{3}-4)^{\frac{1}{3}} \\ &\qquad + (3\sqrt{11}-3\sqrt{3}-4)^{\frac{1}{3}}\} - 2] \end{aligned}\right\},$$

$$g_{126} = \sqrt{\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)}(\sqrt{6}+\sqrt{7})^{\frac{1}{3}}\left\{\sqrt{\left(\frac{3+\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{\sqrt{2}-1}{4}\right)}\right\}^2,$$

$$g_{138}^2 = \sqrt{\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)}\,(78\sqrt{2}+23\sqrt{23})^{\frac{1}{6}}$$
$$\times\left\{\sqrt{\left(\frac{5+2\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{1+2\sqrt{6}}{4}\right)}\right\},$$

$$G_{141}^2 = (4\sqrt{3}+\sqrt{47})^{\frac{1}{4}}\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{\frac{1}{6}}\left\{\sqrt{\left(\frac{18+9\sqrt{3}}{4}\right)}+\sqrt{\left(\frac{14+9\sqrt{3}}{4}\right)}\right\},$$

$$G_{145}^2 = \sqrt{\left\{\frac{(2+\sqrt{5})(5+\sqrt{29})}{2}\right\}}\left\{\sqrt{\left(\frac{17+\sqrt{145}}{8}\right)}+\sqrt{\left(\frac{9+\sqrt{145}}{8}\right)}\right\},$$

$$\frac{1}{G_{147}} = 2^{-\frac{1}{12}}\left[\frac{1}{2}+\frac{1}{\sqrt{3}}\left\{\sqrt{\left(\frac{7}{4}\right)}-(28)^{\frac{1}{6}}\right\}\right],$$

$$G_{153} = \left\{\sqrt{\left(\frac{5+\sqrt{17}}{8}\right)}+\sqrt{\left(\frac{\sqrt{17}-3}{8}\right)}\right\}^2$$
$$\times\left\{\sqrt{\left(\frac{37+9\sqrt{17}}{4}\right)}+\sqrt{\left(\frac{33+9\sqrt{17}}{4}\right)}\right\}^{\frac{1}{2}},$$

$$g_{154}^2 = \sqrt{\left\{(2\sqrt{2}+\sqrt{7})\left(\frac{\sqrt{7}+\sqrt{11}}{2}\right)\right\}}$$
$$\times\left\{\sqrt{\left(\frac{13+2\sqrt{22}}{4}\right)}+\sqrt{\left(\frac{9+2\sqrt{22}}{4}\right)}\right\},$$

$$g_{158}+\frac{1}{g_{158}} = \tfrac{1}{2}\{\sqrt{(9+\sqrt{2})}+\sqrt{(17+13\sqrt{2})}\},$$

$$\left.\begin{aligned} G_{169}+\frac{1}{G_{169}} &= \left(\frac{13}{4}\right)^{\frac{1}{6}}\left\{\left(1+\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(1-\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}\right\}^2 \\ \frac{1}{G_{169}} &= \frac{1}{3}\left[(\sqrt{13}-2)+\left(\frac{13-3\sqrt{13}}{2}\right)^{\frac{1}{3}}\right. \\ &\quad\left.\times\left\{\left(3\sqrt{3}-\frac{11-\sqrt{13}}{2}\right)^{\frac{1}{3}}-\left(3\sqrt{3}+\frac{11-\sqrt{13}}{2}\right)^{\frac{1}{3}}\right\}\right] \end{aligned}\right\},$$

$$g_{198} = \sqrt{(1+\sqrt{2})}\,(4\sqrt{2}+\sqrt{33})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{9+\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{1+\sqrt{33}}{8}\right)}\right\},$$

$$G_{205} = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^{\frac{1}{4}}\left\{\sqrt{\left(\frac{7+\sqrt{41}}{8}\right)}+\sqrt{\left(\frac{\sqrt{41}-1}{8}\right)}\right\},$$

$$G_{213}^2 = (5\sqrt{3}+\sqrt{71})^{\frac{1}{4}}\left(\frac{59+7\sqrt{71}}{4}\right)^{\frac{1}{6}}$$
$$\times\left\{\sqrt{\left(\frac{21+12\sqrt{3}}{2}\right)}+\sqrt{\left(\frac{19+12\sqrt{3}}{2}\right)}\right\},$$

$$G_{217}^2 = \left\{\sqrt{\left(\frac{9+4\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{11+4\sqrt{7}}{2}\right)}\right\}$$
$$\times\left\{\sqrt{\left(\frac{12+5\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{16+5\sqrt{7}}{4}\right)}\right\},$$

$$G_{225} = \left(\frac{1+\sqrt{5}}{4}\right)(2+\sqrt{3})^{\frac{1}{4}}\{\sqrt{(4+\sqrt{15})}+15^{\frac{1}{4}}\},$$

$$g_{238} = \left\{\sqrt{\left(\frac{1+2\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{5+2\sqrt{2}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{1+3\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{5+3\sqrt{2}}{4}\right)}\right\},$$

$$G_{265}^2 = \sqrt{\left\{(2+\sqrt{5})\left(\frac{7+\sqrt{53}}{2}\right)\right\}} \left\{\sqrt{\left(\frac{89+5\sqrt{265}}{8}\right)} + \sqrt{\left(\frac{81+5\sqrt{265}}{8}\right)}\right\},$$

$$G_{280} = \left[\sqrt{\left\{\frac{17+\sqrt{17}+17^{\frac{1}{4}}(5+\sqrt{17})}{16}\right\}} + \sqrt{\left\{\frac{1+\sqrt{17}+17^{\frac{1}{4}}(5+\sqrt{17})}{16}\right\}}\right]^2,$$

$$G_{301}^2 = \left\{(8+3\sqrt{7})\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right)\right\}^{\frac{1}{4}} \times \left\{\sqrt{\left(\frac{46+7\sqrt{43}}{4}\right)} + \sqrt{\left(\frac{42+7\sqrt{43}}{4}\right)}\right\},$$

$$g_{310} = \left(\frac{1+\sqrt{5}}{2}\right)\sqrt{(1+\sqrt{2})}\left\{\sqrt{\left(\frac{7+2\sqrt{10}}{4}\right)} + \sqrt{\left(\frac{3+2\sqrt{10}}{4}\right)}\right\},$$

$$\left.\begin{aligned} G_{325} &= \left(\frac{3+\sqrt{13}}{2}\right)^{\frac{1}{4}} t, \text{ where} \\ & t^3 + t^2\left(\frac{1-\sqrt{13}}{2}\right)^2 + t\left(\frac{1+\sqrt{13}}{2}\right)^2 + 1 \\ & \qquad = \sqrt{5}\left\{t^3 - t^2\left(\frac{1+\sqrt{13}}{2}\right) + t\left(\frac{1-\sqrt{13}}{2}\right) - 1\right\} \end{aligned}\right\},$$

$$G_{333} = \tfrac{1}{2}(6+\sqrt{37})^{\frac{1}{4}}(7\sqrt{3}+2\sqrt{37})^{\frac{1}{6}}\{\sqrt{(7+2\sqrt{3})} + \sqrt{(3+2\sqrt{3})}\},$$

$$\left.\begin{aligned} G_{363} &= 2^{\frac{5}{12}} t, \text{ where} \\ & 2t^3 - t^2\{(4+\sqrt{33}) + \sqrt{(11+2\sqrt{33})}\} \\ & \qquad - t\{1+\sqrt{(11+2\sqrt{33})}\} - 1 = 0 \end{aligned}\right\},$$

$$G_{441}^2 = \left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)(2+\sqrt{3})^{\frac{1}{3}}\left\{\frac{2+\sqrt{7}+\sqrt{(7+4\sqrt{7})}}{2}\right\}\left\{\frac{\sqrt{(3+\sqrt{7})}+(6\sqrt{7})^{\frac{1}{4}}}{\sqrt{(3+\sqrt{7})}-(6\sqrt{7})^{\frac{1}{4}}}\right\},$$

$$G_{445} = \sqrt{(2+\sqrt{5})}\left(\frac{21+\sqrt{445}}{2}\right)^{\frac{1}{4}}\sqrt{\left\{\left(\frac{13+\sqrt{89}}{8}\right)} + \sqrt{\left(\frac{5+\sqrt{89}}{8}\right)}\right\},$$

$$G_{465}^2 = \sqrt{\left\{(2+\sqrt{3})\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{3}+\sqrt{31}}{2}\right)\right\}}(5\sqrt{5}+2\sqrt{31})^{\frac{1}{6}} \times \left\{\sqrt{\left(\frac{2+\sqrt{31}}{4}\right)} + \sqrt{\left(\frac{6+\sqrt{31}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{11+2\sqrt{31}}{2}\right)} + \sqrt{\left(\frac{13+2\sqrt{31}}{2}\right)}\right\},$$

$$G_{505}^2 = (2+\sqrt{5})\sqrt{\left\{\left(\frac{1+\sqrt{5}}{2}\right)(10+\sqrt{101})\right\}} \times \left\{\left(\frac{5\sqrt{5}+\sqrt{101}}{4}\right) + \sqrt{\left(\frac{105+\sqrt{505}}{8}\right)}\right\},$$

$$g_{522} = \sqrt{\left(\frac{5+\sqrt{29}}{2}\right)}(5\sqrt{29}+11\sqrt{6})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\},$$

$$G_{553}{}^2 = \left\{\sqrt{\left(\frac{96+11\sqrt{79}}{4}\right)} + \sqrt{\left(\frac{100+11\sqrt{79}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{141+16\sqrt{79}}{2}\right)} + \sqrt{\left(\frac{143+16\sqrt{79}}{2}\right)}\right\},$$

$$g_{630} = (\sqrt{14}+\sqrt{15})^{\frac{1}{6}} \sqrt{\left\{(1+\sqrt{2})\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right\}} \times \left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+2}{4}\right)} + \sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-2}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+4}{8}\right)} + \sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-4}{8}\right)}\right\},$$

$$G_{765}{}^2 = \left(\frac{3+\sqrt{5}}{2}\right)(16+\sqrt{255})^{\frac{1}{6}} \sqrt{\left\{(4+\sqrt{15})\left(\frac{9+\sqrt{85}}{2}\right)\right\}} \times \left\{\sqrt{\left(\frac{6+\sqrt{51}}{4}\right)} + \sqrt{\left(\frac{10+\sqrt{51}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{18+3\sqrt{51}}{4}\right)} + \sqrt{\left(\frac{22+3\sqrt{51}}{4}\right)}\right\},$$

$$G_{777}{}^2 = \sqrt{\left\{(2+\sqrt{3})(6+\sqrt{37})\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right\}}(246\sqrt{7}+107\sqrt{37})^{\frac{1}{6}} \times \left\{\sqrt{\left(\frac{6+3\sqrt{7}}{4}\right)} + \sqrt{\left(\frac{10+3\sqrt{7}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{15+6\sqrt{7}}{2}\right)} + \sqrt{\left(\frac{17+6\sqrt{7}}{2}\right)}\right\},$$

$$G_{1225} = \left(\frac{1+\sqrt{5}}{2}\right)(6+\sqrt{35})^{\frac{1}{4}}\left\{\frac{7^{\frac{1}{4}}+\sqrt{(4+\sqrt{7})}}{2}\right\}^{\frac{3}{2}} \times \left[\sqrt{\left\{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{(10\sqrt{7})}}{8}\right\}} + \sqrt{\left\{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{(10\sqrt{7})}}{8}\right\}}\right],$$

$$G_{1353}{}^2 = \sqrt{\left\{(3+\sqrt{11})(5+3\sqrt{3})\left(\frac{11+\sqrt{123}}{2}\right)\right\}} \times \left(\frac{6817+321\sqrt{451}}{4}\right)^{\frac{1}{6}} \times \left\{\sqrt{\left(\frac{17+3\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{25+3\sqrt{33}}{8}\right)}\right\} \times \left\{\sqrt{\left(\frac{561+99\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{569+99\sqrt{33}}{8}\right)}\right\},$$

$$G_{1645}{}^2 = (2+\sqrt{5})\sqrt{\left\{(3+\sqrt{7})\left(\frac{7+\sqrt{47}}{2}\right)\right\}}\left(\frac{73\sqrt{5}+9\sqrt{329}}{2}\right)^{\frac{1}{4}} \times \left\{\sqrt{\left(\frac{119+7\sqrt{329}}{8}\right)} + \sqrt{\left(\frac{127+7\sqrt{329}}{8}\right)}\right\} \times \left\{\sqrt{\left(\frac{743+41\sqrt{329}}{8}\right)} + \sqrt{\left(\frac{751+41\sqrt{329}}{8}\right)}\right\}.$$

7. Hence we deduce the following approximate formulæ.

TABLE II.

$$e^{\frac{1}{2}\pi\sqrt{18}} = 2\sqrt{7},\quad e^{\pi\sqrt{22}/12} = 2+\sqrt{2},\quad e^{\frac{1}{4}\pi\sqrt{30}} = 20\sqrt{3}+16\sqrt{6},$$

$$e^{\frac{1}{2}\pi\sqrt{34}} = 12(4+\sqrt{17}),\quad e^{\frac{1}{2}\pi\sqrt{46}} = 144(147+104\sqrt{2}),$$

$$e^{\frac{1}{2}\pi\sqrt{42}} = 84+32\sqrt{6},\quad e^{\pi\sqrt{58}/12} = \frac{5+\sqrt{29}}{\sqrt{2}},$$

$$e^{\frac{1}{2}\pi\sqrt{70}} = 60\sqrt{35}+96\sqrt{14},\quad e^{\frac{1}{2}\pi\sqrt{78}} = 300\sqrt{3}+208\sqrt{6},$$

$$e^{\pi\sqrt{55}/24} = \frac{1+\sqrt{(3+2\sqrt{5})}}{\sqrt{2}},\quad e^{\frac{1}{2}\pi\sqrt{102}} = 800\sqrt{3}+196\sqrt{51},$$

$$e^{\frac{1}{2}\pi\sqrt{130}} = 12(323+40\sqrt{65}),\quad e^{\pi\sqrt{190}/12} = (2\sqrt{2}+\sqrt{10})(3+\sqrt{10}),$$

$$\pi = \frac{12}{\sqrt{130}}\log\left\{\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right\},$$

$$\pi = \frac{24}{\sqrt{142}}\log\left\{\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right\},$$

$$\pi = \frac{12}{\sqrt{190}}\log\{(2\sqrt{2}+\sqrt{10})(3+\sqrt{10})\},$$

$$\pi = \frac{12}{\sqrt{310}}\log\left[\tfrac{1}{4}(3+\sqrt{5})(2+\sqrt{2})\{(5+2\sqrt{10})+\sqrt{(61+20\sqrt{10})}\}\right],$$

$$\pi = \frac{4}{\sqrt{522}}\log\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)^3(5\sqrt{29}+11\sqrt{6}) \times\left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^6\right].$$

The last five formulæ are correct to 15, 16, 18, 22, and 31 places of decimals respectively.

8. Thus we have seen how to approximate to π by means of logarithms of surds. I shall now shew how to obtain approximations in terms of surds only. If

$$n\frac{K'}{K} = \frac{L'}{L},$$

we have

$$\frac{n\,dk}{kk'^2K^2} = \frac{dl}{ll'^2L^2}.$$

But, by means of the modular equation connecting k and l, we can express dk/dl as an algebraic function of k, a function moreover in which all coefficients which occur are algebraic numbers. Again,

$$q = e^{-\pi K'/K},\quad q^n = e^{-\pi L'/L},$$

$$\frac{q^{\frac{1}{12}}(1-q^2)(1-q^4)(1-q^6)\ldots}{q^{\frac{1}{12}n}(1-q^{2n})(1-q^{4n})(1-q^{6n})\ldots} = \left(\frac{kk'}{ll'}\right)^{\frac{1}{6}}\sqrt{\left(\frac{K}{L}\right)}. \quad \ldots\ldots(15)$$

Differentiating this equation logarithmically, and using the formula

$$\frac{dq}{dk} = \frac{\pi^2 q}{2kk'^2K^2},$$

we see that

$$n\left\{1 - 24\left(\frac{q^{2n}}{1-q^{2n}} + \frac{2q^{4n}}{1-q^{4n}} + \dots\right)\right\} - \left\{1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \dots\right)\right\} = \frac{KL}{\pi^2} A(k), \quad \dots(16)$$

where $A(k)$ denotes an algebraic function of the special class described above. I shall use the letter A generally to denote a function of this type.

Now, if we put $k = l'$ and $k' = l$ in (16), we have

$$n\left\{1 - 24\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \dots\right)\right\} - \left\{1 - 24\left(\frac{1}{e^{2\pi/\sqrt{n}}-1} + \frac{2}{e^{4\pi/\sqrt{n}}-1} + \dots\right)\right\} = \left(\frac{K}{\pi}\right)^2 A(k). \quad \dots(17)$$

The algebraic function $A(k)$ of course assumes a purely numerical form when we substitute the value of k deduced from the modular equation. But by substituting $k = l'$ and $k' = l$ in (15) we have

$$n^{\frac{1}{4}} e^{-\pi\sqrt{n}/12}(1 - e^{-2\pi\sqrt{n}})(1 - e^{-4\pi\sqrt{n}})(1 - e^{-6\pi\sqrt{n}})\dots = e^{-\pi/(12\sqrt{n})}(1 - e^{-2\pi/\sqrt{n}})(1 - e^{-4\pi/\sqrt{n}})(1 - e^{-6\pi/\sqrt{n}})\dots.$$

Differentiating the above equation logarithmically we have

$$n\left\{1 - 24\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \dots\right)\right\} + \left\{1 - 24\left(\frac{1}{e^{2\pi/\sqrt{n}}-1} + \frac{2}{e^{4\pi/\sqrt{n}}-1} + \dots\right)\right\} = \frac{6\sqrt{n}}{\pi}. \quad \dots\dots\dots(18)$$

Now, adding (17) and (18), we have

$$1 - \frac{3}{\pi\sqrt{n}} - 24\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \dots\right) = \left(\frac{K}{\pi}\right)^2 A(k). \quad \dots\dots(19)$$

But it is known that

$$1 - 24\left(\frac{q}{1+q} + \frac{3q^3}{1+q^3} + \frac{5q^5}{1+q^5} + \dots\right) = \left(\frac{2K}{\pi}\right)^2(1 - 2k^2),$$

so that

$$1 - 24\left(\frac{1}{e^{\pi\sqrt{n}}+1} + \frac{3}{e^{3\pi\sqrt{n}}+1} + \dots\right) = \left(\frac{K}{\pi}\right)^2 A(k). \quad \dots\dots\dots(20)$$

Hence, dividing (19) by (20), we have

$$\frac{1 - \dfrac{3}{\pi\sqrt{n}} - 24\left(\dfrac{1}{e^{2\pi\sqrt{n}}-1} + \dfrac{2}{e^{4\pi\sqrt{n}}-1} + \dots\right)}{1 - 24\left(\dfrac{1}{e^{\pi\sqrt{n}}+1} + \dfrac{3}{e^{3\pi\sqrt{n}}+1} + \dots\right)} = R, \quad \dots\dots\dots(21)$$

where R can always be expressed in radicals if n is any rational number. Hence we have

$$\pi = \frac{3}{(1-R)\sqrt{n}}, \qquad \text{..........................(22)}$$

nearly, the error being about $8\pi e^{-\pi\sqrt{n}}(\pi\sqrt{n}-3)$.

9. We may get a still closer approximation from the following results. It is known that

$$1 + 240\sum_{r=1}^{r=\infty} \frac{r^3 q^{2r}}{1-q^{2r}} = \left(\frac{2K}{\pi}\right)^4 (1-k^2k'^2),$$

and also that

$$1 - 504\sum_{r=1}^{r=\infty} \frac{r^5 q^{2r}}{1-q^{2r}} = \left(\frac{2K}{\pi}\right)^6 (1-2k^2)(1+\tfrac{1}{2}k^2k'^2).$$

Hence, from (19), we see that

$$\left\{1 - \frac{3}{\pi\sqrt{n}} - 24\sum_{r=1}^{r=\infty}\frac{r}{e^{2\pi r\sqrt{n}}-1}\right\}\left\{1 + 240\sum_{r=1}^{r=\infty}\frac{r^3}{e^{2\pi r\sqrt{n}}-1}\right\}$$

$$= R'\left\{1 - 504\sum_{r=1}^{r=\infty}\frac{r^5}{e^{2\pi r\sqrt{n}}-1}\right\}, \quad \text{......(23)}$$

where R' can always be expressed in radicals for any rational value of n. Hence

$$\pi = \frac{3}{(1-R')\sqrt{n}}, \quad \text{.............................(24)}$$

nearly, the error being about $24\pi(10\pi\sqrt{n}-31)e^{-2\pi\sqrt{n}}$

It will be seen that the error in (24) is much less than that in (22), if n is at all large.

10. In order to find R and R' the series in (16) must be calculated in finite terms. I shall give the final results for a few values of n.

TABLE III.

$$q = e^{-\pi K'/K}, \quad q^n = e^{-\pi L'/L},$$

$$f(q) = n\left(1 - 24\sum_1^\infty \frac{q^{2mn}}{1-q^{2mn}}\right) - \left(1 - 24\sum_1^\infty \frac{q^{2m}}{1-q^{2m}}\right),$$

$$f(2) = \frac{4KL}{\pi^2}(k'+l),$$

$$f(3) = \frac{4KL}{\pi^2}(1+kl+k'l'),$$

$$f(4) = \frac{4KL}{\pi^2}(\sqrt{k'}+\sqrt{l})^2,$$

$$f(5) = \frac{4KL}{\pi^2}(3+kl+k'l')\sqrt{\left(\frac{1+kl+k'l'}{2}\right)},$$

$$f(7) = \frac{12KL}{\pi^2}(1+kl+k'l'),$$

$$f(11)=\frac{8KL}{\pi^2}\{2(1+kl+k'l')+\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\},$$

$$f(15)=\frac{4KL}{\pi^2}[\{1+(kl)^{\frac{1}{2}}+(k'l')^{\frac{1}{2}}\}^2-\{1+kl+k'l'\}],$$

$$f(17)=\frac{4KL}{\pi^2}\sqrt{\{44(1+k^2l^2+k'^2l'^2)+168(kl+k'l'-kk'll')}$$
$$-102(1-kl-k'l')(4kk'll')^{\frac{1}{3}}-192(4kk'll')^{\frac{2}{3}}\},$$

$$f(19)=\frac{24KL}{\pi^2}\{(1+kl+k'l')+\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\},$$

$$f(23)=\frac{4KL}{\pi^2}[11(1+kl+k'l')-16(4kk'll')^{\frac{1}{6}}\{1+\sqrt{(kl)}+\sqrt{(k'l')}\}$$
$$-20(4kk'll')^{\frac{1}{3}}],$$

$$f(31)=\frac{12KL}{\pi^2}[3(1+kl+k'l')+4\{\sqrt{(kl)}+\sqrt{(k'l')}+\sqrt{(kk'll')}\}$$
$$-4(kk'll')^{\frac{1}{4}}\{1+(kl)^{\frac{1}{4}}+(k'l')^{\frac{1}{4}}\}].$$

$$f(35)=\frac{4KL}{\pi^2}[2\{\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\}$$
$$+(4kk'll')^{-\frac{1}{6}}\{1-\sqrt{(kl)}-\sqrt{(k'l')}\}^3].$$

Thus the sum of the series (19) can be found in finite terms, when $n=2, 3, 4, 5, \ldots$, from the equations in Table III. We can use the same table to find the sum of (19) when $n=9, 25, 49, \ldots$; but then we have also to use the equation

$$\frac{3}{\pi}=1-24\left(\frac{1}{e^{2\pi}-1}+\frac{2}{e^{4\pi}-1}+\frac{3}{e^{6\pi}-1}+\ldots\right),$$

which is got by putting $k=k'=1/\sqrt{2}$ and $n=1$ in (18).

Similarly we can find the sum of (19) when $n=21, 33, 57, 93, \ldots$, by combining the values of $f(3)$ and $f(7)$, $f(3)$ and $f(11)$, and so on, obtained from Table III.

11. The errors in (22) and (24) being about

$$8\pi e^{-\pi\sqrt{n}}(\pi\sqrt{n}-3),\quad 24\pi(10\pi\sqrt{n}-31)e^{-2\pi\sqrt{n}},$$

we cannot expect a high degree of approximation for small values of n. Thus, if we put $n=7, 9, 16$, and 25 in (24), we get

$$\frac{19}{16}\sqrt{7}=3\cdot14180\ldots,$$

$$\frac{7}{3}\left(1+\frac{\sqrt{3}}{5}\right)=3\cdot14162\ldots,$$

$$\frac{99}{80}\left(\frac{7}{7-3\sqrt{2}}\right)=3\cdot14159274\ldots,$$

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right)=3\cdot14159265380\ldots,$$

while $$\pi = 3{\cdot}14159265358\ldots.$$
But if we put $n = 25$ in (22), we get only
$$\frac{9}{5} + \sqrt{\left(\frac{9}{5}\right)} = 3{\cdot}14164\ldots.$$

12. Another curious approximation to π is
$$\left(9^2 + \frac{19^2}{22}\right)^{\frac{1}{4}} = 3{\cdot}14159265262\ldots.$$
This value was obtained empirically, and it has no connection with the preceding theory.

The actual value of π, which I have used for purposes of calculation, is
$$\frac{355}{113}\left(1 - \frac{{\cdot}0003}{3533}\right) = 3{\cdot}1415926535897943\ldots,$$
which is greater than π by about 10^{-16}. This is obtained by simply taking the reciprocal of $1 - (113\pi/355)$.

In this connection it may be interesting to note the following simple geometrical constructions for π. The first merely gives the ordinary value 355/113. The second gives the value $(9^2 + 19^2/22)^{\frac{1}{4}}$ mentioned above.

(1) Let AB (Fig. 1) be a diameter of a circle whose centre is O.

Bisect AO at M and trisect OB at T.

Draw TP perpendicular to AB and meeting the circumference at P.

Draw a chord BQ equal to PT and join AQ.

Draw OS and TR parallel to BQ and meeting AQ at S and R respectively.

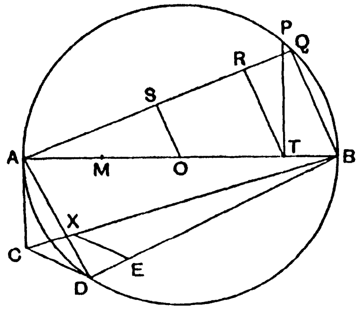


Fig. 1.

Draw a chord AD equal to AS and a tangent $AC = RS$.

Join BC, BD, and CD; cut off $BE = BM$, and draw EX, parallel to CD, meeting BC at X.

Then the square on BX is very nearly equal to the area of the circle, the error being less than a tenth of an inch when the diameter is 40 miles long.

(2) Let AB (Fig. 2) be a diameter of a circle whose centre is O. Bisect the arc ACB at C and trisect AO at T.

Join BC and cut off from it CM and MN equal to AT.

Join AM and AN and cut off from the latter AP equal to AM.

Through P draw PQ parallel to MN and meeting AM at Q.

Join OQ and through T draw TR, parallel to OQ, and meeting AQ at R.

Draw AS perpendicular to AO and equal to AR, and join OS.

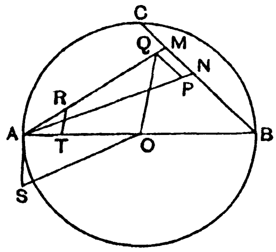


Fig. 2.

Then the mean proportional between OS and OB will be very nearly equal to a sixth of the circumference, the error being less than a twelfth of an inch when the diameter is 8000 miles long.

13. I shall conclude this paper by giving a few series for $1/\pi$.

It is known that, when $k \leqslant 1/\sqrt{2}$,

$$\left(\frac{2K}{\pi}\right)^2 = 1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1.3}{2.4}\right)^3 (2kk')^4 + \dots \quad \dots\dots\dots(25)$$

Hence we have

$$q^{\frac{1}{3}}(1-q^2)^4(1-q^4)^4(1-q^6)^4\dots$$
$$= \left(\frac{1}{4}kk'\right)^{\frac{2}{3}} \left\{1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1.3}{2.4}\right)^3 (2kk')^4 + \dots\right\}. \quad \dots\dots(26)$$

Differentiating both sides in (26) logarithmically with respect to k, we can easily shew that

$$1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \dots\right)$$
$$= (1-2k^2)\left\{1 + 4\left(\frac{1}{2}\right)^3 (2kk')^2 + 7\left(\frac{1.3}{2.4}\right)^3 (2kk')^4 + \dots\right\}. \dots(27)$$

But it follows from (19) that, when $q = e^{-\pi\sqrt{n}}$, n being a rational number, the left-hand side of (27) can be expressed in the form

$$A\left(\frac{2K}{\pi}\right)^2 + \frac{B}{\pi},$$

where A and B are algebraic numbers expressible by surds. Combining (25) and (27) in such a way as to eliminate the term $(2K/\pi)^2$, we are left with a series for $1/\pi$. Thus, for example,

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots,$$
$$(q = e^{-\pi\sqrt{3}},\ 2kk' = \tfrac{1}{2}), \quad \dots\dots\dots\dots\dots(28)$$

$$\frac{16}{\pi} = 5 + \frac{47}{64}\left(\frac{1}{2}\right)^3 + \frac{89}{64^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{131}{64^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots,$$
$$(q = e^{-\pi\sqrt{7}},\ 2kk' = \tfrac{1}{8}), \quad \dots\dots\dots\dots\dots(29)$$

$$\frac{32}{\pi} = (5\sqrt{5}-1) + \frac{47\sqrt{5}+29}{64}\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^8$$
$$+ \frac{89\sqrt{5}+59}{64^2}\left(\frac{1.3}{2.4}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^{16} + \ldots,$$
$$\left[q = e^{-\pi\sqrt{15}},\ 2kk' = \frac{1}{8}\left(\frac{\sqrt{5}-1}{2}\right)\right]; \quad \ldots\ldots\ldots(30)$$

here $5\sqrt{5}-1$, $47\sqrt{5}+29$, $89\sqrt{5}+59$, ... are in arithmetical progression.

14. The ordinary modular equations express the relations which hold between k and l when $nK'/K = L'/L$, or $q^n = Q$, where

$$q = e^{-\pi K'/K}, \quad Q = e^{-\pi L'/L},$$
$$K = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \ldots.$$

There are corresponding theories in which q is replaced by one or other of the functions

$$q_1 = e^{-\pi K_1'\sqrt{2}/K_1}, \quad q_2 = e^{-2\pi K_2'/(K_2\sqrt{3})}, \quad q_3 = e^{-2\pi K_3'/K_3},$$

where

$$K_1 = 1 + \frac{1.3}{4^2}k^2 + \frac{1.3.5.7}{4^2.8^2}k^4 + \frac{1.3.5.7.9.11}{4^2.8^2.12^2}k^6 + \ldots,$$
$$K_2 = 1 + \frac{1.2}{3^2}k^2 + \frac{1.2.4.5}{3^2.6^2}k^4 + \frac{1.2.4.5.7.8}{3^2.6^2.9^2}k^6 + \ldots,$$
$$K_3 = 1 + \frac{1.5}{6^2}k^2 + \frac{1.5.7.11}{6^2.12^2}k^4 + \frac{1.5.7.11.13.17}{6^2.12^2.18^2}k^6 + \ldots.$$

From these theories we can deduce further series for $1/\pi$, such as

$$\frac{27}{4\pi} = 2 + 17\,\frac{1}{2}\,\frac{1}{3}\,\frac{2}{3}\left(\frac{2}{27}\right)$$
$$+ 32\,\frac{1.3}{2.4}\,\frac{1.4}{3.6}\,\frac{2.5}{3.6}\left(\frac{2}{27}\right)^2 + \ldots, \quad \ldots\ldots(31)$$

$$\frac{15\sqrt{3}}{2\pi} = 4 + 37\,\frac{1}{2}\,\frac{1}{3}\,\frac{2}{3}\left(\frac{4}{125}\right)$$
$$+ 70\,\frac{1.3}{2.4}\,\frac{1.4}{3.6}\,\frac{2.5}{3.6}\left(\frac{4}{125}\right)^2 + \ldots, \ldots\ldots(32)$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1 + 12\,\frac{1}{2}\,\frac{1}{6}\,\frac{5}{6}\left(\frac{4}{125}\right)$$
$$+ 23\,\frac{1.3}{2.4}\,\frac{1.7}{6.12}\,\frac{5.11}{6.12}\left(\frac{4}{125}\right)^2 + \ldots, \ldots(33)$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8 + 141\,\frac{1}{2}\,\frac{1}{6}\,\frac{5}{6}\left(\frac{4}{85}\right)^3$$
$$+ 274\,\frac{1.3}{2.4}\,\frac{1.7}{6.12}\,\frac{5.11}{6.12}\left(\frac{4}{85}\right)^6 + \ldots, \ldots(34)$$

$$\frac{4}{\pi}=\frac{3}{2}-\frac{23}{2^3}\frac{1}{2}\frac{1.3}{4^2}+\frac{43}{2^5}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}-\ldots, \quad\ldots\ldots\ldots\ldots\ldots(35)$$

$$\frac{4}{\pi\sqrt{3}}=\frac{3}{4}-\frac{31}{3.4^3}\frac{1}{2}\frac{1.3}{4^2}+\frac{59}{3^2.4^5}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}-\ldots, \quad\ldots\ldots\ldots(36)$$

$$\frac{4}{\pi}=\frac{23}{18}-\frac{283}{18^3}\frac{1}{2}\frac{1.3}{4^2}+\frac{543}{18^6}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}-\ldots, \quad\ldots\ldots\ldots(37)$$

$$\frac{4}{\pi\sqrt{5}}=\frac{41}{72}-\frac{685}{5.72^3}\frac{1}{2}\frac{1.3}{4^2}+\frac{1329}{5^2.72^5}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}-\ldots,\ldots\ldots(38)$$

$$\frac{4}{\pi}=\frac{1123}{882}-\frac{22583}{882^3}\frac{1}{2}\frac{1.3}{4^2}+\frac{44043}{882^6}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}-\ldots,\ldots(39)$$

$$\frac{2\sqrt{3}}{\pi}=1+\frac{9}{9}\frac{1}{2}\frac{1.3}{4^2}+\frac{17}{9^2}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}+\ldots, \quad\ldots\ldots\ldots\ldots\ldots\ldots(40)$$

$$\frac{1}{2\pi\sqrt{2}}=\frac{1}{9}+\frac{11}{9^3}\frac{1}{2}\frac{1.3}{4^2}+\frac{21}{9^5}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}+\ldots, \quad\ldots\ldots\ldots\ldots\ldots(41)$$

$$\frac{1}{3\pi\sqrt{3}}=\frac{3}{49}+\frac{43}{49^3}\frac{1}{2}\frac{1.3}{4^2}+\frac{83}{49^5}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}+\ldots, \quad\ldots\ldots\ldots\ldots(42)$$

$$\frac{2}{\pi\sqrt{11}}=\frac{19}{99}+\frac{299}{99^3}\frac{1}{2}\frac{1.3}{4^2}+\frac{579}{99^6}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}+\ldots, \quad\ldots\ldots\ldots\ldots(43)$$

$$\frac{1}{2\pi\sqrt{2}}=\frac{1103}{99^2}+\frac{27493}{99^6}\frac{1}{2}\frac{1.3}{4^2}+\frac{53883}{99^{10}}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2}+\ldots.\ \ldots(44)$$

In all these series the first factors in each term form an arithmetical progression; e.g. 2, 17, 32, 47, ..., in (31), and 4, 37, 70, 103, ..., in (32). The first two series belong to the theory of q_2, the next two to that of q_3, and the rest to that of q_1.

The last series (44) is extremely rapidly convergent. Thus, taking only the first term, we see that

$$\frac{1103}{99^2}=\cdot 11253953678\ldots,$$

$$\frac{1}{2\pi\sqrt{2}}=\cdot 11253953951\ldots.$$

15. In concluding this paper I have to remark that the series

$$1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\frac{3q^6}{1-q^6}+\ldots\right),$$

which has been discussed in §§ 8—13, is very closely connected with the perimeter of an ellipse whose eccentricity is k. For, if a and b be the semi-major and the semi-minor axes, it is known that

$$p=2\pi a\left\{1-\frac{1}{2^2}k^2-\frac{1^2.3}{2^2.4^2}k^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}k^6-\ldots\right\}, \quad\ldots\ldots\ldots(45)$$

where p is the perimeter and k the eccentricity. It can easily be seen from (45) that

$$p = 4ak'^2\left\{K + k\frac{dK}{dk}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46)$$

But, taking the equation

$$q^{\frac{1}{12}}(1-q^2)(1-q^4)(1-q^6)\ldots = (2kk')^{\frac{1}{6}}\sqrt{(K/\pi)},$$

and differentiating both sides logarithmically with respect to k, and combining the result with (46) in such a way as to eliminate dK/dk, we can shew that

$$p = \frac{4a}{3K}\left[K^2(1+k'^2) + (\tfrac{1}{2}\pi)^2\left\{1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \ldots\right)\right\}\right]. \quad (47)$$

But we have shewn already that the right-hand side of (47) can be expressed in terms of K if $q = e^{-\pi\sqrt{n}}$, where n is any rational number. It can also be shewn that K can be expressed in terms of Γ-functions if q be of the forms $e^{-\pi n}$, $e^{-\pi n\sqrt{2}}$ and $e^{-\pi n\sqrt{3}}$, where n is rational. Thus, for example, we have

$$\left.\begin{aligned}
&k = \sin\frac{\pi}{4}, \quad q = e^{-\pi},\\
&\qquad p = a\sqrt{\left(\frac{\pi}{2}\right)}\left\{\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} + \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\right\},\\
&k = \tan\frac{\pi}{8}, \quad q = e^{-\pi\sqrt{2}},\\
&\qquad p = a\sqrt{\left(\frac{\pi}{4}\right)}\left\{\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} + \frac{\Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})}\right\},\\
&k = \sin\frac{\pi}{12}, \quad q = e^{-\pi\sqrt{3}},\\
&\qquad p = a\sqrt{\left(\frac{\pi}{\sqrt{3}}\right)}\left\{\left(1 + \frac{1}{\sqrt{3}}\right)\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})} + 2\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}\right\},\\
&\frac{b}{a} = \tan^2\frac{\pi}{8}, \quad q = e^{-2\pi},\\
&\qquad p = (a+b)\sqrt{\left(\frac{\pi}{2}\right)}\left\{\frac{1}{2}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} + \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\right\},
\end{aligned}\right\} \ldots\ldots(48)$$

and so on.

16. The following approximations for p were obtained empirically:

$$p = \pi\left[3(a+b) - \sqrt{\{(a+3b)(3a+b)\}} + \epsilon\right], \ldots\ldots\ldots\ldots\ldots(49)$$

where ϵ is about $ak^{12}/1048576$;

$$p = \pi\left\{(a+b) + \frac{3(a-b)^2}{10(a+b) + \sqrt{(a^2 + 14ab + b^2)}} + \epsilon\right\}, \ldots\ldots\ldots(50)$$

where ϵ is about $3ak^{20}/68719476736$.

THE MARQUIS AND THE LAND-AGENT; A TALE OF THE EIGHTEENTH CENTURY.

Presidential Address to the Mathematical Association, 1933.

BY G. N. WATSON, Sc.D., F.R.S.

THERE is a well-known story about the late Archbishop Temple, that he once had to listen to a sermon by a youthful and inexperienced clergyman, and to dine with him afterwards; the young man, by way of making conversation during the meal, ventured to remark, "I think, my lord, that I chose a good text for my sermon". Instantly there came the grim reply, "There was nothing wrong with the text". It may be that the consequence of my having selected a seductive title which does not possess a very close connection with the actual subject of my address will be that, when we adjourn presently, I may get the impression that my audience has consisted entirely of archbishops.

It seems to me that, whenever a Pure Mathematician is called upon to deliver a lecture of a more or less general nature outside the walls of his usual lecture room, the necessity of choosing a topic of general interest at once confronts him with a difficult problem; my choice of a title has been dictated by my desire to ensure that my address should not be entirely devoid of interest. An Applied Mathematician, of course, labours under no similar difficulty; it is always open to him to be irrelevant and to choose as his theme (if I may borrow a phrase of one of my predecessors) "what has been humorously called the real world, the world of physics and sensation, of sight and hearing, heat and cold, earthquakes and eclipses; and earthquakes and eclipses are plainly not constituents of the world of mathematics". It is, however, extremely difficult to give an interesting oral exposition of any modern developments in Pure Mathematics, and I personally have not the courage to attempt to do so; I can imagine few ways of rapidly emptying this room which would be more effective than an attempt on my part to expound the work on Singular Moduli which has occupied all my spare time during the last couple of years. I consider, too, that subjects of a pedagogical character are equally barred to me; I agree with Professor Hardy when he said on a similar occasion some years ago that he considered such questions to be of secondary importance.

Like many of the problems which confront the Pure Mathematician, the particular problem which I am considering always has one method of solution available; it is not in my opinion a particularly good solution, but I think that it is a perfectly valid procedure; and I have chosen the path of least resistance in adopting it, and therefore I have selected a topic of an historical nature; I propose to give you an account of some of the comparatively elementary work on arcs of ellipses and other curves which seems to me to have led up to the idea, familiar to you all, which came simultaneously to Abel and Jacobi 105 years ago, of inverting an elliptic integral, and so laying the foundations of the theory of elliptic functions and doubly periodic functions generally. It may be a little rash for me to use the word historical; a distinguished historian has described the scope of his subject by saying, "Modern history began when Abraham came out of Ur of the Chaldees". I do not propose to go so far back as Abraham; but I would begin by referring you to 1 Kings vii. 23, where we are told that Solomon "made a molten sea, ten cubits from the one brim to the other; it was round all about, and his height was five cubits; and a line of thirty cubits did compass it round about". It has been suggested, I think rather too ingeniously, that this passage does not bear the obvious interpretation that a comparatively primitive people regarded π as being equal to 3, but that the word round was used in a vague sense and that the vessel in question was elliptical, its major axis being ten cubits and its minor axis being about 9·53 cubits.

The earliest indubitable reference to the length of an elliptic arc that I have been able to discover occurs in Kepler's *Astronomia nova* (Prag, 1609); his discovery, announced in the same work, that the orbit of the planet Mars is an ellipse naturally drew attention to the length of the perimeter of an ellipse. In modern notation, Kepler's theorem is that the perimeter of an ellipse with semi-axes a and b exceeds $2\pi\sqrt{(ab)}$; his proof consists of an application of the theorem that, of figures which enclose the same area, the circle has the least perimeter; and the perimeter of the circle which has the same area as the ellipse is $2\pi\sqrt{(ab)}$. This theorem is followed by the naïf remark that the perimeter of the ellipse is nearly equal to the arithmetic mean of the circumferences of the major and minor auxiliary circles, for the somewhat slender reason that both the perimeter and the arithmetic mean exceed the geometric mean of the circumferences. Things which are greater than the same thing are nearly equal to each other!

Passing over half a century, I come to 1659, in which year Pascal published his essay on *Dimensions des lignes courbes de toutes les Roulettes*, and Wallis published two *Tracts*; Pascal's work and the second tract of Wallis both contain the theorem that the span of the curtate cycloid traced by a point distant b from the centre of a rolling circle of radius a is equal to the semi-perimeter of an ellipse whose semi-axes are $2(b+a)$ and $2(b-a)$. The proof given by Wallis is apparently due to Wren, and I am convinced that Wren discovered both the theorem and his proof of it before July 1658, while I am doubtful whether Pascal knew of the theorem until about the time that he completed his work on Roulettes in October 1658. I regard Wren's proof as elegant and direct, but Pascal's reasoning seems to me to be obscure, like other writings of his that I have read. The second tract of Wallis is also remarkable for containing an expression for the differential element of arc of an ellipse in terms of the abscissa, and the construction of a curve about which he says "the figure so constructed is no more capable of quadrature than an ellipse of rectification".

By applying the binomial theorem to the differential element of arc and integrating the series so obtained term by term, Newton obtained an expression for the length of a quadrant of an ellipse as the sum of two infinite series in 1676 ; the second series is convergent only when the eccentricity does not exceed $\frac{2}{3}$.

The series for the length of a quadrant in powers of the eccentricity, namely

$$a\int_0^{\frac{1}{2}\pi} \sqrt{(1-\epsilon^2 \sin^2\theta)}\, d\theta$$
$$=\tfrac{1}{2}\pi a\left[1-\frac{1}{2\,.\,2}\epsilon^2-\frac{1\,.\,1\,.\,3}{2\,.\,2\,.\,4\,.\,4}\epsilon^4-\frac{1\,.\,1\,.\,3\,.\,3\,.\,5}{2\,.\,2\,.\,4\,.\,4\,.\,6\,.\,6}\epsilon^6-\dots\right],$$

seems to have been discovered by Maclaurin in 1742, and it is probably the best known formula of its kind.

The source of the main topic of my address is not, however, to be found in the discoveries which I have just described. It consists of an observation made by Jacob Bernoulli in 1691, namely that, in the expression for the arc of the parabolic spiral

$$(a-r)^2=2ab\theta$$

as an integral $$s=\int\sqrt{\left\{1+\frac{r^2(a-r)^2}{a^2b^2}\right\}}\,dr,$$

the integrand is an *even function* of $\frac{1}{2}a-r$, so that $\int_{\frac{1}{2}a-c}^{\frac{1}{2}a}=\int_{\frac{1}{2}a}^{\frac{1}{2}a+c}$, that is to say, the arc of the curve joining the points for which r has the values $\frac{1}{2}a-c$ and $\frac{1}{2}a$ is equal to the arc of the curve joining the points for which r has the values $\frac{1}{2}a$ and $\frac{1}{2}a+c$, *although these arcs are not congruent*.

This observation, which nowadays would be regarded as almost a truism, inspired an Italian mathematician Giulio Carlo di Fagnano, to carry out the researches on arcs of equal lengths which resulted in his being loaded with honours in his lifetime and which constitute his claim to mathematical immortality.

Before I proceed to describe some of Fagnano's discoveries, I extract a few biographical details from Gambioli's account of his life contained in the third of the sumptuous volumes of Fagnano's *Opere matematiche* which were published in 1911-12.

Fagnano's family derived from the ancient lords of the Castle of Fagnano between Bologna and Imola ; to this family belonged Honorius II, elected pope in 1124. Later the family flourished in Bologna under the surname of Toschi ; about 1341 they settled in Senigallia, a little town on the shores of the Adriatic, about half-way between Pesaro and Ancona, now a sea-bathing resort of 12,000 inhabitants.

Giulio Carlo, son of Francesco Fagnano and his wife Camilla Bartoli, was born at Senigallia on 26th September 1682, and died there on 18th May 1766. From the age of ten he enjoyed poetry and composed sonnets ; later he published a philosophical poem on the resurrection of the dead. At the age of fourteen he entered the Clementine College at Rome, where he studied for three years ; he quickly attained distinction in the College, and completed his philosophy course in two years and the theological course in one year. He abandoned the study of the Aristotelian philosophy for the works of Gassendi and Descartes ; he got into communication with Malebranche and gave him a new explanation of the mystery of transubstantiation, receiving an alternative explanation from him in return. He proceeded to the philosophy of Leibniz and Newton, and derived much profit from studying the famous work of Malebranche, *La recherche de la verité*. While a student he held mathematics in contempt, in spite of advice to study the subject ; but later he decided to abandon philosophy and to devote himself entirely to mathematics. According to Gambioli, he was entirely self-taught, so far as mathematics were concerned, and Gambioli emphasizes the mathematical ability which he must have possessed in being able to study the subject so effectively after attaining years of maturity, in a small town like Senigallia, without a library and remote from any centre of learning.

In 1743 he was consulted with regard to the safety of St. Peter's, which was threatened with collapse ; and, as a reward for his services, Benedict XIV commissioned the publication of his collected works, though the two volumes composing them did not actually appear until 1750. He had been created a Count by Louis XV in 1721, and in 1745 the Pope created him a Marquis ; when the Roman nobility was reconstituted in the following year, he was included in the 101 noble families. He was also created Marquis of Sant' Onorio by Charles III, King of the two Sicilies. He was nominated to the Berlin Academy in 1750, and he sent the Academy a copy of his *Produzioni matematiche* which reached Euler's hands on 23rd December 1751, a day described by Jacobi as " Der Geburtstag der elliptischen Funktionen ".

He married Francesca Conciatti of Sassoferrato, by whom he had six sons and six daughters ; she died in 1726, and but four of his children survived him ; one of the sons was a mathematician, but he was less able than his father ; the family is now extinct.

Gambioli gives a lively account of his personality ; he describes him as cheerful and lively in conversation ; and relates, as an instance of his equanimity, a story of how, one evening, he was driven to shooting two robbers and, meeting some friends shortly afterwards, he conversed with them as though nothing had happened.

He died on 18th May 1766, and is buried in the church of Santa Maria Maddalena at Senigallia ; the municipality of Senigallia possesses his portrait, representing him with a diagram of a lemniscate in his hand ; below is an elaborate inscription describing him as " Julius Carolus de Tuschis de Fagnano, S. Honorii Marchio, in Ordine Constantin. S. Georgii Marchiae Prior, Patricius Romanus et Senogalliensis ". He is described in a similar manner on the title-page of his *Produzioni matematiche* and below his titles is a diagram of a lemniscate with the words " Multifariam divisa atque dimensa. Deo veritatis gloria ".

I come now to a brief description of Fagnano's work; his investigations on the geometrical theory of proportion, which occupy a large part of the first volume of his *Produzioni matematiche*, are, of course, irrelevant, and I do not hesitate to say that I naturally consider them rather dull reading. I pass over, also, his work on the rectification of parabolas of higher order which was a preliminary to his more famous discoveries published from 1714 onwards, and connected, as might be anticipated from some hints which I have given, with the rectification of the lemniscate.

Taking the equation of the lèmniscate in the usual form

$$r^2 = 2a^2 \cos 2\theta,$$

and writing $\quad \cos 2\theta = t, \quad r = a\sqrt{(2t)},$

we obtain the parametric representation

$$x = a\sqrt{(t+t^2)}, \quad y = a\sqrt{(t-t^2)},$$

whence the expression for the arc in the form of an integral

$$s = \frac{a}{\sqrt{2}}\int \frac{dt}{\sqrt{\{t(1-t^2)\}}}$$

is immediately derivable.

Fagnano's first discovery was that, if two parameters t and z are connected by the relation

$$(t+1)(z+1) = 2,$$

then
$$\frac{dt}{\sqrt{\{t(1-t^2)\}}} = -\frac{dz}{\sqrt{\{z(1-z^2)\}}},$$

and hence
$$\int_0^z \frac{du}{\sqrt{\{u(1-u^2)\}}} = \int_t^1 \frac{du}{\sqrt{\{u(1-u^2)\}}}.$$

In modern terminology, he had discovered a homographic transformation under which the integrand is invariant, its singularities being permuted among themselves.

I remark that the points with parameters 0 and 1 are the centre C and the end of the transverse axis A; so that, using an obvious notation, we see that the arcs Cz and tA are equal. If now we take $t = z$, it follows from the homographic relation that each is equal to $\sqrt{2}-1$, so that *the point with parameter* $\sqrt{2}-1$ *bisects the arc forming the positive quadrant of the lemniscate.*

To describe Fagnano's second discovery, it is convenient to take a slightly different parametric representation

$$x = az\sqrt{(1+z^2)}, \quad y = az\sqrt{(1-z^2)},$$

by writing z^2 for t in the former representation.

The formula for the arc is now

$$\text{Arc } Cz = a\sqrt{2}\int_0^z \frac{du}{\sqrt{(1-u^4)}},$$

and the first theorem becomes:

$$\textit{If} \quad \zeta^2 = \frac{1-z^2}{1+z^2} \quad \textit{then} \quad \int_0^z = \int_\zeta^1.$$

Now consider the transformation

$$\zeta=\frac{1-w^2}{1+w^2},$$

so that

$$\left(\frac{1-w^2}{1+w^2}\right)^2=\frac{1-z^2}{1+z^2}.$$

It is not very difficult to verify that this may be written in the equivalent forms

$$z=\frac{w\surd 2}{\surd(1+w^4)},\quad \frac{dz}{\surd(1-z^4)}=\surd 2\frac{dw}{\surd(1+w^4)}.$$

The form of this relation is the key to Fagnano's subsequent work; the method by which he was led to discover the relation is not clear, and I think that it must be regarded as one of those happy chances which, as Legendre once remarked in another connection, occur only to those who know how to produce them.

Unlike the first transformation, the second does not give an invariant integral; but, if we take

$$w=\frac{v\surd 2}{\surd(1-v^4)},$$

we get

$$\frac{dw}{\surd(1+w^4)}=\surd 2\frac{dv}{\surd(1-v^4)},$$

and hence

$$\frac{dz}{\surd(1-z^4)}=2\frac{dv}{\surd(1-v^4)}.$$

This is a rudimentary form of what is now known as complex multiplication; the relation between z and w is of the same form as the relation between $w\eta$ and $v\eta$, where η is a fourth root of -1. Whether Fagnano discovered the transformation in this way or not, I do not know; it is by no means impossible that this was his method, because he had made an intensive study of the technique of the algebra of complex numbers, and, in fact, he was the discoverer of the well known formula

$$\pi=2i\log\frac{1-i}{1+i}.$$

The last differential equation may be written in the equivalent forms

$$\text{Arc } Cz=2\text{ Arc } Cv,\quad v=\frac{z}{\surd[\{1+\surd(1-z^2)\}\{1+\surd(1+z^2)\}]}.$$

Take now a new parameter u, defined by the equation

$$u^2=\frac{1-v^2}{1+v^2},$$

and then, by the first result,

$$\text{Arc } Cv=\text{Arc } uA,\qquad \text{Arc } Cz=2\text{ Arc } uA,$$

where the equation connecting u and z can be worked out without much difficulty. If, in this equation, we take $u=z$, the positive quadrant is trisected at v and u; and then the equation for z reduces to

$$z^8 + 6z^4 - 3 = 0,$$

whence
$$z = \sqrt[4]{(2\sqrt{3} - 3)},$$

so that the point with parameter $\sqrt[4]{(2\sqrt{3}-3)}$ *is one of the points of trisection of the arc forming the positive quadrant of the lemniscate.*

The problem of the quinquesection of the quadrant was attacked by Fagnano in a similar manner. It is necessary to take four parameters, t, z, v, u, representing points on the lemniscate such that

$$\text{Arc } Ct = 2 \text{ Arc } Cz, \quad \text{Arc } Cz = 2 \text{ Arc } Cv, \quad \text{Arc } Cv = \text{Arc } uA,$$

so that consecutive parameters are connected either by the first or by the second of Fagnano's relations, whence an algebraic equation equivalent to the formula

$$\text{Arc } Ct = 4 \text{ Arc } uA,$$

connecting t and u, is obtainable.

If, in this equation, we take $t = u$, we get

$$\text{Arc } Ct = \tfrac{4}{5} \text{ Arc } CA,$$

and the equation for t (which was not worked out by Fagnano) reduces to

$$t^{24} + 50t^{20} - 125t^{16} + 300t^{12} - 105t^8 - 62t^4 + 5 = 0.$$

When we discard from this equation factors which may be proved to be irrelevant, we find that

$$t^8 - (12\sqrt{5} - 26)t^4 + 9 - 4\sqrt{5} = 0,$$

and hence the problem of the quinquesection of the quadrantal arc of the lemniscate is reduced ultimately to the solution of quadratic equations.

As I said, Fagnano did not work out this result to the bitter end; but he clearly realized that the work was possible, for he went so far as to state the general theorem that, if m is any integer, it is possible to find, *algebraically*, a point with parameter z such that the arc Cz is any one of the following three fractions of the length of the quadrant of the lemniscate:

$$\frac{1}{2^m}, \quad \frac{1}{3 \cdot 2^m}, \quad \frac{1}{5 \cdot 2^m};$$

and then comes the remark which I think must have been uttered with a pride that was fully justifiable: " E questa è una nuova, e singolare proprietà della mia curva ".

I now temporarily leave the work of Fagnano and turn to that of John Landen. There is a short biography of Landen in the *D.N.B.* He was born near Peterborough on 23rd January, 1719, and brought up to the business of a surveyor. He acted as land-agent to William Wentworth, Earl Fitzwilliam, from 1762 to 1788. He published a small volume not very happily entitled *Mathematical Lucubrations* in 1755, about which I shall have something to say in a moment. His most important work from my present point of view is contained in some later papers published in *Phil. Trans. Roy. Soc.* He had been elected a Fellow of that Society in 1766. In addition to his work on elliptic arcs he investigated certain dynamical problems, about which the *D.N.B.* merely states that his results differed from those of Euler and d'Alembert. He is not the only writer on dynamical problems whose results are different from those of recognised authorities ; a similar instance is to be found in the *Quarterly J. of Math.*, 48, published as recently as 1920. Landen was not to be convinced of the incorrectness of his views and defended them in a volume of memoirs which he prepared for press while suffering from a painful disease ; a copy of the volume was placed in his hands the day before his death, which occurred on 15th January 1790.

The *Mathematical Lucubrations* contain one passage which is relevant here. Landen gives the following definitions : " e denotes one fourth of the periphery of an Ellipsis whose semiaxes are $\sqrt{2}$ and 1. f denotes one fourth of the periphery of a Circle whose radius is 1 ". He then makes a number of extremely odd-looking statements of which I quote two as typical :

$$\int_0^1 \frac{x^{-\frac{1}{2}}\,dx}{\sqrt{(1-x^2)}} = e + \sqrt{(e^2-2f)}, \quad \int_0^1 \frac{x^{\frac{1}{2}}\,dx}{(1-x^2)^{\frac{1}{2}}} = \frac{e-\sqrt{(e^2-2f)}}{\sqrt{2}}.$$

The clue to these theorems is found by taking Maclaurin's formula (already quoted) in the form

$$e = \int_0^{\frac{1}{2}\pi} \sqrt{(2-\sin^2\theta)}\,d\theta,$$

and making the substitutions

$$\sin\theta = \sqrt{2}\,.\,\sin\phi, \quad \cos^2 2\phi = y,$$

whence it follows that

$$e = \frac{1}{4}\left[\int_0^1 y^{-\frac{3}{4}}(1-y)^{-\frac{1}{2}}\,dy + \int_0^1 y^{-\frac{1}{4}}(1-y)^{-\frac{1}{2}}\,dy\right].$$

Landen had previously obtained a modification of the well-known product given by Wallis for π in the form

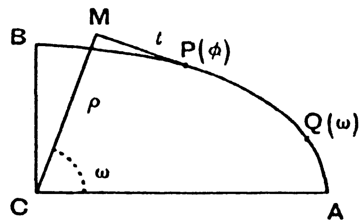
$$\int_0^1 y^{-\frac{3}{4}}(1-y)^{-\frac{1}{2}}\,dy \times \int_0^1 y^{-\frac{1}{4}}(1-y)^{-\frac{1}{2}}\,dy = 4\pi$$

by a very ingenious piece of analysis (I remark that the truth of Landen's result is evident when the integrals are expressed in terms of gamma functions), and it is an immediate consequence of the last two equations that

$$\sqrt{(e^2-2f)} = \frac{1}{4}\left[\int_0^1 y^{-\frac{3}{4}}(1-y)^{-\frac{1}{2}}\,dy - \int_0^1 y^{-\frac{1}{4}}(1-y)^{-\frac{1}{2}}\,dy\right].$$

The whole set of Landen's formulae may now be obtained by quite elementary transformations.

I take next a very beautiful theorem, first discovered in an algebraical form by Fagnano, given a geometrical interpretation by Euler, and then modified and developed by Landen. The investigation which I give is due to Legendre; it seems to me much more elegant than the earlier work on the theorem. (I remark in parenthesis that it has been my experience—I do not know to what extent my impressions are shared with others—that the technique of eighteenth century mathematicians in dealing with problems of pure calculus was considerably superior to their technique in analytical geometry, and that frequently the algebraical work which they found necessary in solving problems of analytical geometry can only be described as clumsy when compared with their other work.)



To return to the theorem, a point P with eccentric angle $\frac{1}{2}\pi - \phi$ is taken on an ellipse, M is the foot of the central perpendicular on the tangent at P, p and ω are the polar coordinates of M, Q is the point with eccentric angle $\frac{1}{2}\pi - \omega$, and finally t and s are defined as the length MP and the arc BP.

The coordinates of P being (x, y), the truth of the equations

$$\frac{dx}{ds} = \sin \omega, \quad \frac{dy}{ds} = -\cos \omega,$$

$$p = x \cos \omega + y \sin \omega, \quad t = x \sin \omega - y \cos \omega = -\frac{dp}{d\omega}$$

is then fairly evident. Hence

$$\frac{d(s-t)}{d\omega} = \frac{(dx)^2 + (dy)^2}{ds\, d\omega} - \frac{dt}{d\omega} = \sin \omega \frac{dx}{d\omega} - \cos \omega \frac{dy}{d\omega} - \frac{dt}{d\omega} = -p.$$

It then follows that

$$\Big[s - t \Big]_{\omega}^{\frac{1}{2}\pi} = -\int_{\omega}^{\frac{1}{2}\pi} p d\omega,$$

$$s - t = \int_{\omega}^{\frac{1}{2}\pi} p\, d\omega = \int_{\omega}^{\frac{1}{2}\pi} \sqrt{(a^2 \cos^2 \omega + b^2 \sin^2 \omega)}\, d\omega;$$

and, by Maclaurin's formula, the integral on the right is equal to QA. *Hence*

$$\text{Arc } BP - \text{Arc } QA = MP.$$

The coordinates of Q being called (x', y'), we have

$$x = \frac{a^2 \cos \omega}{\sqrt{(a^2 \cos^2 \omega + b^2 \sin^2 \omega)}}, \quad x' = a \sin \omega,$$

and hence

$$x^2 x'^2 (a^2 - b^2) - (x^2 + x'^2) a^4 + a^6 = 0,$$

a result which may be expressed much more elegantly in the form

$$\tan\phi \tan\omega = \frac{a}{b}.$$

Landen's most important discovery was a consequence of taking the corresponding formula for a hyperbola; the arc being now measured from the end of the major axis and ω being negative, we have

$$t - s = \int_\omega^0 \surd(a^2\cos^2\omega - b^2\sin^2\omega)\,d\omega$$

$$= \int_0^\tau \sqrt{\left(\frac{(a^2-\tau^2)}{a^2+b^2-\tau^2}\right)}\,d\tau,$$

where $\qquad \tau = \surd(a^2 - p^2).$

To deal with the last integral, Landen assumed an ellipse of arbitrary semi-axes, α and β, and took the point of it for which the projection of the radius vector on the tangent is τ. If σ is the arc from the end of the minor axis to the point, and ψ the inclination of the perpendicular on the tangent, then

$$\frac{d(\sigma-\tau)}{d\psi} = -\surd(\alpha^2\cos^2\psi + \beta^2\sin^2\psi).$$

Now τ is expressible as a function of ψ; and, by taking τ as independent variable in place of ψ, Landen found, with the help of some heavy algebra, that

$$2\frac{d\sigma}{d\tau} = 1 + \frac{\alpha^2+\beta^2-\tau^2}{\alpha^2-\beta^2}\left[1 - \frac{2(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)^2}\tau^2 + \frac{\tau^4}{(\alpha^2-\beta^2)^2}\right]^{-\frac{1}{2}}.$$

He then chose α and β in terms of a and b, so that the expression in [ ] factorized into

$$\left(1-\frac{\tau^2}{a^2}\right)\left(1-\frac{\tau^2}{a^2+b^2}\right)$$

the connection of which with the original integrand is evident. It turns out that this choice makes

$$\alpha, \beta = \frac{\pm a + \surd(a^2+b^2)}{2}.$$

It then follows that

$$2\frac{d\sigma}{d\tau} = 1 + \frac{(a^2-\tau^2)+(a^2+b^2-\tau^2)}{2\surd\{(a^2-\tau^2)(a^2+b^2-\tau^2)\}}$$

$$= 1 + \frac{d(t-s)}{2\,d\tau} + \frac{1}{2}\sqrt{\left(\frac{a^2+b^2-\tau^2}{a^2-\tau^2}\right)},$$

and hence, on integration,

$$s = t + 2\tau - 4\sigma + \sigma_1,$$

where σ_1 is an arc of an ellipse with semi-axes $\surd(a^2+b^2)$ and b.

Pairs of ellipses whose semi-axes are related in the manner of the two ellipses of this problem are said to be connected by Landen's transformation. In the hands of Legendre, the transformation became a most powerful method for computing elliptic integrals;

and the transformation made possible the theory of more general transformations, leading up to the theories of modular equations, complex multiplication, and singular moduli.

A few words are permissible about Legendre's applications. In his notation

$$F(\phi, k)=\int_0^\phi \frac{d\phi}{\surd(1-k^2\sin^2\phi)}=\int_0^\phi \frac{d\phi}{\Delta(\phi, k)};$$

$$F(\tfrac{1}{2}\pi, k)=K, \quad F(\pi, k)=2K; \quad k'=\surd(1-k^2);$$

and the change of variable which effects Landen's transformation is

$$\sin\phi_1=\frac{(1+k')\sin\phi\cos\phi}{\Delta(\phi, k)}.$$

We then have

$$F(\phi, k)=\frac{1}{1+k'}F(\phi_1, k_1)=\frac{1+k_1}{2}F(\phi_1, k_1),$$

where $$k_1=\frac{1-k'}{1+k'}, \quad k_1'=\frac{2\surd k'}{1+k'},$$

and so $$K=(1+k_1)K_1.$$

Legendre then got the idea of repeating the transformation so as to have a sequence or *scale*, $k_1, k_2, k_3, \ldots$, such that

$$k_{m+1}=\frac{1-k_m'}{1+k_m'}=\frac{k_m^2}{(1+k_m')^2},$$

so that $k_{m+1}<k_m^2$ and, by induction, $k_m<k^{2^m}$. Further

$$K=K_\infty\prod_{m=1}^{\infty}(1+k_m)=\frac{\pi}{2}\prod_{m=1}^{\infty}(1+k_m),$$

and the convergence of the sequence k_m to zero is most extraordinarily rapid. Even if k is as large as 0·99990, then

$$k_4=0{\cdot}00367\,06, \quad k_5=0{\cdot}00000\,34.$$

For some purposes it is better to follow Gauss and work with the semi-axes of ellipses rather than with the eccentricity in the manner of Legendre. If *

$$\sin\phi_1=\frac{(a+b)\sin\phi\cos\phi}{\surd(a^2\cos^2\phi+b^2\sin^2\phi)},$$

then $$\frac{d\phi_1}{\surd\{(a+b)^2\cos^2\phi_1+4ab\sin^2\phi_1)\}}=\frac{d\phi}{\surd(a^2\cos^2\phi+b^2\sin^2\phi)},$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi_1}{\surd(a_1^2\cos^2\phi_1+b_1^2\sin^2\phi_1)}=\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\surd(a^2\cos^2\phi+b^2\sin^2\phi)},$$

where $$a_1=\tfrac{1}{2}(a+b), \quad b_1=\surd(ab).$$

Repetitions of the operation give a double sequence (a_m, b_m), whose members converge with similar rapidity to a common limit, called,

* Gauss actually made a different transformation, but one which led to the same result.

for obvious reasons, the arithmetic-geometric mean of a and b; if it is denoted by $\mathfrak{M}(a, b)$, then

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{\surd(a^2\cos^2\phi + b^2\sin^2\phi)} = \frac{\frac{1}{2}\pi}{\mathfrak{M}(a, b)}.$$

It is not irrelevant to quote here the investigation given in 1796 by Ivory (better known for his work on gravitating ellipsoids) on the perimeter of an ellipse, since it is connected with Maclaurin's formula by Landen's transformation.

If x denotes $(a-b)/(a+b)$, the perimeter is equal to

$$\begin{aligned}
4&\int_0^{\frac{1}{2}\pi} \surd(a^2\cos^2\phi + b^2\sin^2\phi)\,d\phi \\
&= 2(a+b)\int_0^{\frac{1}{2}\pi} \sqrt{\left\{1 + 2\frac{a-b}{a+b}\cos 2\phi + \left(\frac{a-b}{a+b}\right)^2\right\}}\,d\phi \\
&= 2(a+b)\int_0^{\frac{1}{2}\pi} \surd\{(1+xe^{2i\phi})(1+xe^{-2i\phi})\}\,d\phi \\
&= 2(a+b)\int_0^{\frac{1}{2}\pi} \left(1 + \tfrac{1}{2}xe^{2i\phi} - \frac{1.1}{2.4}x^2e^{4i\phi} - \frac{1.1.3}{2.4.6}x^3e^{6i\phi} - \ldots\right) \\
&\qquad \times \left(1 + \tfrac{1}{2}xe^{-2i\phi} - \frac{1.1}{2.4}x^2e^{-4i\phi} - \frac{1.1.3}{2.4.6}x^3e^{-6i\phi} - \ldots\right)d\phi \\
&= \pi(a+b)\left[1 + (\tfrac{1}{2})^2x^2 + \left(\frac{1.1}{2.4}\right)^2x^4 + \left(\frac{1.1.3}{2.4.6}\right)^2x^6 + \ldots\right],
\end{aligned}$$

and this is Ivory's series.

To return to Gauss, it would be absurd for me to stop short with the description of the arithmetic-geometric mean without mentioning a much more noteworthy discovery of his, and one which is not so generally known. Although Gauss published little during his lifetime, he kept a fairly full diary for a number of years, and under the date 8th January 1797 is to be found the concept of " inverting " the integral which occurs in Fagnano's work on the lemniscate, thirty years before the inversions due to Abel and Jacobi.

As samples of the definitions and the simpler formulae discovered by Gauss I quote:

$$\phi = \int_x^1 \frac{dx}{\surd(1-x^4)}, \quad \tfrac{1}{2}\varpi = \int_0^1 \frac{dx}{\surd(1-x^4)} = 1{\cdot}31102\,878\ldots,$$

$$x = \text{coslemn}\,\phi = \text{cl}\,\phi,$$

$$\text{sinlemn}\,\phi = \text{sl}\,\phi = \text{cl}\,(\tfrac{1}{2}\varpi - \phi),$$

$$\text{sl}^2\,\phi = \frac{1-\text{cl}^2\,\phi}{1+\text{cl}^2\,\phi}.$$

The connection between the Gaussian functions and the Jacobian functions is expressed by the formulae

$$\text{sl}\,\phi = \frac{1}{\surd 2}\,\text{sd}\left(\phi\surd 2, \frac{1}{\surd 2}\right), \quad \text{cl}\,\phi = \text{cn}\left(\phi\surd 2, \frac{1}{\surd 2}\right).$$

It is well known that the reluctance of Gauss to publish his discoveries was due to the rejection of his *Disquisitiones arithmeticae* by the French Academy, the rejection being accompanied by a sneer which, as Rouse Ball has said, would have been unjustifiable even if the work had been as worthless as the referees believed. It is the irony of fate that, but for this sneer, the *Traité des fonctions elliptiques*, the work of a Frenchman, might have assumed a different and vastly more valuable form, and Legendre might have been spared the pain of realizing that many years of his life had been practically wasted, had the method of inversion come to be published when Legendre's age was fifty instead of seventy-six.

I have now completed my tale of the work of the eighteenth century; but, just as I began by quoting an empirical formula of a hundred years earlier, I shall end by quoting some empirical formulae of a hundred years later; the first of these formulae for the perimeter of an ellipse is due to Peano (1887), namely

$$\pi[a+b+\tfrac{1}{2}(\sqrt{a}-\sqrt{b})^2];$$

it contrasts rather curiously with the type of work for which he is best known. It is not to be expected that I should so restrain myself as to refrain from mentioning the Indian mathematician Ramanujan, and the last two formulae are due to him (1914):

$$\pi[3(a+b)-\sqrt{\{(a+3b)(3a+b)\}}],$$

$$\pi\left[a+b+\frac{3(a-b)^2}{10(a+b)+\sqrt{(a^2+14ab+b^2)}}\right].$$

I have proved that in these formulae the errors (which are zero in the limiting case of the circle) are in defect and that they increase when the eccentricity is increased, though they are extremely small even in the extreme case of the parabola. Ramanujan has mentioned that, for small values of the eccentricity ϵ, the errors are about

$$\pi a\epsilon^{12}/10\,48576, \quad 3\pi a\epsilon^{20}/6\,87194\,76736.$$

My final task, and a very pleasant one, is to express most cordially my gratitude to Professor Neville for having made the extremely happy suggestion of an exhibition of books on elliptic functions so as to give some indication of the growth of the subject during the last century. Many of the books come from his own library; and my appreciation of his idea and of the energy with which he has carried out the task of making the exhibition complete will be shared by all who see the collection. G. N. W.

[The exhibition consisted of 97 volumes, not counting a few duplicates, and included all but a very few of the existing books on elliptic functions. For this measure of completeness members of the Association are indebted to Dr. J. R. Airey, Dr. L. J. Comrie, and Mr. F. Robbins, and to the Libraries of the University of Birmingham; University College, London; University College, Bangor; and St. John's and Trinity Colleges, Cambridge, as well as to their own President and Librarian and the Editor of the *Gazette*. The Librarian wishes to record his appreciation of the courtesy and helpfulness with which his requests were met, not only by those who actually lent books, but also by others in commercial as well as in academic circles who found they had nothing to add by the time they were approached.

Before the collection is dispersed, the most careful bibliographical record will be compiled, and it is hoped that this will become of permanent value to librarians and scholars.]

THE BEST (?) FORMULA FOR COMPUTING π TO A THOUSAND PLACES

J. P. BALLANTINE, University of Washington

In the December 1938 issue of this MONTHLY, D. H. Lehmer gave a very comprehensive list of formulas for computing π. He rightly chose formulas (23) and (32) as the best self checking pair, with (18) a good substitute for (23).

I shall give some suggestions on the use of formula (32) which will strengthen Lehmer's conclusion.

If [18], which means arc tan 1/18, and [57] are evaluated by the Gregory series, their measures 0.7966 and 0.5695, are correctly stated by Lehmer. However, Euler also developed a formula for $[x]$, namely

$$\operatorname{arc\,cot} x = x\left(\frac{1}{S} + \frac{2}{3S^2} + \frac{2\cdot 4}{3\cdot 5S^3} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7S^4} + \cdots\right),$$

where $S = x^2 + 1$. Lehmer's remarks about the best way of computing the terms of the Gregory series apply equally well to the Euler series, because

$$u_{n+1}(x) = \frac{2nu_n(x)}{(2n+1)S},$$

and it is apparent that for machine computation one series is about as good as the other. The Euler series has the slight advantage that all the terms are positive.

Now, to return to formula (32), we have seen that nothing can be lost by evaluating [18] and [57] by the Euler series instead of the Gregory series. In fact there is a great saving, due to a lucky accident, for we have

$$[18] = 18\left(\frac{1}{325} + \frac{2}{3\cdot 325^2} + \frac{2\cdot 4}{3\cdot 5\cdot 325^3} + \cdots\right),$$

$$[57] = 57\left(\frac{1}{3250} + \frac{2}{3\cdot 3250^2} + \frac{2\cdot 4}{3\cdot 5\cdot 3250^3} + \cdots\right).$$

Thus the successive terms of the series for [57] can be obtained from those of [18] by shifting the decimal point. In fact, they do not even have to be copied, because they can be added on the bias.

Thus, the measure of formula (32) is reduced from 1.7866 to 1.2171 by the practical elimination of [57]. This leaves formulas (32) and (23) about tied for first place, and certainly making the best self checking pair, with Machin's formula (18) close behind. Moreover, all the other low measure formulas mentioned in Lehmer's paper are found on closer examination to have higher measure.

Before once and for all eliminating formula (18), one important feature should be mentioned. Suppose it is desired to compute both π and $M=\log_{10}e=.43429\cdots$ to 1000 decimal places. Then formula (18) is better than (23), because

$$\log\frac{3}{2}=2\left[\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\cdots\right],$$

and the terms of the series are the same as of [5]. Furthermore, log 120/119 will be useful to find log 17.

For the readers convenience, I restate formulas (18), (23), and (32) discussed here, with revised measures:

$$[1]=4[5]-[239]\quad(1.3511). \tag{18}$$

$$[1]=8[10]-[239]-4[515]\quad(1.2892). \tag{23}$$

$$[1]=12[18]+8[57]-5[239]\quad(1.2171). \tag{32}$$

In summary, the best formulas now known for computing π to a large number of places are:

$$\frac{[18]}{18}=\left(\frac{1}{325}+\frac{2}{3\cdot 325^2}+\frac{2\cdot 4}{3\cdot 5\cdot 325^3}+\cdots\right),$$

$$\frac{[57]}{57}=\left(\frac{1}{3250}+\frac{2}{3\cdot 3250^2}+\frac{2\cdot 4}{3\cdot 5\cdot 3250^3}+\cdots\right),$$

$$[239]=\left(\frac{1}{239}-\frac{1}{3\cdot 239^3}+\frac{1}{5\cdot 239^5}-\cdots\right),$$

$$\pi=864\,\frac{[18]}{18}+1824\,\frac{[57]}{57}-5[239].$$

For checking, we may use

$$[10]=\frac{1}{10}-\frac{1}{3\cdot 10^3}+\frac{1}{5\cdot 10^5}\cdots,$$

$$[515]=\frac{1}{515}-\frac{1}{3\cdot 515^3}+\frac{1}{5\cdot 515^5}\cdots,$$

$$\pi=32[10]-4[239]-16[515].$$

Editorial Note. The following observation is due to Dr. J. B. Rosser. The labor of computing the reciprocals of the first thousand integers can be very much expedited by the use of either the table given in vol. ii, pp. 412–434, of Gauss's Werke or the much more comprehensive table given in *A Table of the Circles arising from the Division of a Unit or any other Whole Number by all the Integers from 1 to 1024*, by Henry Goodwyn, London, 1823. (A description of this table is given by Glaisher, Proceedings of the Cambridge Philosophical Society, 1878, 3, p. 185.) For this reason, the measure of [10] should probably be taken as much less than 0.5; possibly 0.2 or 0.1.—R. J. W.

AN ALGORITHM FOR THE CONSTRUCTION OF ARCTANGENT RELATIONS

R. H. BIRCH*.

An important source of solutions for the quadrature of the circle has been the use of Gregory's series

$$\arctan x^{-1} = x^{-1} - \tfrac{1}{3}x^{-3} + \tfrac{1}{5}x^{-5} - \dots$$

in certain identities, for instance in those expressing $\frac{1}{4}\pi$ as a sum of arctangents, of which the following, due to Euler, is typical:

$$\tfrac{1}{4}\pi = 4 \arctan \tfrac{1}{5} - \arctan \tfrac{1}{70} + \arctan \tfrac{1}{99}.$$

Gauss† spent a lot of effort on this topic and among his unfinished work were large tables constructed to aid him in his search for such identities. He does not appear to have developed any systematic method for their discovery except to express arctangents of large numbers in terms of those of smaller ones and to eliminate the latter. One of the simplest of his identities is:

$$\tfrac{1}{4}\pi = 12 \arctan \tfrac{1}{18} + 8 \arctan \tfrac{1}{57} - 5 \arctan \tfrac{1}{239}.$$

The question of the economy and efficiency of these relations in the computation of π has been discussed by D. H. Lehmer and others‡. Lehmer has given a list of some 33 such relations with numerical "measures" of their efficiency. It is clear that identities expressing $\frac{1}{4}\pi$ as a sum of multiples of arctangents of powers of 10^{-1} will be very convenient. The object of this note is to show how such relations can be obtained at will. The idea of using complex numbers goes back to Gauss and has also been exploited by Størmer§.

We use the fact that from the identity

$$a+bi = (c+di) \times \{(a+bi)(c-di)\} \times (c^2+d^2)^{-1}$$

there follows the relation

$$\arg(a+bi) = \arg(c+di) + \arg\{(a+bi)(c-di)\};$$

* Received 16 May, 1946; read 20 June, 1946.

† Gauss, Werke II (1863), *Tafel zur Cyclotechnie*, 476–495, 523–525.

‡ D. H. Lehmer, *American Math. Monthly*, 45 (1938), 657–664; G. W. Wrench, Jnr., *American Math. Monthly*, 45 (1938), 108–109; J. P. Ballantine, *American Math. Monthly*, 46 (1937), 499–501.

§ C. Størmer, *Archiv. for Math. og Naturvidenskab*, 19 No. 3 (1896), 1–96.

and by repeated use of this relation, beginning with $a = b = 1$, with c always a suitable power of 10 and $d = \pm 1$, we obtain results of the form desired. The details of the first eleven steps are as follows:

$$\begin{aligned}
\arg(1+i) &= \arg(10+i)+\arg(11+9i)\\
&= 2\arg(10+i)+\arg(119+79i)\\
&= 3\arg(10+i)+\arg(1269+671i)\\
&= 4\arg(10+i)+\arg(13361+5441i)\\
&= 5\arg(10+i)+\arg(1,39051+41049i)\\
&= 6\arg(10+i)+\arg(14,31559+2,71439i)\\
&= 7\arg(10+i)+\arg(145,87029+12,82831i)\\
&= 8\arg(10+i)+\arg(1471,53121-17,58719i)\\
&= 8\arg(10+i)+\arg(100-i)+\arg(1,47170,70819-287,18779i)\\
&= 8\arg(10+i)+\arg(100-i)+\arg(1000-i)\\
&\qquad +\arg(1471,70995,37779-1,40017,08181i)\\
&= 8\arg(10+i)+\arg(100-i)+2\arg(1000-i)\\
&\qquad +\arg(14,71711,35394,87181+71,53913,56779i).
\end{aligned}$$

Thus we have

$$\tfrac{1}{4}\pi = 8\arctan\tfrac{1}{10}-\arctan\tfrac{1}{100}-2\arctan\tfrac{1}{1000}+\arctan\epsilon,$$

where $\epsilon = 0{\cdot}00004,86094,88189\ldots.$

We can now use Gregory's series, remembering that the absolute value of the remainder at any stage is less than that of the first term omitted; and we obtain estimates for $\frac{1}{4}\pi$. Replacing the $\arctan\epsilon$ by ϵ, we find, with an error not greater than $0{\cdot}4\times 10^{-13}$, that

$$\tfrac{1}{4}\pi = 0{\cdot}78539,81633,97487,\ \ldots.$$

The correct value is

$$\tfrac{1}{4}\pi = 0{\cdot}78539,81633,97448,\ \ldots.$$

33 Woodland Way,
Morden, Surrey.

A SIMPLE PROOF THAT π IS IRRATIONAL

IVAN NIVEN

Let $\pi = a/b$, the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a - bx)^n}{n!},$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x),$$

the positive integer n being specified later. Since $n!f(x)$ has integral coefficients and terms in x of degree not less than n, $f(x)$ and its derivatives $f^{(i)}(x)$ have integral values for $x=0$; also for $x=\pi=a/b$, since $f(x)=f(a/b-x)$. By elementary calculus we have

$$\frac{d}{dx}\{F'(x) \sin x - F(x) \cos x\} = F''(x) \sin x + F(x) \sin x = f(x) \sin x$$

and

$$(1) \qquad \int_0^\pi f(x) \sin x dx = [F'(x) \sin x - F(x) \cos x]_0^\pi = F(\pi) + F(0).$$

Now $F(\pi)+F(0)$ is an *integer*, since $f^{(i)}(\pi)$ and $f^{(i)}(0)$ are integers. But for $0<x<\pi$,

$$0 < f(x) \sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (1) is *positive, but arbitrarily small* for n sufficiently large. Thus (1) is false, and so is our assumption that π is rational.

PURDUE UNIVERSITY

Received by the editors November 26, 1946, and, in revised form, December 20, 1946.

An ENIAC Determination of π and e to more than 2000 Decimal Places

Early in June, 1949, Professor JOHN VON NEUMANN expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of π and e to many decimal places with a view toward obtaining a statistical measure of the randomness of distribution of the digits, suggesting the employment of one of the formulas:

$$\pi/4 = 4 \arctan 1/5 - \arctan 1/239$$
$$\pi/4 = 8 \arctan 1/10 - 4 \arctan 1/515 - \arctan 1/239$$
$$\pi/4 = 3 \arctan 1/4 + \arctan 1/20 + \arctan 1/1985$$

in conjunction with the GREGORY series

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} x^{2n+1}.$$

Further interest in the project on π was expressed in July by Dr. NICHOLAS METROPOLIS who offered suggestions about programming the calculation.

Since the possibility of official time was too remote for consideration, permission was obtained to execute these projects during two summer holiday week ends when the ENIAC would otherwise stand idle, and the planning and programming of the projects was undertaken on an extra-curricular basis by the author.

The computation of e was completed over the July 4th week end as a practice job to gain experience and technique for the more difficult and longer project on π. The reciprocal factorial series was employed:

$$e = \sum_{n=0}^{\infty} (n!)^{-1}.$$

The first of the above-mentioned formulas was employed for the computation of π; its advantage over the others will be explained later. The computation of π was completed over the Labor-Day week end through the combined efforts of four members of the ENIAC staff: CLYDE V. HAUFF (who checked the programming for π), Miss HOMÉ S. MCALLISTER (who checked the programming for e), W. BARKLEY FRITZ and the author, taking turns on eight-hour shifts to keep the ENIAC operating continuously throughout the week end.

While the programming for e is valid for a little over 2500 decimal places and, with minor alterations, can be extended to much greater range, and while the programming for π is valid for around 7000 decimal places, the arbitrarily selected limit of 2000+ was a convenient stopping point for e and about all that could be anticipated for a week end's operation for π.

While the details of the programming for each project were completely different, the general pattern of procedure was roughly the same, and both projects will be discussed together. In both projects the ENIAC'S divider was employed to determine a chosen number i of digits of each successive term of the series being computed, the remainder after each division being stored in the ENIAC'S memory and the digits of each term being added to (or subtracted from) the cumulative total. After performing this operation for as many successive terms as practicable, the remainders for these terms were printed on an I.B.M. card (the standard input-output vehicle for the ENIAC), and the process was repeated, continuing through some term beyond which the digits of and remainders for all further terms would be zeros. At this point was printed the cumulative total of the digits of the individual terms, which yielded (after adjustment for carry-over) the actual digits of the series being determined.

The cards bearing the remainders then were fed into the ENIAC reader, and the entire process was repeated for the next i digits, the ENIAC reading each remainder in turn and placing it before the digits of the appropriate term. Each deck of cards bearing remainders was then employed to determine the "next" i digits and the "next" deck of "remainder" cards continuing through the first stopping point beyond the 2000th decimal place. The cards bearing the cumulative totals of sets of i digits of the terms were then adjusted for carry-over into each preceding set of i digits. In the case of e this yielded the final result; in the case of π all the above described operations were performed once for each inverse tangent series, so that each set of "cumulative total" cards, adjusted for carry-over, yielded the digits of one of the series, the final result being determined by the combination of these series in appropriate manner.

The number of places i chosen for each interval of computation, the maximum magnitude of each remainder, the amount of memory space available, and the details of divider operation (the number of places to which division can be performed to yield a positive remainder, and the necessary conditions of relative and absolute positioning of numerator and denominator) all were interrelated, and where opportunity for selection existed, that selection was made which provided maximum efficiency of computation. In the case of π there was imposed the additional requirement that identical programming apply for all series employed, and for this reason the formula:

$$\pi/4 = 4 \arctan 1/5 - \arctan 1/239$$

was superior to the other two.

In order to insure absolute digital accuracy, the programming was arranged so that one half applied to computation and the other half to checking. Before any deck of "remainder" cards was employed to determine the next i digits, the cards were reversed and employed in the checking sequence to confirm each division by a multiplication and each addition by a subtraction and vice versa, reproducing the previous deck of "remainder" cards and insuring that the cumulative total reduced to zero. (In the case of e this was a simple inversion of the computation; in the case of π the factor $(2n + 1)^{-1}$ in each term made it a more complicated affair.) After the correctness of each deck was established through this checking, the "remainder" cards were rereversed, and the computation proceeded for the next i digits.

Since the determination of each i digits was not begun until the determination of the previous i digits had been confirmed by checking, the ENIAC stood idle during the reversals and rereversals and comparisons of the decks in the computation of e; in the case of π, however, the ENIAC was never idle, for operation on each series was alternated with operation on the other, card-handling on either being accomplished while the other was being operated upon by the ENIAC. In the case of e, insurance against any undiscovered accidental misalignment of cards was provided by rerunning the entire computation without checking, i.e., without card reversals, confirming the original results; in the case of π, the same assurance was provided by a programmed check upon the identification numbers of each successive card in both computation and checking.

In the case of e, there was printed (in addition to each "remainder" card) a card containing the current i digits of $(n!)^{-1}$ for $n = 20K; K = 1, 2, 3 \cdots$; in the case of π only remainder and final total cards were printed.

The ENIAC determinations of both π and e confirm the 808—place determination of e published in *MTAC*, v. 2, 1946, p. 69, and the 808—place determination of π published in *MTAC*, v. 2, 1947, p. 245, as corrected in *MTAC*, v. 3, 1948, p. 18–19.

Only the following minor observation is offered at this time concerning the randomness of the distribution of the digits. Publication on this subject will, however, be forthcoming soon. A preliminary investigation has indicated that the digits of e deviate significantly from randomness (in the sense of staying closer to their expectation values than a random sequence of this length normally would) while for π no significant deviations have so far been detected.

The programming was checked and the first few hundred decimal places of each constant were determined on a Sunday before each holiday week end mentioned above, the principal effort being made on the longer week end. The actual required machine running time for both computation and checking in the case of e was around 11 hours, though card-handling time approximately doubled this, and the recomputation without checking added about 6 hours more; actual required machine running time (including card-handling time) for π was around 70 hours.

The following values of π and e have been rounded off to 2035D and 2010D respectively.

π = 3.14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
44288	10975	66593	34461	28475	64823	37867	83165	27120	19091
45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724	89122	79381	83011	94912
98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
50244	59455	34690	83026	42522	30825	33446	85035	26193	11881
71010	00313	78387	52886	58753	32083	81420	61717	76691	47303
59825	34904	28755	46873	11595	62863	88235	37875	93751	95778
18577	80532	17122	68066	13001	92787	66111	95909	21642	01989
38095	25720	10654	85863	27886	59361	53381	82796	82303	01952
03530	18529	68995	77362	25994	13891	24972	17752	83479	13151
55748	57242	45415	06959	50829	53311	68617	27855	88907	50983
81754	63746	49393	19255	06040	09277	01671	13900	98488	24012
85836	16035	63707	66010	47101	81942	95559	61989	46767	83744
94482	55379	77472	68471	04047	53464	62080	46684	25906	94912
93313	67702	89891	52104	75216	20569	66024	05803	81501	93511
25338	24300	35587	64024	74964	73263	91419	92726	04269	92279
67823	54781	63600	93417	21641	21992	45863	15030	28618	29745
55706	74983	85054	94588	58692	69956	90927	21079	75093	02955
32116	53449	87202	75596	02364	80665	49911	98818	34797	75356
63698	07426	54252	78625	51818	41757	46728	90977	77279	38000
81647	06001	61452	49192	17321	72147	72350	14144	19735	68548
16136	11573	52552	13347	57418	49468	43852	33239	07394	14333
45477	62416	86251	89835	69485	56209	92192	22184	27255	02542
56887	67179	04946	01653	46680	49886	27232	79178	60857	84383
82796	79766	81454	10095	38837	86360	95068	00642	25125	20511
73929	84896	08412	84886	26945	60424	19652	85022	21066	11863
06744	27862	20391	94945	04712	37137	86960	95636	43719	17287
46776	46575	73962	41389	08658	32645	99581	33904	78027	59009
94657	64078	95126	94683	98352	59570	98258			

e = 2.71828	18284	59045	23536	02874	71352	66249	77572	47093	69995
95749	66967	62772	40766	30353	54759	45713	82178	52516	64274
27466	39193	20030	59921	81741	35966	29043	57290	03342	95260
59563	07381	32328	62794	34907	63233	82988	07531	95251	01901
15738	34187	93070	21540	89149	93488	41675	09244	76146	06680
82264	80016	84774	11853	74234	54424	37107	53907	77449	92069
55170	27618	38606	26133	13845	83000	75204	49338	26560	29760

67371	13200	70932	87091	27443	74704	72306	96977	20931	01416
92836	81902	55151	08657	46377	21112	52389	78442	50569	53696
77078	54499	69967	94686	44549	05987	93163	68892	30098	79312
77361	78215	42499	92295	76351	48220	82698	95193	66803	31825
28869	39849	64651	05820	93923	98294	88793	32036	25094	43117
30123	81970	68416	14039	70198	37679	32068	32823	76464	80429
53118	02328	78250	98194	55815	30175	67173	61332	06981	12509
96181	88159	30416	90351	59888	85193	45807	27386	67385	89422
87922	84998	92086	80582	57492	79610	48419	84443	63463	24496
84875	60233	62482	70419	78623	20900	21609	90235	30436	99418
49146	31409	34317	38143	64054	62531	52096	18369	08887	07016
76839	64243	78140	59271	45635	49061	30310	72085	10383	75051
01157	47704	17189	86106	87396	96552	12671	54688	95703	50354
02123	40784	98193	34321	06817	01210	05627	88023	51930	33224
74501	58539	04730	41995	77770	93503	66041	69973	29725	08868
76966	40355	57071	62268	44716	25607	98826	51787	13419	51246
65201	03059	21236	67719	43252	78675	39855	89448	96970	96409
75459	18569	56380	23637	01621	12047	74272	28364	89613	42251
64450	78182	44235	29486	36372	14174	02388	93441	24796	35743
70263	75529	44483	37998	01612	54922	78509	25778	25620	92622
64832	62779	33386	56648	16277	25164	01910	59004	91644	99828
93150	56604	72580	27786	31864	15519	56532	44258	69829	46959
30801	91529	87211	72556	34754	63964	47910	14590	40905	86298
49679	12874	06870	50489	58586	71747	98546	67757	57320	56812
88459	20541	33405	39220	00113	78630	09455	60688	16674	00169
84205	58040	33637	95376	45203	04024	32256	61352	78369	51177
88386	38744	39662	53224	98506	54995	88623	42818	99707	73327
61717	83928	03494	65014	34558	89707	19425	86398	77275	47109
62953	74152	11151	36835	06275	26023	26484	72870	39207	64310
05958	41166	12054	52970	30236	47254	92966	69381	15137	32275
36450	98889	03136	02057	24817	65851	18063	03644	28123	14965
50704	75102	54465	01172	72115	55194	86685	08003	68532	28183
15219	60037	35625	27944	95158	28418	82947	87610	85263	98139
55990	06738								

Values of the auxiliary numbers arccot 5 and arccot 239 to 2035D are in the possession of the author and also have been deposited in the library of Brown University and the UMT FILE [1] of *MTAC*.

GEORGE W. REITWIESNER

Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland

[1] See *MTAC*, v. 4, p. 29.

THE CHRONOLOGY OF PI

Herman C. Schepler

3000 B. C. The Pyramids (Egypt). $3\frac{1}{7}$ (3.142857.....). The sides and heights of the pyramids of Cheops and of Sneferu at Gizeh are in the ratio 11:7, which makes the ratio of half the perimeter to the height $3\frac{1}{7}$. In 1853, H. C. Agnew, Esq., London, published a letter from Alexandria on evidence of this ratio found in the pyramids being connected or related to the problem of the quadrature of the circle.

There is considerable disagreement in the dates given for the construction of the pyramids, particularly that of the Great Pyramid[1]. Dates for pyramids as taken from various sources range from 1800 B. C. to 4750 B. C. [7], 328; [11], 43; Encyclopedia Britannica; Americana Encyclopedia; Chronology of World Events.

2000 B. C. Rhind Papyrus (Egypt). $(\frac{16}{9})^2$ (3.1604.....). Also called the Ahmes Papyrus after Ahmes, its writer. The rule given for constructing a square having the same area as a given circle is: *Cut $\frac{1}{9}$ off the circle's diameter and construct a square on the remainder.*

The Rhind Papyrus is the oldest mathematical document in existence and was founded on an older work believed by Birch to date back as far as 3400 B. C. A. Henry Rhind, an Egyptologist, brought the manuscript to England in the middle of the 19th century. It was deciphered by Eisenlohr in 1877 and is part of the Rhind collection of the British Museum.

Dates from various sources for the origin of the Rhind Papyrus range from c. 1500 B. C. to c. 2000 B. C. [5], 9; [11], 261; [18], 14; [20], 3, 188; [24], 392.

950 B. C. Bible (Hebrews). 3. I Kings vii 23: ''And he made a molten sea, ten cubits from the one brim to the other; it was round all about..... and a line of thirty cubits did compass it round about''. See also, II Chron. iv 2. The first book of Kings which deals with Samuel quite some time before Solomon, was probably written about 932-800 B. C.[2]

The Babylonians, Hindus, and Chinese used the value 3. It is probable that the Hebrews adopted this value from the Semites (Babylonian predecessors). No definite statement for the value of π has yet been found on the Babylonian cylinders (1600 to 2300 B. C.).

The value 3 is still more plainly given in the Talmud, where we read that ''that which measures three lengths in circumference is

[1] The Pyramid of Cheops, called Khufu.
[2] From personal communications with Brothers Reginald and Edmund, University of Notre Dame.

one length across.'' The Talmud was not put into final form until about 200 A. D., but it was patterned after the Old Testament, and hence the value 3 was retained.[2] [5], 88; [11], 58, 261; [15], 50; [18], 13; [22], 105; [24], 391.

460 B. C. Hippocrates of Chios (Greece) attempted to square the circle. He actually squared the lune and was the first to square a curvilinear figure in attempting to square the circle. That he really committed the fallacy of applying his lune-quadratures to the quadrature of the circle is not generally accepted. Euclid's *Elements* was probably founded on the first elementary textbook on geometry, written by Hippocrates. [5], 22; [8], 310; [19], 41, 44; [20], 258; [24], 393.

440 B. C. Anaxagoras (499-428 B. C.) (Athens, Greece), a contemporary of Hippocrates, ''drew the quadrature of the circle'' while in prison. No solution was offered, however. This is the first mention found of the quadrature problem as such. Anaxagoras was the last and most famous philospher of the Ionian School. He was accused of heresy because of his hypothesis that the moon shone only by reflected light and that it was made of some earthly substance, while the sun, on the contrary was a red hot stone that emitted its own light; born in Asia Minor near Smyrna, he spent the greater part of his life in Athens. [5], 17; [8], 299; [20], 256, 258; [22], 105.

430 B. C. Antiphon (born 480 B. C.) of Rhamnus (Attica, Greece) thought he had squared the circle. By inscribing polygons of ever increasing numbers of sides, he pioneered the invention of the modern calculus by approximately exhausting the difference between the polygon and the circle, thus approximating the area of the circle. Bryson of Heraclea, a contemporary of Antiphon, advanced the problem of the quadrature considerably by circumscribing polygons at the same time that he inscribed them. He erred, however, in assuming that the area of a circle was the arithmetical mean between circumscribed and inscribed polygons. [5], 23; [19], 41; [20], 259; [24] 393.

425 B. C. Hippias (born 460 B. C.) of Elis, (Greece), mathematician, astronomer, natural scientist, and a contemporary of Socrates; invented the quadratrix which he used to trisect the angle. Dinostratus used this curve in 350 B. C. to square the circle. See 350 B. C., Dinostratus, [5], 20, 42, 50; [8], 310; [20], 259; [24], 392.

414 B. C. Aristophanes (c.448-c.385 B. C.) (Athens, Greece), the comic poet, in his play, *The Birds*, refers to a geometer who announces his intention to make a square circle. Since that time the term ''circle-squarers'' has been applied to those who attempt the impossible. The Greeks had a special word τετραγωνιζειν, which meant ''to busy oneself with the quadrature.'' [5], 17; [18], 12; [20], 257; [22], 99.

390 B. C. (?) Plato (429-348 B. C.) (Athens, Greece), called solutions

of the famous geometrical problems mechanical and not geometrical when they required the use of instruments other than ruler and compasses. He solved the duplication mechanically; founded the Academy and contributed to the philosophy of mathematics. [5], 27; [8], 316.

370 B. C. Eudoxus (c.408 B. C.-c.355 B. C.) of Cnidus, (Greece) carried Antiphon's method of exhaustion further by considering both the inscribed and circumscribed polygons. He was a pupil of Archytas and Plato. In his astronomical observations he discovered that the solar year is longer than 365 days by six hours. Vitruvius credits him with inventing the sun-dial. See 430 B. C., Antiphon. [8], 306; [20]. 259.

350 B. C. Dinostratus (Greece) used the quadratrix invented by Hippias to square the circle. See 425 B. C., Hippias. Menaechmus (c. 350 B. C.), brother of Dinostratus and a friend of Plato, invented the conic sections with which he solved the duplication. [5], 20, 27, 282; [8], 305; [24], 392.

300 B. C. Euclid (Greece) offered no solution to the quadrature. He was the most successful textbook writer the world has ever known. Over 1000 editions of his geometry have appeared in print since 1482 and manuscripts of this work had dominated the teaching of the subject for 1800 years preceding that time. He was a pupil of Plato and studied in Athens; was a Greek mathematician and founder of the Alexandrian School. See c. 470 B. C., Hippocrates. [1], 340; [18], 21; [20], 268.

240 B. C. Archimedes (278-212 B. C.) (Greece) gave between $3\frac{10}{71}$ (3.1408.....) and $3\frac{1}{7}$ (3.1428.....). His *Measurement of the Circle* contains but three propositions. (1) He proves that the area of a circle = πr^2; (2) shows that $\pi r^2:(2r)^2::11:14$, very nearly; (3) shows that the true value of π lies between $3\frac{10}{71}$ and $3\frac{1}{7}$. For the latter, inscribed and circumscribed polygons of 96 sides each were used. He also gave the quadrature of the parabola; was an engineer, architect, physicist, and one of the world's greatest mathematicians.[3] [8], 299; [11], 261; [18], 14; [19], 100; [20], 11, 188.

213 B. C. Chang T'sang (China). 3. Given in the *Chiu-chang Suan-shu (Arithmetic in Nine Sections)*, commonly called the *Chiu-chang*, the most celebrated Chinese text on arithmetic. Neither its authorship nor the time of its composition is known definitely. By an edict of the despotic emperor Shih Hoang-ti of the Ch'in Dynasty "all books were burned and all scholars were buried in the year 213 B. C." After the death of this emperor, learning revived again. Chang T'sang, a scholar,

[3]The achievements of Archimedes were indeed remarkable since neither our present numerals nor the Arabic numerals, nor any system equivalent to our decimal system was known in his time. Try multiplying XCVII and MDLVIII in the Roman numeral system without using Arabic or common numerals. This will indicate the difficulties under which Archimedes labored.

found some old writings upon which he based the famous treatise, the *Chiu-chang*. [5], 71.

100 B. C. Heron (Hero) of Alexandria, sometimes used the value $3\frac{1}{7}$ (3.1428.....) for purposes of practical measurement, and sometimes even the rougher approximation 3. He contributed to mensuration and was called the father of surveying. This date is also given as c. 50 A. D. Heath, however, believes that Hero may have lived considerably later, perhaps even in the fourth century A. D. [1], 340; [8], 309; [20], 232; [22], 108.

20 B. C. Vitruvius (Marcus Vitruvius Pollio) (Rome, Italy) $3\frac{1}{8}$ (3.125). Roman architect and engineer; first described the direct measurement of distances by the revolution of a wheel. [8], 321; [20], 188, 238.

25 A. D. Liu Hsiao (China). 3.16. He was of the Imperial House of the Han Dynasty and one of the most prominent "circle squarers" of his day. His son, Liu Hsing, devised a new calendar. [11], 261.

97 A. D. Frontius (Sextus Julius Frontinus) (c. 40-103 A. D.) (Rome, Italy) used $3\frac{1}{7}$ (3.1428.....) for water pipes and areas. He said: "The square digit is greater than the circle digit by $\frac{3}{14}$th of itself; the circle digit is less than the square digit by $\frac{3}{11}$". A Roman author, soldier, surveyor, and engineer; was successively city preator of Rome, governor of Britain, and superintendent of the aqueducts at Rome. [8], 306; [15], 50.

125 A. D. Ch'ang Höng (78-139 A. D.) (China). $\sqrt{10}$ (3.162.....). This was one of the earliest uses of this approximation. Höng was chief astrologer and minister under the emperor An-ti'; constructed an armillary sphere and wrote on astronomy and geometry. [23], 553; [24], 394.

150 A. D. Prolemy (Claudius Ptolemaeus) (87-165 A. D.) (Alexandria) $3\frac{17}{120}$ (3.141666.....). This value is found in his astronomical treatise, the *Syntaxis* (*Almagest* in Arabian[4]) in 13 books. The result was expressed in sexagesimal system as 3°8'30", *i.e.*, $3 + \frac{8}{60} + \frac{30}{60^2}$, which reduces to $3\frac{17}{120}$. This is the most notable result after Archimedes. Ptolemy was a geographer, mathematician, and one of the greatest of Greek astronomers. [1], 340; [5], 46; [8], 317; [18], 15; [20], 188; [24], 394.

250 A. D. Wang Fan (Wang Fun) (229-267 A. D.) (China) $\frac{142}{45}$ (3.155...). Chinese astronomer. [11], 261; [23], 553; [24], 394.

263. Liu Hui (China). $\frac{157}{50}$ (3.1400). Calculated the perimeters of

[4] *Almagest* is the usual nomenclature for the *Syntaxis*, derived from an Arabic term signifying "the greatest".

regular inscribed polygons up to 192 sides in the same way as Antiphon (430 B. C.); made a commentary on Chang T'sang's Chiu-chang (See 213 B. C., Chang T'sang); contemporary writer of Wang Fan and best-known Chinese mathematician of the 3rd century. [5], 71; [24], 394.

450. Wo (China) 3.1432+. Geometer. [23], 553.

480. Tsu Ch'ung-chih (430-501 A. D.) (China). Between 3.1415926 and 3.1415927. Expert in mechanics and interested in machinery; gave $\frac{22}{7}$ (3.1428.....) as an ''inaccurate'' value and $\frac{355}{113}$ (3.1415929.....) as the ''accurate'' value. The latter is often attributed to Adriaen Anthonisz. See 1585, Adriaen Anthonisz, and 1573, Valentin Otto. [11], 261; [20], 73; [23], 553; [24], 394.

500. Arya-Bhata (Aryabhata) (c. 475-c. 550) (India) gave $3\frac{177}{1250}$ (3.1416) and $\frac{62832}{20000}$ (3.1416). The latter was calculated from the perimeter of an inscribed polygon of 384 sides[5]. Arya-Bhata was a Hindu mathematician who wrote chiefly on algebra, including quadratic equations, permutations, indeterminate equations, and magic squares. [1], 341; [5], 85, 87; [6], 10; [8], 299; [11], 261; [20], 16; [23], 553; [24], 394.

505. Varāhamihira Pancha Siddhāntikā (India) $\sqrt{10}$ (3.162.....). Hindu astronomer; the most celebrated of astronomical writers in early India; taught the sphericity of the earth and was followed in this respect by most other Hindu astronomers of the middle ages. [5], 96.

510. Anicius Boethius (c. 480-524 A. D.) (Rome) said the circle had been squared in the period since Aristotle's time but noted that the proof was too long for him to give. A philosopher, statesman, and founder of medieval scholasticism; wrote on arithmetic and translated and revised many Greek writings on mathematics, mechanics, and physics. His *Consolations of Philosophy* were composed while in prison. He was executed at Pavia in 524. [8], 301; [20], 261.

530. Baudhayana (India) $\frac{49}{16}$ (3.062.....). He lived before 530 A. D. His value and other of his works were published in England in 1875. [1], 341.

628. Brahmagupta (Born 598 A. D.) (India). $\frac{22}{7}$ (3.1428.....). He gave 3 as the ''practical value'', and $\sqrt{10}$ (3.162.....) as the ''exact value'', perhaps because of the common approximate formula $\sqrt{a^2 + r} = a + \frac{1}{2a + r}$ which leads to $\sqrt{10} = 3 + \frac{1}{7}$, or the common Archimedean value. The value $\sqrt{10}$ was extensively used in medieval times. Brahmagupta, the most prominent Hindu mathematician of the 7th century, was the first Indian writer to apply algebra extensively to astronomy. [5], 85; [8], 302; [20], 188; [24] 394.

This article will be concluded in the March-April issue.

[5]This may be due to a later writer by the same name.

REFERENCES

(1) Ball, W. W. R. Mathematical Recreations and Problems. (11th ed.) Rev. by H. S. M. Coxeter, New York, Macmillan, 1942.

(2) Ball, W. W. R. A short account of the History of Mathematics. London, Macmillan & Co., limited 1919.

(3) Bell, Eric Temple. Development of Mathematics. McGraw Hill, 1945.

(4) Cajori, Florian. History of Mathematical Notations. Vol. II. Open Court Publ. Co., 1929.

(5) Cajori, Florian. History of Mathematics. New York, Macmillan, 1938.

(6) Condon, Edward. Curiosities of Mathematics. Girard, Kansas, Haldeman-Julius, 1925.

(7) De Morgan, Augustus. A Budget of Paradoxes. Chicago, Open Court Pub. Co., 1940.

(8) Fink, Karl. Brief History of Mathematics. (Trans.) Chicago, Open Court Pub. Co., 1940.

(9) Gould, Sylvester C. What is the value of Pi. Manchester, N. H., "Notes and Queries", 1888.

(10) Hobson, E. W. Squaring the Circle. Cambridge Univ. Press, 1913.

(11) Hogben, Lancelot. Mathematics for the Million. Norton, 1943.

(12) Kasner, Edward. Mathematics and the Imagination. Simon and Schuster, 1940.

(13) Klein, Felix. Famous Problems in Elementary Geometry. Ginn and Co., 1897.

(14) Larrett, Denham. The Story of Mathematics. New York, Greenberg, 1926.

(15) Licks, H. E. Recreations in Mathematics. New York, Van Nostrand, 1931.

(16) Merrill, Helen A. Mathematical Excursions. Boston, Bruce Humphries, Inc., 1934.

(17) Myers, William A. Quadrature of the Circle. (Translation of Montucla). 2nd Ed. Cincin., 1874.

(18) Phin, John. Seven Follies of Science. Van Nostrand, 1911.

(19) Rupert, William W. Famous Geometrical Theorems and Problems. Boston, D. C. Heath, 1901.

(20) Sanford, Vera. Short History of Mathematics. Houghton Mifflin, 1930.

(21) Schubert, Hermann. Mathematical Essays and Recreations. (Trans. by T. J. McCormack.) Chicago, Open Court Pub. Co., 1910.

(22) Schubert, Hermann. Squaring of the circle. Smithsonian Institution Annual Report, 1890.

(23) Smith, David Eugene. History of Mathematics. Boston, New York, Ginn & Co. 1923.

(24) Smith, David Eugene. History and Transcendence of Pi. Monograms on Modern Mathematics. W. J. A. Young, Longmans Green, 1911.

(25) Steinhaus, H. Mathematical Snapshots. Warsaw, New York, G. E. Stechert. 1938.

(26) Sullivan, John William Navin. History of Mathematics in Europe. Oxford University, 1925.

THE CHRONOLOGY OF PI

Herman C. Schepler

(Continued from the January-February issue)

825. Muhammed Ibn Musa al-Khowarizmi (Alchwarizmi, Alkarism) (813-833) (Arabia) gave (used ?) 22/7 (3.1428 ···), $\sqrt{10}$ (3.162277 ···). and 62832/20000 (3.1416). The first he described as an approximate value, the second as used by geometricians, and the third as used by astronomers. He was an astronomer and the greatest mathematician at the court of al-Mâmûm; wrote Liber Algorism[6] (the book of al-Khowarizmi) the title of which gave the name to algebra. His name means Mohammed, the son of Musa (or Moses), from Khwarizm (Khiva), a locality south of the Black Sea. It was an Arab custom to give a man the name of his city. [1], 342; [5], 94, 102; [8], 298; [20], 28, 144; [22], 111; [26], 17.

850. Mahāvīra (India). Used $\sqrt{10}$. (3.162277 ···). His rule for the sphere is interesting, the approximate value being given as $\frac{9}{2}\left(\frac{1}{2}d\right)^3$ and the accurate value as $\frac{9}{10}\cdot\frac{9}{2}\left(\frac{1}{2}d\right)^3$, which means that π must be taken as 3.0375. He is perhaps the most noteworthy Hindu contributor to mathematics, possibly excepting Bhāskara, who lived three centuries later.

First half of 11th century. Frankos Von Luttich (Germany). No value available. He is claimed to have "contributed the only important work in the Christian era on squaring the circle." His works are published in six books, but only preserved in fragments. [22], 111.

1150. Bhāskara (Bhāskara Acharya) (1114 - c. 1185) (India) gave 3927/1250 (3.14160) as an "accurate" value, 22/7 (3.1428 ···) as an "inaccurate" value, and $\sqrt{10}$ (3.162277 ···) for ordinary work. The value 3927/1250 was possibly copied from Arya-Bhata, but is said to have been calculated afresh by Archimedes' method from the perimeter of a regular polygon of 384 sides. He also gave 754/240 (3.141666 ···), the origin of which is uncertain. Bhāskara wrote chiefly on astronomy and mathematics. He and Srīndhara are the only outstanding writers in the history of Hindu mathematics from 1000 to 1500 A.D. [1], 341; [5], 85, 87; [8], 301; [20], 18.

1220. Leonardo Pisano (Leonardo of Pisa) (c. 1170 - c. 1250) (Italy), called Fibonacci, i.e., son of Bonaccio. 1440/(458 1/3) (3.1418 ···). His *Practica geometriae* refers to Euclid, Archimedes, Heron, and Ptolemy.

[6] The 14th century was characterized by the production of "algorisms", works devoted to the exposition of the uses of Hindu-Arabic numerals. The name is a corruption of al-Khowarizmi, but the books had no other connection with his work.

Without making use of Archimedes, he determined 1440/(458 1/5) (3.1427 ···) and 1440/(458 4/9) (3.1410 ···) from the regular polygon of 96 sides, of which he took the mean 1440/(458 1/3) (3.1418 ···). He was considered by some as the greatest mathematical genius of the Middle Ages. He traveled extensively and brought back to Italy a knowledge of the Hindu numerals and the general learning of the Arabs, which he set forth in many of his writings. [1], 342; [5], 120; [8], 218; [20], 24, 188; [24], 395.

1260. Johannes Campanus (Rome) 22/7 (3.1428571 ···). Sometime chaplain to Urban IV who reigned as pope from 1261 to 1264; published an edition of Euclid's *Elements*. [7], 31; [20], 25; [26], 25.

1430. Al Kashi (Arab) of Samerkand (Persia). 16 places; 3.1415926535898732. Wrote a short treatise in Persian on arithmetic and geometry; was assistant to Ulugh Beg, the Royal Astronomer of Persia. [5], 108; [11], 261.

1460. Georg von Peurbach (Purbach) (1423-1461) (Austria). 62832/20000 (3.141600). Published in 1541. Studied under Nicholas de Cusa and other great teachers. Interested in astronomy and trigonomy and wrote an arithmetic. [1], 342; [5], 131; [8], 316; [20], 27.

1464. Nicolaus de Cusa (Nicolaus Cusanus) (1401-1464) (Germany). $(3/4) \times (\sqrt{3} + \sqrt{6}) = (3.1423 \cdots)$. Letters from Regiomontanus in 1464 and 1465, and published in 1533, rigidly demonstrated that de Cusa's quadrature was incorrect. This date was taken from these letters and may well have been earlier. (See 1464, Regiomontanus). de Cusa was a theologian, physicist, astronomer and geometer. The son of a fisherman, he rose rapidly in the Church, was made a cardinal, and became governor of Rome in 1448. His name was derived from his birthplace, the small town of Cues. [1], 342; [5], 143; [7], 47; [8], 315; [22], 111; [26], 28.

1464. Regiomontanus (Johann Müller) (1436-1476) (Germany). 3.14343. From his letters to de Cusa, published in 1533, proving de Cusa's results wrong. (See 1464, Nicolaus de Cusa). He was a pupil and associate of Peurbach; was a mathematician, astronomer, geographer, translator of Greek mathematics, and author of the first textbook of trigonometry. Pope Sixtus IV summoned him to Rome to aid in the reform of the calendar. He died there in 1476. Johannes Muller was commonly called Regiomontanus for his birthplace at Konigsberg, that is, king's mountain, or in Latin, regius mons. [1], 342; [8], 317; [20], 27, 296; Encyclopedia Brittanica.

1503. Tetragonismus. (Italy?). 22/7 (3.1428571 ···). *Circuli quadratura per Campanum, Archimedem Syracusanum* ··· . This book and the one mentioned in the next item, are probably the earliest in print on the subject of the quadrature. The quadrature of Campanus takes the ratio of Archimedes, (3.1428571···) to be absolutely correct. (See c. 1260, Campanus). [7], 31.

1503. Charles Bouelles (Paris, France) (3.1423 ···) announced anew the construction of de Cusa. (See c. 1464, Nicolaus de Cusa). The publication is: *In hoc opere contenta Epitome ··· Liber de quadratura Circuli ···*. Paris, 1503, folio. Was professor of theology at Noyon. Wrote on geometry and the theory of numbers, and was the first to give scientific consideration to the cycloid. [7], 31, 44; [18], 16; [22], 112.

1525. Stifel (1486-1567) (Germany). 3 1/8. *Underweysung,* etc. "The quadrature of the circle is obtained when the diagonal of the square conatins 10 parts of which the diameter of the circle contains 8." (Stated to be only approximate). Stifel's work had wide circulation in Germany. He was a minister who predicted the end of the world to occur October 3, 1533. Many of his followers who believed him spent and disposed of all their worldly goods and were ruined. He was imprisoned. [2], 220; [8], 221.

1544. Oronce Fine (Orontius Finaeus) (1494-1555) (Paris, France). His quadrature was disproved by Portugese Petrus Nonius (Pedro Nunes) (1502-1578) and also by Jean Buteo (1492-1572). He became professor of mathematics at College de France in 1532. [5], 142; [7], 50; [22], 112.

1573. Valentin Otto (Valentine Otho) (Germany). 355/113 (3.14159292035). He was an engineer. (See 480 A.D., Tsu Ch'ung chih, and 1585, Adriaen Anthonisz). [5], 73, 132; [23], 553.

1579. Francois Vieta (1540-1603) (Paris, France). Between 3.1415926535 and 3.1415926537. The first to give an infinite series:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

Following the Greek method, he considered polygons of $6 \cdot 2^{16}$ sides, i.e. 393,216 sides, and found π correct to nine places. He disproved the quadrature of Joseph Scaliger. Vieta was the greatest French mathematician of the 16th century. In 1580 he became master of requests at Paris, and later a member of the king's privy council. He wrote chiefly on algebra but was interested in the calendar and mathematics in general. [1], 342; [5], 143; [8], 321; [11], 261; [18], 16; [20], 38; [24], 395.

1580. Tycho Brahe (1546-1601) (Copenhagen, Denmark). $88/\sqrt{785}$ (3.1408 ···). Danish astronomer, his observatory being near Prague. Taught Johann Kepler. Brahe lost his nose in a duel with Passberg and adopted a golden one which he attached to his face with a cement that he always carried with him. [5], 159; [20], 188.

1585. Simon van der Eycke (Netherlands?) 3.1416055. In 1584 he gave 1521/484 equals 3.14256···. In 1585, van Ceulen gave $\pi<3.14205<1521/484$, found by calculating a regular polygon of 192 sides. In his reply, van der Eycke determined $\pi = 3.1416055$, whereupon van Ceulen in 1586 computed π between 3.142732 and 3.14103, and finally computed it correctly to 35 places. (See 1596, van Ceulen, and 1610, van Ceulen). [8], 222; [22], 112.

1585. Adriaen Anthonisz (Peter Metius) (1527-1607) (France) 355/113 (3.14159292 ⋯). Rediscovered this Chinese value, also known to the Japanese, when he was obliged to seek a still more accurate value than 3 1/7 to disprove the quadrature of van der Eycke. (See 1585, Simon van der Eycke). He proved that π lay between 377/120 and 333/106 from which he concluded that the true fractional value would be obtained by taking the mean of the numerators and the mean of the denominators of these fractions, which gave 355/113. The "rediscovery" was perhaps a lucky guess. The value was published in 1625 by his son Adriaen (1571-1635), who, from the fact that his family was originally from Metz, took the name of Metius. Adriaen Anthonisz also disproved the quadrature of Quercu. (See 480 A.D., Tsu Ch'ung-chih, and 1573, Velentin Otto). [1], 342; [5], 73, 80, 143; [8], 314; [18], 16; [22], 112; [24], 395.

1593. Adriaen van Rooman (Adrianus Romanus) (1561-1615) (Antwerp, Netherlands). He carried the computation to 17 places (15 correct), by computing the circumference of a regular circumscribed polygon of 2^{30} sides, i.e., 1,073,741,824 sides. He, Vieta and Clavius each disproved the quadrature of Joseph Scaliger. Romanus was successively professor of medicine and mathematics in Louvain, professor of mathematics at Würzburg, and royal mathematician (astrologer) in Poland; was the first to prove the usual formula for $\sin(A+B)$. [1], 342; [2], 230; [5], 143; [22], 113; [24], 395.

1596. Ludolph Van Ceulen (1540-1610) (Germany). To 20 places. This result was calculated from finding the perimeters of the inscribed and circumscribed regular polygons of $60 \cdot 2^{33}$ sides, i.e., 515,396,075,520 sides, obtained by the repeated use of a theorem of his discovery equivalent to the formula: $1 - \cos A = 2\sin^2 \frac{1}{2}A$. (See 1610, Ludolph Van Ceulen). [1], 342.

1610. Ludolph Van Ceulen (1540-1610) (Germany). To 35 places. π has since been called the Ludolphian Number in Germany. Van Ceulen devoted a considerable part of his life to the subject. His work was considered so extraordinary that the numbers were cut on his tombstone (now lost) in St. Peter's churchyard, at Leyden.

His post-humous arithmetic, published in Leyden, 1615, contains the result to 32 places, calculated from the perimeter of a polygon of 2^{62} sides, i.e., 4,611,686,018,427,387,904 sides. He also compiled a table of the perimeters of various regular polygons. His investigations led Snellius, Huygens, and others to further studies. (See 1585, Simon van der Eycke, and 1596, Ludolph Van Ceulen). [1], 342; [5], 143; [6], 10; [8], 222, 321; [18], 16; [20], 188; [24], 395.

1621. Snell (Willebrod Snellius) (1580-1626) (Leyden, Germany). To 34 places, published in his *Cyclometricus*, Leyden, 1621. He showed how narrower limits may be obtained for π without increasing the number of sides of the polygons by constructing two other lines from their sides which gave closer limits for the corresponding arcs. His method was so superior to that of Van Ceulen that the 34 places were obtained from a polygon of 2^{30} sides, i.e., 1,073,741,824 sides, from which Van Ceulen had obtained only 14 or perhaps 16 places. Similarly, the value for π obtained correct to two places by Archimedes from a polygon of 96 sides was obtained by Snell from a hexagon, while he determined the value correct to seven places from a polygon of 96 sides. Snell was a physicist, astronomer, and contributor to trigonometry. He discovered the law of refraction in optics in 1619. [1], 344; [2], 256; [5], 143; [7], 75; [8], 319.

1630. Grienberger. (Rome). To 39 places. Was among the last to make a calculation by the method of Archimedes. (See 1654, Christian Huygens). [1], 345; [6], 11.

1647. Gregory St. Vincent (1584-1667) (Belgium) a Jesuit, proposed four methods of squaring the circle in his book, *Opus geometricum quadraturae circuli et sectionum coni*, Antwerp, 1647, but they were not actually carried out. The fallacy in the quadrature was pointed out by Huygens and the work was attacked by many others. He published another book on the subject in 1668. Montucla remarks that "no one ever squared the circle with so much ability or (except for his principal object) with so much success". His *Theoremata Mathematica*, published in 1624, contains a clear account of the method of exhaustions, which is applied to several quadratures, notably that of the hyperbola. He discovered the property of the area of the hyperbola which led to Napier's logarithms being called hyperbolic. [2], 309; [5], 181; [8], 318.

1647. William Oughtred (1574-1660) (England), designated the ratio of circumference of a circle to its diameter by π/δ. π and δ were not separately defined, but undoubtedly π stood for periphery and δ for diameter. Oughtred's notation was adopted by Isaac Barrow (publication date, 1860) and by David Gregory (1697) except that he writes π/ρ, ρ being the radius. Oughtred wrote on arithmetic and trigonometry. [4], 8.

1650. John Wallis (1616-1703). (England). Unlimited series. By an extremely difficult and complicated method he arrived at the interesting expression for π:

$$\frac{4}{\pi} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots} \text{ (7)}$$

(7) It is often given in another form:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}$$

He showed this value to Lord Brouncker (1620-1684), first president of the Royal Society, who brought it into the form of a continued fraction:

$$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}}$$

Wallis was Savilian professor of geometry at Oxford and published many mathematical works. In his *Arithmetica infinitorum*, the square □ stands for 4/3.14149 ⋯ . In 1685 he represented by π the "periphery" described by the center of gravity in a revolution. [1], 345; [4], 8; [5], 186; [6], 11; [8], 321; [11], 261; [12], 76; [20], 188; [26], 74.

1654. Christian Huygens (1629-1695). (Hague, Netherlands). 9 places. Used only the inscribed polygon of 60 sides. Last noteworthy attempt made by Greek methods. He proved Snell's theorems and made the greatest refinements in the use of the geometrical method of Archimedes. (See 1621, Snell). With his labors the ancient methods may be said to close. Famous physicist and astronomer, and made numerous contributions to mathematics, particularly to the study of curves. [5], 143; [8], 310; [23], 424; [24], 395.

1666. Thomas Hobbes. (Malmsbury and London, England.) 3 1/5 (3.2) Refuted by Huygens and Wallis. In 1678 he published ⋯ *Proportion of a Straight Line to Half the Arc of a Quadrant* in London, in which $\sqrt{10}$ (3.16227 ⋯) was given. Celebrated English philosopher. [7], 109; [9], 15; [21], 136; [22], 114.

1666. Satō Seikō (Japan). 3.14. Given in his *Kongenki*. First Japanese work in which the ancient Chinese method of solving numerical higher equations appears. [5], 79.

1668. James Gregory (1638-1675) (Scotland). Unlimited series: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. When $x = 1$, the series becomes $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. This series was discovered independently by Leibnitz in 1673. See 1673, Leibnitz.[8]

Gregory proved the geometrical quadrature of the circle impossible in 1668. In 1661 he invented, but did not practically construct the telescope bearing his name. He originated the photometric method of estimating the stars. After living in Italy for some years he returned

[8] The series is also given as:

$$\frac{\pi}{4} = 1 - 2\left(\frac{1}{3 \times 5} + \frac{1}{7 \times 9} + \frac{1}{11 \times 13} + \frac{1}{15 \times 17} \cdots\right)$$

to Scotland in 1668 where he became professor of mathematics at St. Andrews; and at Edinburgh in 1674. [5], 143; [6], 11; [7], 118; [8], 308; [11], 261; [24], 396.

1673. Gottfried Wilhelm/von Leibniz (Leibnitz) (1646-1716) (Leipzig, Germany) Series: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \cdots$. This series had previously been discovered by Gregory. (See 1668, Gregory). Leibniz was the only first-class pure mathematician produced by Germany during the 17th century. He shares credit with Newton in developing the differential and integral calculus. [5], 206; [8], 312; [20], 188.

1680. Isaac Newton (1642-1727) (England) significantly gave no value for π. Began his work on the calculus in 1665 and was considered the most promising mathematician in England in 1668. In 1669 he succeeded Barrow as Lucasian professor of mathematics at Cambridge. Contributed extensively to all branches of mathematics then known, particularly to the theories of series, equations, and curves. [5], 202; [8], 315.

1685. Father Kochansky (Poland) 3.1415333 ··· by geometrical construction. *An Approximate Geometrical Construction for Pi*, Leipsiger Berichte, 1685. This construction passed into many geometrical textbooks. Kochansky was librarian of the Polish King John III. In Figure 1, let AB be the diameter of the circle. Construct perpendiculars to this diameter at extremities A and B. Construct the 30° angle DOB; D is the intersection of OD with the perpendicular at B. Mark off AC equal to three times the radius of the circle. Line CD is the approximate semicircumference. The error is –.0000593. [6], 14; [18], 22; [21], 237; [25], 40.

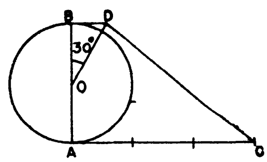


Fig. 1

1690. Takebe (Takebe Hikojirō Kenkō) (1664-1739) (Japan). Unlimited series. Correct to 41 figures. [11], 261; [23], 553.

1699. Abraham Sharp (1651-1742) (England) calculated π to 72 decimal places (71 correct) under instructions from E. Halley. The value was obtained by taking $x = \sqrt{\frac{1}{3}}$ in Gregory's series, giving

$$\frac{\pi}{6} = \sqrt{\frac{1}{3}}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{3^2\cdot 5} - \frac{1}{3^3\cdot 7} + \frac{1}{3^4\cdot 9} - \cdots\right)$$

which is more usable than the form of the series giving $\frac{\pi}{4}$ when $x = 1$. (See 1668, James Gregory). [1], 346; [5], 206; [12], 77; [18], 16; [24], 397.

18th century. Oliver de Serres. (France). 3. With a pair of scales he determined that a circle weighed as much as the square upon the side of the equilateral triangle inscribed in it, which gives $\pi = 3$. [22], 115.

1706. John Machin (1680-1752) (England). 100 places. Obtained by substituting Gregory's infinite series for arc tan (1/5) and arc tan (1/239) in the expression: $\frac{\pi}{4} = 4 \text{ arc tan } (1/5) - \text{arc tan } (1/239)$. He was a mathematician and professor of astronomy at Gresham College, London. [5], 206; [18], 16; [24], 397.

1706. William Jones (1675-1749) (England) designated the ratio of the circumference of a circle to its diameter by π. This is the first occurrence of the sign π for the ratio. [4], 9.

1713. Anonymous. (China). 19 figures. In the *Su-li Ching-yün,* compiled by Imperial order in 1713. [24], 394.

1719. Thomas Fantet de Lagny (Lagny) (1660-1734) (France) 127 places (112 correct). He was a French mathematician. [1], 346; [5], 203; [24], 397.

1720. Matsunaga (Matsunaga Ryōhitsu) (Japan). Correct to 50 figures in our notation. [11], 261; [23], 553.

1728. Malthulon (France) offered solutions to squaring the circle and to perpetual motion. He offered 1000 crowns reward in legal form to anyone proving him wrong. Nicoli, who proved him wrong, collected the reward and abandoned it to the Hotel Dieu of Lyons. Later the courts gave the money to the poor. [6], 14; [17], 18; [22], 114.

1728. Alexander Pope (1688-1744) (England) Poet. In his *Dunciad* is mentioned, "The mad Mathesis, now, running round the circle, finds it square." A note explains that this "regards the wild and fruitless attempts of squaring a circle." [18], 12; Americana Encyclopedia.

1753. M. de Causans, of the Guards (France) 4. He cut a circular piece of turf, squared it, and deduced original sin and the Trinity from the result; found that the circle was equal to the square in which it is inscribed, making $\pi = 4$. He offered a reward for the detection of any error and actually deposited 10,000 francs as earnest of 300,000, but the courts would not allow anyone to recover. [18], 11.

1754. Montucla (1725-1799) (France). *Histoire des rechurches sur la quadrature du cercle* ···. Paris. This was *the* history of the subject in its time and is still a classic on its early history. [7], 159; [23], 540.

1755. The French Academy of Sciences (France) refused to examine any more quadratures or problems of similar nature. De Morgan called this "the official blow to circle-squarers." The Royal Society, London, followed suit a few years later. [5], 246; [7], 163.

1760. Count (Comte) de Buffon (1707-1788) derived his famous *Needle Problem* in which π is determined from probability.(9) A number of equidistant parallel straight lines, distance a apart, are ruled upon a plane surface. A stick of length L, which is less than a, is dropped on the plane. The probability of it falling so as to lie across one of the lines is $2L/\pi a$. π may be evaluated by repeating the experiment a large number of times. When $L = a$, the probability has the simplified value of $2/\pi$. Of the many experiments that have been performed by this method, perhaps the most accurate determination of π was made by Lazzerini, an Italian mathematician, in 1901. He made 3,408 tosses of a needle across parallel lines, giving a value for π of 3.1415929, an error of only .000,000,3. Other similar methods of approximating the value of π have been employed at various times. [1], 348; [5], 243; [6], 17; [7], 170; [12], 246; [22], 118.

1761. Johann Heinrich Lambert (1728-1777) (Germany) proved that π is irrational, i.e., not expressible as an integer or as the quotient of two integers. A physicist, mathematician and astronomer; founder of hyperbolic trigonometry. [5], 2, 246; [8], 312.

1776. Hesse (Berlin, Germany). 3 14/99 (3.14141). Wrote an arithmetic in which this value was given. [21], 136.

1776. Hutton. (1737-1823) (England). Suggested the use of the formula:

$$1/4\ \pi = \tan^{-1} 1/2 + \tan^{-1} 1/3, \quad \text{or}$$

$$1/4\ \pi = 5\ \tan^{-1} 1/7 + 2\ \tan^{-1} 3/79.$$

This formula was also suggested by Euler in 1779, but neither Hutton nor Euler carried the approximation as far as had been done previously by other formulae. Hutton was one of the best known English writers on mathematics at the close of the 18th century; was professor of mathematics at the Military Academy at Woolrich (1772-1807).

1779. Leonard Euler (Leonhard Euler) (1707-1783) (Basel, Switzerland). Series. Published in 1798. $\pi = 20$ arc tan $(1/7) + 8$ arc tan $(3/79)$. Euler used p for the circumference-to-diameter ratio in 1734, and c in 1736. He first used π in 1742. After the publication of his *Introductio in analysin infinitorum* in 1748, the use of the symbol π for the ratio became favorable for wider adoption. Euler used π in most of his later

(9) *Essai d'Arithmetique Moral* by Buffon, appearing in Vol. IV of the *Supplement a l'Histoire Naturelle,* in 1777.

publications. He taught mathematics and physics in Petrograd and was one of the greatest physicists, astronomers and mathematicians of the 18th century. [1], 346; [4], 11; [5], 227, 232; [8], 306; [20], 189; [24], 398.

1788. M. de Vausenville (France). Laboring under the erroneous impression that a reward had been offered for the solution of the quadrature, he brought action against the French Academy of Sciences to recover a reward to which he felt himself entitled. [18], 9.

1789. Georg Vega (1756-1802) (Austria). 143 places (126 correct). [1], 346.

1794. Georg Vega (1756-1802) (Austria). 140 places (136 correct) (See 1789, Georg Vega). [1], 346.

1794. Adrien Marie Legendré (1752-1833) (France) used π in his *Éléments de géométrie* as a symbol for the circumference/diameter ratio. This was the earliest elementary French schoolbook to contain π in regular use. The proofs of the irrationality of π and π^2 are also given in this publication. Legendré was a celebrated mathematician who contributed to the theory of elliptic functions, the theory of numbers, least squares, and geometry. [4], 13.

1797. Lorenzo Mascheroni (1750-1800) (Italy) proved that all constructions possible with ruler and compasses are possible with compasses alone; claimed that constructions with compasses are more accurate than those with ruler. [5], 268; [8], 313.

End of 18th Century. Anonymous. F. X. von Zach (in England) saw a manuscript by an unknown author in the Radcliffe Library, Oxford, which gives the value of π to 154 places (152 correct). [1], 346.

1825. Malacarne. (Italy). Less than 3. One of the last to attempt the geometrical quadrature. His *Solution Géométrique* was published at Paris, 1825. [7], 118.

1828. Specht (Germany) $3.141591953\cdots$ by geometrical construction. Crelle's Journal,[10] Vol. III, page 83. "The rectangle with sides equal to AE and half the radius r is very approximately equal in area to the circle". In Figure 2, on the tangent to the circle at A, let $AB = 2\ 1/5$ radius and $BC = 2/5$ radius. On the diameter through A take $AD = OB$, and draw DE parallel to OC. Then $AE/AD = AC/AO = 13/5$. Therefore, $AE = r \cdot \frac{13}{5}\sqrt{1+\left(\frac{11}{5}\right)^2} = r \cdot \frac{13}{25}\sqrt{146}$. Thus, AE =

[10] Complete title of the publication is *Fur die Reine und Angewandte Mathematik*, (Berlin).

$r \cdot 6.283183906 \cdots$, which is smaller than the circumference of the circle by less than two millionths of the radius. The error is $-.000000700 \cdots$. [10], 34.

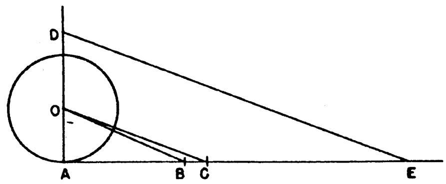


Fig. 2

1833. William Baddeley (England) 3.202216 24/361. ***Mechanical Quadrature of the Circle,*** London Mechanics' Magazine, August, 1833. "From a piece of carefully rolled sheet brass was cut out a circle 1 9/10 inches in diameter, and a square 1 7/10 inches in diameter. On weighing them they were found to be of exactly the same weight, which proves that, as each are of the same thickness, the surfaces must also be precisely similar. The rule, therefore, is that the square is to the circle as 17 to 19." This would make his ratio, 3.202216 24/361. [9], 5.

1836. LaComme (Paris, France). 3 1/8 (3.125). A French well sinker, requesting information from a mathematics professor regarding the amount of stone required to pave the circular bottom of a well, was told that a correct answer was impossible since the exact ratio of the diameter of a circle to its circumference had never been determined. This true but impractical statement set LaComme to thinking. After a haphazard study of mathematics he announced his discovery that 3 1/8 was the exact ratio. Although π had been accurately determined to 152 decimal places in the 18th century (see End of 18th Century, F. X. von Zach), LaComme was honored for his profound discovery with several medals of the first class, bestowed by Parisian societies. [7], 46; [18], 27.

1837. J. F. Callet (Paris, France). 154 places (152 correct). The result of this calculation of π was published in J. F. Callet's Tables in 1837. The calculation may have been made by someone else. [1], 346.

1841. William Rutherford. (England). 208 places (152 correct). He used the formula: $\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$. [1], 346; [18], 16.

1844. Zacharias Dase (1824-1861) (Hamburg, Germany). To 205 places (200 correct) using the formula: $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$. Dase was a lightning calculator employed by Gauss. [1], 346; [12], 77; [20], 188; [24], 398.

1847. Thomas Clausen (1801-1885) (Germany). 250 places (248 correct). His calculation was made independently by the formula:

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

[1], 346.

1849. Jakob de Gelder (Germany). 3.14159292035 ··· by geometrical construction. Grunert's Archiv., Vol. VII, 1849, p. 98. Since $\frac{355}{113} = 3 + \frac{4^2}{7^2+8^2}$, it can easily be constructed. In Figure 3, let $CD = 1$; $CE = 7/8$; $AF = 1/2$; and let FG be parallel to CD and FH parallel to EG. Then $AH = \frac{4^2}{7^2+8^2} = 0.14159292035\cdots$. The error is $+.000000266\cdots$ [5], 73; [10], 34.

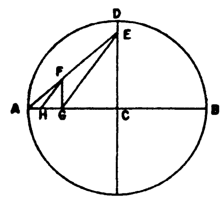


Fig. 3

1851. John Parker. 20612/6561 (3.141594269 ···). Published in his book *Quadrature of the Circle.*

1853. William Rutherford. (England). 440 places (all correct). (See 1841, Rutherford). [1], 346.

1853. William Shanks (1812-1882). (England). 607 places, using Machin's formula. (See 1706, John Machin). Shanks, English mathematician, was assisted by Dr. William Rutherford in the verification of the first 440 decimals of π. (See 1853, William Rutherford). [1], 346; [6], 12; [9], 20; [18], 16.

1853. Richter. (Germany?). 333 places (330 correct). This was presumably done in ignorance of what had been done in England. [1], 346.

1854. Richter. (Died, 1854) (Germany?). 500 places. This value was published in 1855. [1], 346; [12], 77; [20], 188; [23], 311; [24], 398.

1855. Athenaeum (publication, Oxford University Press). 4. A correspondent makes the square equal to a circle by making each side equal to a quarter of the circumference. [18], 12.

1860. James Smith (Liverpool, England). 3 1/8 (3.125). Published several books and pamphlets arguing for the accuracy of his value, and even attempted to bring it before the British Association for the Advancement of Science. Professors De Morgan and Whewell, and even the famous mathematician, Sir William Rowan Hamilton, tried unsuccessfully to convince him of his error. In a letter to Smith, Professor Whewell gave the following demonstration: "You may do this. Calculate the side of a polygon of 24 sides inscribed in a circle. You will find that if the radius of the circle be one, the side of the polygon is .264. Now the arc which this side subtends is, according to your proposition, 3.125/12 = .2604, and therefore, the chord is greater than its arc, which, you will allow, is impossible". [7], 46; [18], 28.

1862. Lawrence Sluter Benson (U.S.A.). 3.141592 ⋯. Endeavored to demonstrate that the area of the circle is equal to $3R^2$, or the arithmetical square between the inscribed and circumscribed squares. His theorem is: "The $\sqrt{12} = 3.4641016+$ is the ratio between the diameter of a circle and the perimeter of its equivalent square". It was his belief that the ratio between the diameter and circumference is not a function of the area of the circle, but that the area of the circle is $\triangleq 3R^2$, or .75. He accepted the value of $\pi = 3.141592+$. He published some 20 pamphlets on the area of the circle, three volumes on philosophic essays, and one geometry. [9], 6.

1863. S. M. Drach (England). 3.14159265000 ⋯ Suggested an approximation for finding the circumference of a circle. Phil. Mag., Jan., 1863. The solution is extremely accurate but awkward to construct. From three diameters, deduct 8/1000 and 7/1,000,000 of a diameter, and add .5 per cent to the result. This gives a length which is smaller than the circumference by about 1 1/60 inch in 14,000 miles. The error is −.00000000358 ⋯. [9], 10; [18], 24.

1868. Cyrus Pitt Grosvenor, Rev. (New York). 3.142135 ⋯ by geometrical construction. Published in a pamphlet, *The Circle Squared*, New York, 1868. The following rule is given for the area: Square the diameter of the circle; multiply the square by two; extract the square root of the product; from the root subtract the diameter of the circle; square the remainder; multiply this square by five-fourths; subtract the product from the square of the diameter of the circle. In algebraic terms the rule is:

$$\text{Area} = D^2 - \frac{5}{4}(\sqrt{2D^2} - D)^2 = D^2 - \frac{5}{4}D^2(\sqrt{2} - 1)^2$$
$$= D^2[1 - \frac{5}{4}(\sqrt{2} - 1)^2] = D^2(0.7855339706472 \cdots) = \frac{\pi D}{4}.$$

The error for the value of π from this construction is +.000543. [9], 14.

This article will be concluded in the May-June issue.

THE CHRONOLOGY OF PI

Herman C. Schepler

(Continued from the March-April issue)

1872. Augustus De Morgan (1806-1871), born in Madras, was a severe critic of the would-be circle squarer. Educated at Trinity College, Cambridge, and became professor at the University of London in 1828; celebrated teacher who also contributed to algebra and the theory of probability. Most of his references to circle squaring appear in his *Budget of Paradoxes* which was edited and published by his wife in 1872, after his death. Among De Morgan's many antics regarding the circle-squarer, he ordained St. Vitus as the patron saint of the circle-squarer and suggested that Mr. James Smith (see 1860, James Smith) was seized with the *morbus cyclometricus,* defined as the *circle-squaring disease.* He tells of many of his experiences with cyclometers, such as the Jesuit who came from South America in about 1844, bringing a quadrature and a newspaper clipping announcing that a reward was ready for the discovery in England. [5], 330; [7]; [8], 305.

1873. William Shanks, (1812-1882) (England). 707 places, using Machin's formula. English mathematician. (See 1853, William Shanks). The value of π to 707 places is:

3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944
59230 78164 06286 20899 86280 34825 34211 70679 82148 08651 32823 06647
09384 46095 50582 23172 53594 08128 48111 74502 84102 70193 85211 05559
64462 29489 54930 38196 44288 10975 66593 34461 28475 64823 37867 83165
27120 19091 45648 56692 34603 48610 45432 66482 13393 60726 02491 41273
72458 70066 06315 58817 48815 20920 96282 92540 91715 36436 78925 90360
01133 05305 48820 46652 13841 46951 94151 16094 33057 27036 57595 91953
09218 61173 81932 61179 31051 18548 07446 23798 34749 56735 18857 52724
89122 79381 83011 94912 98336 73362 44193 66430 86021 39501 60924 48077
23094 36285 53096 62027 55693 97986 95022 24749 96206 07497 03041 23668
86199 51100 89202 38377 02131 41694 11902 98858 25446 81639 79990 46597
00081 70029 63123 77381 34208 41307 91451 18398 05709 85 &c.

[5], 206; [11], 261; [12], 77; [18], 16; [20], 188.

1876. Alick Carrick (England). 3 1/7 (3.1428571 ···). Proposed in his book, *The Secret of the Circle, Its Area Ascertained.* Nature 15: 1876.

1879. Pliny E. Chase, LLD (Haverford, Penna.) 3.14158499 ··· by geometrical construction. Pamphlet, *Approximate Quadrature of the*

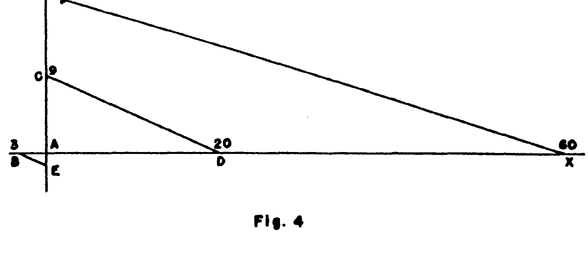


Fig. 4

Circle, Haverford, Penn., June 16, 1879. On the rectangular coordinates X, Y, lay off from a scale of equal parts, $AB = 3$; $AC = 9$; $AD = 20$; and $AX = 60$. Join C and D and through B draw BE parallel to CD intersecting the Y-axis at E. Take $EY = AD$, and join X and Y. Then $XY:AD::$Circum:Diam., nearly. The error is $-.00000766 \cdots$. [9], 9.

1882. Ferdinand Lindemann (1852-1939) (Germany) proved π to be transcendental, i.e., cannot be represented as the root of any algebraic equation with rational coefficients. He also proved the quadrature impossible. [5], 446; [19], 41.

1888. Sylvester Clark Gould (1840-1909). U.S.A. Editor of *Notes and Queries*, Manchester, New Hampshire. Due to popular demand for information on the subject of the quadrature, Gould was provoked to compile a bibliography on the subject which he titled, *What is The Value of Pi*, and published in 1888. 100 titles are presented which give the results of 63 writers. With the exception of two items the titles are all dated in the 19th century. [9].

1892. New York Tribune. (U.S.A.). 3.2 A writer announced this ratio as the rediscovery of a long lost secret which consisted in the knowledge of a certain "Nicomedean line". This announcement caused considerable discussion, and even near the beginning of the Twentieth Century 3.2 had its advocates as against the accepted ratio $3.14159 \cdots$. [18], 29.

1906. A. C. Orr. (U.S.A.). Literary Digest, Vol. 32. A mnemonic sentence is given for π. The number of letters in the words give the digits in the number π to 30 places.

"Now I, even I, would celebrate
In rhymes inapt, the great
Immortal Syracusan, rivaled nevermore,
Who in his wondrous lore,
Passed on before
Left men his guidance
How to circles mensurate".

A sentence giving π to 31 places appears in [14], 67; a French verse to 30 places may be found in [16], 89. [6], 19; [15], 50.

1913. E. W. Hobson (England). $3.14164079 \cdots$. By a geometrical construction. Given in reference [10], by Hobson. Probably developed by another author at an earlier date.

Let r be the radius of a given circle whose diameter is AOB. Let $OD = (3/5)r$; $OF = (3/2)r$; $OE = (1/2)r$. With DE and AF as diameters, describe semicircles DGE and AHF. Let the perpendicular to AB through O cut these semicircles in G and H respectively. GH is the side of a square whose area is approximately equal to that of the given circle. The error is $+.0000481 \cdots$. [10], 35.

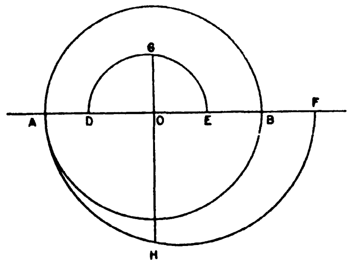


Fig. 5

1913. Srinivasa Ramanujan (Ramanujacharya) (1887-1920) (India) 3.1415926525826 $\cdots$. A mathematician who contributed several notweorthy approximations for evaluating π, both in empirical formulae and geometrical constructions. See *Approximate Geometrical Constructions for Pi*, Quarterly Journal of Math., *XLV*, 1914, pp. 350-374. See also, *Collected Papers of Srinivasa Ramanujacharya*, edited by G. H. Hardy, P. V. Seshu Aiyer and B. M. Wilson, published by University Press, Cambridge, England, 1927. A curious approximation to π obtained empiracally is:

$$\left[9^2 + \frac{(19)^2}{22}\right]^{\frac{1}{4}} = 3.1415926525826 \cdots.$$

The following construction is given for this value: Let *AB*, Figure 6, be a diameter of a circle whose center is *O*. Bisect the arc *ACB* at *C* and trisect *AO* at *T*. Draw *BC* and on it lay off *CM* and *MN* equal to *AT*. Draw *AM* and *AN* and on *AN* from *A*, lay off *AP* equal to *AM*. Through *P* draw *PQ* parallel to *MN* and meeting *AM* at *Q*. Draw *OQ* and through *T* draw *TR* parallel to *OQ*, and meeting *AQ* at *R*. Draw *AS* perpendicular to *AO* and equal to *AR*, and draw *OS*. Then the mean proportional between *OS* and *OB* will be very nearly equal to a sixth of the circumference, the error being less than 1/12 inch when the diameter is 8000 miles. The error is -.0000000010072 $\cdots$.

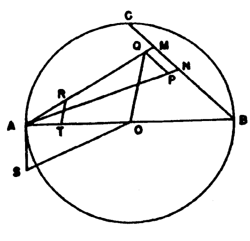


Fig. 6

1914. T. M. P. Hughes, (England) 3.14159292035 ⋯ by geometrical construction. *The Diameter of a Circle Equal in Area to any Given Square,* Nature, *93*:1914, page 110. This is a construction for the ratio 355/113. Make a circle with diameter = 11.3. Let AX = 8 7/8. Draw a perpendicular from X to cut the circle in Y. Join AY, and extend the line to Z. The construction is derived from the approximation: $AX{:}AB = \pi/4 = 355/4{\cdot}113 = 710/8{\cdot}113 = (8\ 7/8)/11.3$. Any circle with its diameter upon AB and one extremity of that diameter at A, will cut the line AZ (or AZ produced) in a point Y', making AY' the side of a square equal in area to the circle. Also, a line from Y' perpendicular to AB will cut the diameter in a point X', making AX' equal to 1/4 the circumference of the circle. Again, any square with its base upon AB and a corner at A, will cut AZ with side $X'W$ in a point Y', making AY' the diameter of an equal circle. The error of this quadrature is −.000000266 ⋯.

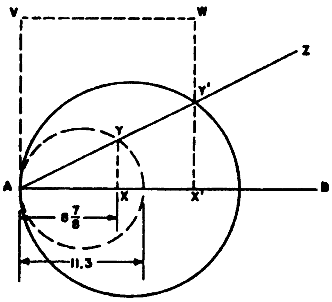


Fig. 7

1914. Scientific American, Mar. 21, 1914. (U.S.A.). The following mnemonic sentence is given for pi: "See I have a rhyme assisting My feeble brain its tasks sometimes resisting". The number of letters in the words give the digits in the number π to twelve places.

1928. Gottfried Lenzer. U.S.A. 3.1378 ⋯, an approximation obtained geometrically. In March 1928, Lenzer bequethed the University of Minnesota with 60 drawings dating from 1911 to 1927, regarding the three classical problems. His geometrical construction for squaring the circle gave π = 3.1378 ⋯. *Some Constructions for the Classical Problems of Geometry* Amer. Math. Mo., *37*, Aug.-Sept., 1930. p. 343.

1933. Helen A. Merrill (U.S.A.) 3.141591953 ⋯ by geometrical construction. Given in reference [16], by Merrill. Probably developed by another author at an earlier date. In Figure 8, let AB, the diameter of a circle, be 1. Construct radius OE perpendicular to it. Tangents drawn at A and E intersect at F. On AB produced, lay off BC = 1/10, and BD = 2/10. At D draw DH perpendicular to AD and meeting FE extended at *H*. Join *A* and *H*. From *F* on *FA* extended, lay off *FG* equal to *FC*. Through *G* draw a line parallel to *AH*, meeting *FH* extended in *P*. Solving from the relationships set up in the construction, $GP = 3.141591953\cdots$. The error is $-.000000700\cdots$. [16], 86.

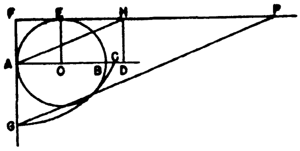


Fig. 8

1934. Carl Theadore Heisel (U.S.A.). 3 13/81 (3.1604938 ···). Proposed in his book, *Mathematical and Geometrical Demonstrations*. This book is 1¼" thick by 6½" x 10". It was printed at the author's expense and distributed to libraries without charge.

1941. Miff Butler. U.S.A. *Geomath*, General Engineering Co., Casper, Wyoming, 1941. Claimed discovery of a new relationship between π and e. Stated his work to be the first basic mathematical principle ever developed in the U.S.A. He got his congressman to read it into the Congressional Record on June 5, 1940.

1949. U.S. Army (U.S.A.). 2,035 places. Yielding to an irresistible temptation, some mathematical machine operators presented the problem of evaluating π to Eniac, the all-electronic calculator at the Army's Ballistic Research Laboratories in Aberdeen, Maryland. The machine's 18,800 electron tubes went into action and computed π to 2,035 places in about 70 hours. In 1873, William Shanks gave the value of π to 707 decimal places (527 correct). The computation took him more than 15 years. Scientific American, Dec., 1949, p. 30 and Feb., 1950, p. 2.

Stanford Research Institute,
Stanford, California.

MATHEMATICS

ON THE APPROXIMATION OF π

BY

K. MAHLER

(Communicated by Prof. J. F. KOKSMA at the meeting of November 29, 1952)

The aim of this paper is to determine an explicit lower bound free of unknown constants for the distance of π from a given rational or algebraic number.

1. In my paper *"On the approximation of logarithms of algebraic numbers"*, which is to appear in the Transactions of the Royal Society, the following result was proved:

Lemma: Let x be a real or complex number different from 0 *and* 1; *let* log x *denote the principal value of the natural logarithm of x; and let m and n be two positive integers such that*

(1) $$m + 1 \geqslant 2\,|\log x|.$$

There exist $(m+1)^2$ *polynomials*

$$A_{hk}(x) \qquad (h, k = 0, 1, \ldots, m)$$

in x with rational integral coefficients, of degrees not greater than n, and with the following further properties:

(a) *The determinant*

$$D(x) = \| A_{hk}(x) \|$$

does not vanish.

(b) $A_{hk}(x) << m!\, 2^{m-(3n/2)} (n+1)^{2m+1} (\sqrt{32})^{(m+1)n} (1 + x + \ldots + x^n).$

(c) *The* $m+1$ *functions*

$$R_h(x) = \sum_{k=0}^{m} A_{hk}(x) (\log x)^k \qquad (h = 0, 1, \ldots, m)$$

satisfy the inequalities

$$|R_h(x)| \leqslant m!\, 2^{-(3n/2)} (e\sqrt{n})^{m+1} e^{(2n+1)\,|\log x|} \left(\frac{\sqrt{8}\,|\log x|}{m+1}\right)^{(m+1)n}.$$

Denote by y a further real or complex number, and put

$$S_h(x,y) = \sum_{k=0}^{m} A_{hk}(x)\, y^k, \quad T_h(x,y) = \sum_{k=1}^{m} A_{hk}(x) \frac{(\log x)^k - y^k}{\log x - y} \quad (h = 0, 1, \ldots, m),$$

so that

(2) $$R_h(x) - S_h(x,y) = T_h(x,y)(\log z - y),$$

identically in x and y. This identity will enable us to find a measure of irrationality for π.

2. For this purpose, substitute in the last formulae the values

$$x = i, \ \log x = \pi \frac{i}{2}, \ y = \frac{p}{q}\frac{i}{2}$$

for x, $\log x$, and y; here p and q may be any two positive integers for which

(3) $$p < 4q.$$

Then

$$|\log x| < 2, \ |y| < 2,$$

so that

$$\left|\frac{(\log x)^k - y^k}{\log x - y}\right| = |(\log x)^{k-1} + (\log x)^{k-2}\, y + \ldots + (\log x)\, y^{k-2} + y^{k-1}| < 2^{k-1}\, k$$

and

$$\sum_{k=1}^{m} \left|\frac{(\log x)^k - y^k}{\log x - y}\right| < \sum_{k=1}^{m} 2^{k-1}\, k \leqslant \sum_{k=1}^{m} 2^{k-1}\, m < 2^m\, m.$$

Hence

(4) $$|T_h(x,y)| < 2^m\, m \cdot \max_{h,k=0,1,\ldots,m} |A_{hk}(x)|.$$

3. From now on assume that

$$m = 10 \text{ and } n \geqslant 50.$$

This choice of m satisfies the condition (1) of the lemma. The lemma may then be applied, and we find, first, that

$$\max_{h,k=0,1,\ldots,m} |A_{hk}(x)| \leqslant 10!\, 2^{10-(3n/2)}\, (n+1)^{21}\, 2^{(55/2)n}\, (1 + |x| + \ldots + |x|^n) = $$
$$= 10!\, 2^{10}\, (n+1)^{22}\, 2^{26n},$$

whence, by (4),

(5) $$|T_h(x,y)| < 10.10!\, 2^{20}\, (n+1)^{22}\, 2^{26n}.$$

Secondly,

(6) $$|R_h(x)| \leqslant 10!\, 2^{-(3n/2)}\, e^{11}\, n^{11/2}\, e^{n\pi + (\pi/2)} \left(\frac{\sqrt{2}\,\pi}{11}\right)^{11n} = 10!\, e^{11+(\pi/2)}\, n^{11/2} \left(\frac{16\pi^{11}\, e^{\pi}}{11^{11}}\right)^{n}.$$

Thirdly, $D(x) \neq 0$. Hence the index h, $= h_0$ say, can be chosen such that $S_{h_0}(x, y) \neq 0$. Now $(2q)^m\, S_{h_0}(x, y)$ evidently is an integer in the Gaussian field $K(i)$. Its absolute value is therefore not less than unity, whence, by the choice of m,

(7) $$|S_{h_0}(x,y)| \geqslant 2^{-10}\, q^{-10}.$$

4. Assume now that $n \geqslant 50$ can be selected so as to satisfy the inequality

(8) $$10!\, e^{11+(\pi/2)}\, n^{11/2} \left(\frac{16\pi^{11}\, e^{\pi}}{11^{11}}\right)^{n} \leqslant \tfrac{1}{2}\, 2^{-10}\, q^{-10}.$$

By (6) and (7), this inequality implies that

$$|R_{h_0}(x)| \leqslant \tfrac{1}{2} |S_{h_0}(x,y)|,$$

and so, by (2),

$$\tfrac{1}{2}\,|S_{h_0}(x,y)| \leqslant |T_{h_0}(x,y)\,(\log x - y)|.$$

It follows then from (5) and (7) that

$$(9)\qquad \left|\pi - \frac{p}{q}\right| = 2\,|\log x - y| \geqslant \left|\frac{S_{h_0}(x,y)}{T_{h_0}(x,y)}\right| \geqslant$$
$$\geqslant 2^{-10} q^{-10} \{10.10!\, 2^{20} (n+1)^{22}\, 2^{26n}\}^{-1}.$$

The two inequalities (8) and (9) are equivalent to

$$(10)\qquad \left(\frac{11^{11}}{16\pi^{11} e^{\pi}}\right)^{n} \geqslant 2^{11}\, 10!\, e^{11+(\pi/2)}\, n^{11/2}\, q^{10},$$

and

$$(11)\qquad \left|\pi - \frac{p}{q}\right| \geqslant \{10.10!\, 2^{30}\, (n+1)^{22}\, 2^{26n}\}^{-1} q^{-10},$$

respectively. Here

$$\frac{11^{11}}{16\pi^{11} e^{\pi}} > 10^{3.4181}, \qquad 2^{26} < 10^{7.8268},$$

and also, on account of $n \geqslant 50$,

$$2^{11}\, 10!\, e^{11+(\pi/2)} < 10^{15.3306} < 10^{0.3067n}, \quad 10.10!\, 2^{30} < 10^{16.5907} < 10^{0.3319n}.$$

Further, on denoting by Log N the decadic logarithm of N,

$$n^{11/2} = 10^{11/2\,(\mathrm{Log}\, n/n)n} \leqslant 10^{11/2\,(\mathrm{Log}\, 50/50)n} < 10^{0.1869n}$$

and

$$(n+1)^{22} = 10^{22\,(\mathrm{Log}\,(n+1)/n)n} \leqslant 10^{22\,(\mathrm{Log}\, 51/50)n} < 10^{0.7514n}.$$

These numerical formulae show that the inequality (10) certainly holds if

$$10^{3.4181n} > 10^{0.3067n+0.1869n}\, q^{10},$$

i.e., if

$$10^{2.9245n} > q^{10},$$

and they further give

$$10.10!\, 2^{30}\, (n+1)^{22}\, 2^{26n} < 10^{0.3319n+0.7514n+7.8268n} = 10^{8.9101n}.$$

We thus have proved the following result:

"*Let p and q be two positive integers such that $p < 4q$, and let n be an integer for which*

$$(12)\qquad n \geqslant 50, \qquad 10^{2.9245n} > q^{10}.$$

Then

$$(13)\qquad \left|\pi - \frac{p}{q}\right| > 10^{-8.9101n}\, q^{-10}."$$

5. This result be further simplified. Define n as function of q by the inequalities

$$10^{2.9245(n-1)} \leqslant q^{10} < 10^{2.9245n}.$$

This choice of n is permissible provided q is so large that

$$q^{10} \geqslant 10^{2.9245\times 49} = 10^{143.3005}.$$

It suffices then to make the further assumption that

$$q \geqslant 2.14 \times 10^{14}, \tag{14}$$

because then

$$q^{10} > 10^{143.304}.$$

Since $n \geqslant 50$ and therefore $n - 1 \geqslant \frac{49}{50}n$, we have now

$$q^{10} \geqslant 10^{2.9245\times 0.98n} > 10^{2.8661n},$$

hence, by (13),

$$\left|\pi - \frac{p}{q}\right| > q^{-(8.9101/2.8661)\times 10-10} > q^{-41.09} > q^{-42}. \tag{15}$$

The proof assumed, as we saw, that $p < 4q$ and that (14) is satisfied. If (14) holds, but $p \geqslant 4q$, then trivially

$$\left|\pi - \frac{p}{q}\right| \geqslant 4 - \pi > q^{-42},$$

and (15) remains true.

6. It is now of greater interest that the remaining condition (14) can be replaced by a more natural one.

Theorem 1: *If p and $q \geqslant 2$ are positive integers, then*

$$\left|\pi - \frac{p}{q}\right| > q^{-42}.$$

Proof: By what has already been shown, it suffices to verify that there are no pairs of positive integers p, q for which

$$2 \leqslant q < 2.14 \times 10^{14}, \qquad \left|\pi - \frac{p}{q}\right| \leqslant q^{-42}.$$

If such pairs of integers exist, they necessarily have the additional property that

$$\left|\pi - \frac{p}{q}\right| < \frac{1}{2q^2},$$

because otherwise

$$\frac{1}{2q^2} \leqslant \left|\pi - \frac{p}{q}\right| \leqslant q^{-42}, \qquad q^{40} \leqslant 2, \qquad q < 2,$$

which is false. It follows then, by the theory of continued fractions, that p/q must be one of the convergents p_n/q_n of the continued fraction

$$\pi = b_0 + \frac{1|}{|b_1} + \frac{1|}{|b_2} + \ldots = [b_0; b_1, b_2, \ldots]$$

for π; here the incomplete denominators $b_0, b_1, b_2, \ldots$ are positive integers. According to J. Wallis, the development begins as follows:

$$\pi = [3;\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 14,\ 2,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 1,\ 84,\ 2,\ 1,\ 1,\ 15,\ 3,\ 13,\ 1,\ 4,\ 2,\ 6,\ 6,\ 1, \ldots].$$

A trivial computation shows that the convergent belonging to the incomplete denominator 13 is already greater than 2.14×10^{14}. The largest of the preceding incomplete denominators is 292. Hence, by the theory of continued fractions, we find that

$$\left|\pi - \frac{p_n}{q_n}\right| > \frac{1}{q_n(q_{n+1}+q_n)} =$$

$$= \frac{1}{q_n\{(b_{n+1}+1)q_n+q_{n-1}\}} > \frac{1}{(b_{n+1}+2)q_n^2} \geqslant \frac{1}{294q_n^2} > q_n^{-42}$$

for every convergent the denominator of which lies in the range we are considering. There are therefore no pairs of integers p, q of the required kind. This completes the proof.

The theorem required that $q \geqslant 2$. If one is satisfied with an estimate for $|\pi-(p/q)|$ valid when q is greater than *some* large value q_0, then the exponent 42 can be replaced by 30. No new ideas being involved, the proof may be omitted.

7. As a second application of the lemma in § 1 we study now the approximation of π by arbitrary algebraic numbers.

Let ω be a real or complex algebraic number of degree ν over the Gaussian field $K(i)$, and let

$$f(z) = 0, \text{ where } f(z) = a_0z^\nu + a_1z^{\nu-1} + \ldots + a_\nu$$

and where further the coefficients $a_0 \neq 0, a_1, \ldots, a_\nu$ are integers in $K(i)$, be an irreducible equation for ω over this field. Denote by

$$a = \max(|a_0|, |a_1|, \ldots, |a_\nu|)$$

the height of this equation and by

$$\omega_0 = \omega, \omega_1, \ldots, \omega_{\nu-1}$$

its roots. These roots are all different, and it is well known that

$$(16) \qquad |\omega_j| \leqslant a + 1 \qquad (j = 0, 1, \ldots, \nu - 1).$$

8. In the case when ω is a real algebraic number, the defining equation $f(z) = 0$ may be assumed to have *rational* integral coefficients. For let

$$F(z) = 0, \quad \text{where} \quad F(z) = A_0z^N + A_1z^{N-1} + \ldots + A_N,$$

and where $A_0 \neq 0, A_1, \ldots, A_N$ are rational integers, be an equation for ω irreducible over the rational field. It suffices to show that this equation is also irreducible over $K(i)$, hence that $F(z)$ differs from $f(z)$ only by a constant factor different from zero.

Let the assertion be false. Then $F(z)$ can be written as

$$F(z) = \{A(z) + iB(z)\}\,\{C(z) + iD(z)\}$$

where $A(z)$, $B(z)$; $C(z)$, and $D(z)$ are polynomials with rational coefficients such that neither $A(z) + iB(z)$ nor $C(z) + iD(z)$ is a constant. Since $F(z)$ is a real polynomial, also

$$F(z) = \{A(z) - iB(z)\}\,\{C(z) - iD(z)\}$$

and therefore, on multiplying the two equations,

$$F(z)^2 = \{A(z)^2 + B(z)^2\}\,\{C(z)^2 + D(z)^2\}.$$

Since unique factorization holds for polynomials in one variable over the rational field, this formula implies that

$$F(z) = c\,\{A(z)^2 + B(z)^2\}$$

where $c \neq 0$ is a rational constant.

Put now $z = \omega$. Then $F(z)$ and therefore $A(z)^2 + B(z)^2$ vanish, hence also both $A(z)$ and $B(z)$. This means that $A(z)$ and $B(z)$ are divisible by $z - \omega$, thus $F(z)$ by $(z - \omega)^2$. This is impossible because $F(z)$ is irreducible, so that it cannot have multiple linear factors.

9. Substitute now

$$x = i, \qquad \log x = \pi \frac{i}{2}, \qquad y = \omega \frac{i}{2}$$

for x, $\log x$, and y in the identity

$$R_h(z) - S_h(x, y) = T_h(x, y)\,(\log x - y) \tag{2}$$

of § 1, and assume further that

$$|\omega| < 4, \ m \geqslant 3.$$

One proves just as in § 2 and § 3 that

$$|R_h(x)| \leqslant m!\, 2^{-(3n/2)}\, (e\sqrt{n})^{m+1}\, e^{n\pi + (\pi/2)} \left(\frac{\sqrt{2}\,\pi}{m+1}\right)^{(m+1)n} \tag{17}$$

and

$$|T_h(x,y)| < 2^m\, m \cdot m!\, 2^{m-(3n/2)}\, (n+1)^{2m+2}\, (\sqrt{32})^{(m+1)n} \tag{18}$$

On the other hand, the then given lower bound for $S_{h_0}(x, y)$ is no longer valid and must be replaced by a more involved expression.

10. Since the determinent $D(x)$ does not vanish, there is again an index $h = h_0$ such that

$$S_{h_0}(x,y) = S_{h_0}\left(i, \omega\,\frac{i}{2}\right) \neq 0.$$

This means that also the $\nu - 1$ numbers

$$S_{h_0}\left(i, \omega_1 \frac{i}{2}\right), \quad S_{h_0}\left(i, \omega_2 \frac{i}{2}\right), \ldots, \quad S_{h_0}\left(i, \omega_{\nu-1} \frac{i}{2}\right)$$

obtained from $S_{h_0}(i, \omega i/2)$ on replacing ω by its conjugates $\omega_1, \omega_2, \ldots, \omega_{\nu-1}$ with respect to $K(i)$ do not vanish. For let z be a variable. The expression $S_{h_0}(i, zi/2)$ is a polynomial in z with coefficients in $K(i)$ which does not vanish at $z = \omega$. Therefore the polynomial cannot be divisible by the irreducible polynomial $f(z)$ of which ω is a root, and so it admits none of its other roots ω_j.

It follows then that the product

$$\sigma = \prod_{j=0}^{\nu-1} S_{h_0}\left(i,\, \omega_j \frac{i}{2}\right)$$

does not vanish. This product is a symmetric polynomial in $\omega, \omega_1, \ldots, \omega_{\nu-1}$ which is in each ω_j of degree m; moreover, the coefficients of this polynomial are elements of $K(i)$, and their common denominator is a divisor of $2^{m\nu}$. Therefore σ itself lies in the Gaussian field, and its denominator is in absolute value not greater than

$$2^{m\nu}\, |a_0|^m \leqslant 2^{m\nu}\, a^m.$$

Since σ is not zero, the inequality

$$2^{m\nu}\, a^m\, |\sigma| \geqslant 1,$$

holds, and we find that

$$|S_{h_0}(x, y)| \geqslant \left\{2^{m\nu}\, a^m \prod_{j=1}^{\nu-1} \left|S_{h_0}\left(i,\, \omega_j \frac{i}{2}\right)\right|\right\}^{-1}. \tag{19}$$

11. By definition,

$$S_{h_0}\left(i, \omega_j \frac{i}{2}\right) = \sum_{k=0}^{m} A_{h_0 k}(i) \left(\omega_j \frac{i}{2}\right)^k.$$

Here, by (16),

$$|\omega_j| \leqslant a + 1,$$

so that

$$\sum_{k=0}^{m} \left|\omega_j \frac{i}{2}\right|^k \leqslant \sum_{k=0}^{m} \left(\frac{a+1}{2}\right)^k \leqslant (m+1) \left(\frac{a+1}{2}\right)^m \leqslant (m+1)\, a^m$$

since $a \geqslant 1$. Therefore

$$\left|S_{h_0}\left(i, \omega_j \frac{i}{2}\right)\right| \leqslant (m+1)\, a^m \max_{h,k=0,1,\ldots,m} |A_{hk}(i)|,$$

whence, by the lemma in 1.),

$$\left|S_{h_0}\left(i, \omega_j \frac{i}{2}\right)\right| \leqslant a^m\, (m+1)!\; 2^{m-(3n/2)}\, (n+1)^{2m+2} (\sqrt{32})^{(m+1)n}.$$

Therefore, from (19),

$$|S_{h_0}(x, y)| \geqslant \{2^{m\nu}\, a^m\, (a^m\, (m+1)!\, 2^{m-(3n/2)}\, (n+1)^{2m+2} (\sqrt{32})^{(m+1)n})^{\nu-1}\}^{-1}. \tag{20}$$

12. From now on we proceed in a similar way as in 4.). Let again $m \geqslant 3$ and n be chosen such that

$$|R_{h_0}(x)| \leqslant \tfrac{1}{2} |S_{h_0}(x, y)|; \tag{a}$$

then from the identity (2),

$$|S_{h_0}(x,y)| \leqslant 2|T_{h_0}(x,y)\, (\log x - y)|, \tag{b}$$

so that a lower bound for

$$2|\log x - y| = |\pi - \omega|$$

is obtained.

By (17) and (20), the condition (a) is certainly satisfied if

$$m!\,2^{-(3n/2)}\,(e\sqrt{n})^{m+1}\,e^{n\pi+(\pi/2)}\left(\frac{\sqrt{2}\,\pi}{m+1}\right)^{(m+1)n}\leqslant$$
$$\leqslant \tfrac{1}{2}\{2^{m\nu}\,a^m\,(a^m(m+1)!\,2^{m-(3n/2)}\,(n+1)^{2m+2}\,(\sqrt{32})^{(m+1)n})^{\nu-1}\}^{-1},$$

or, what is the same, if

$$(21)\quad \left(\frac{4\,(m+1)}{2^{5\nu/2}\,\pi}\right)^{(m+1)m}\geqslant$$
$$\geqslant \frac{(m+1)!^{\nu}}{m+1}\,2^{(2\nu-1)m-(3n\nu/2)+1}\,e^{m+n\pi+(\pi/2)+1}\,(\sqrt{n}\,(n+1)^{2(\nu-1)})^{m+1}\,a^{m\nu}.$$

Under this hypothesis, we find from (b), by (18) and (20), that

$$|\pi-\omega|>\{2^{m\nu}\,a^m\,(a^m\,(m+1)!\,2^{m-(3n/2)}\,(n+1)^{2m+2}\,(\sqrt{32})^{(m+1)n})^{\nu-1}\}^{-1}\times$$
$$\times\{2^m\,m\cdot m!\,2^{m-(3n/2)}\,(n+1)^{2m+2}\,(\sqrt{32})^{(m+1)n}\}^{-1}.$$

whence, after some trivial simplification,

$$(22)\quad |\pi-\omega|>\left\{\frac{m}{m+1}(m+1)!^{\nu}\,2^{(2\nu+1)m-(3n\nu/2)}\,(n+1)^{2(m+1)\nu}\,(\sqrt{32})^{(m+1)n\nu}\,a^{m\nu}\right\}^{-1}.$$

In order to put (21) and (22) into a more convenient form, we now apply the well-known inequality

$$(m+1)!\leqslant e\sqrt{m+1}\,(m+1)^{m+1}\,e^{-(m+1)}.$$

It follows that (21) is satisfied if

$$\left(\frac{4(m+1)}{2^{5\nu/2}\,\pi}\right)^{(m+1)n}\geqslant e^{\nu}\,(m+1)^{(\nu/2)-1}(m+1)^{(m+1)\nu}e^{-(m+1)\nu}\,2^{2(m+1)\nu-2\nu-(m+1)-(3n\nu/2)+2}\times$$
$$\times e^{(m+1)+n\pi+(\pi/2)}\left(\frac{\sqrt{n}}{(n+1)^2}\,(n+1)^{2\nu}\right)^{m+1}a^{m\nu},$$

and so even more if

$$(23)\quad \begin{cases}\left(\dfrac{4(m+1)}{2^{5\nu/2}\,\pi}\right)^{(m+1)n}\geqslant\dfrac{4e^{\pi/2}(m+1)^{(\nu/2)-1}(e/4)^{\nu}}{(n+1)^{(m+1)/2}}\cdot\dfrac{(e/2)^{m+1}\,(4/e)^{(m+1)\nu}}{(n+1)^{m+1}}\cdot(e^{\pi}\cdot 2^{-(3\nu)/2})^n\times\\ \qquad\times\,(m+1)^{(m+1)\nu}\,(n+1)^{2(m+1)\nu}\,a^{m\nu}.\end{cases}$$

Therefore, assuming that (23) holds, by (22)

$$|\pi-\omega|^{-1}<\frac{m}{m+1}\,e^{\nu}\,(m+1)^{\nu/2}\,(m+1)^{(m+1)\nu}\,e^{-(m+1)\nu}\,2^{2\nu(m+1)-2\nu+(m+1)-(3n\nu/2)-1}\times$$
$$\times\,(n+1)^{2(m+1)\nu}\,(\sqrt{32})^{(m+1)n\nu}\,a^{m\nu},$$

whence

$$(24)\quad \begin{cases}|\pi-\omega|^{-1}<\left(\dfrac{e}{4}\right)^{\nu}(m+1)^{\nu/2}\left(\dfrac{4}{e}\right)^{(m+1)\nu}2^{m+1}\,2^{-\frac{3}{2}n\nu-1}\cdot(m+1)^{(m+1)\nu}(n+1)^{2(m+1)\nu}\times\\ \qquad\times\,(\sqrt{32})^{(m+1)n\nu}\,a^{m\nu}.\end{cases}$$

13. So far $m\geqslant 3$ and n are restricted solely by the condition (23). In order further to simplify (23) and (24), assume from now on that

$$(25)\qquad m+1\geqslant 20\cdot 2^{\frac{5}{2}(\nu-1)},\qquad n\geqslant(m+1)\log(m+1).$$

Since $\frac{5}{2}\log 2 > 1$, by the first of these conditions,

$$m + 1 \geqslant 20\, e^{\nu-1} \geqslant 20\,(1 + (\nu - 1)) = 20\nu > 3.$$

The second condition implies then that

$$n \geqslant 20\nu \log (20\nu).$$

Now 20 log 20 > 59, 20 log 40 > 73, and so

$$n \geqq 60\nu,$$

both when $\nu = 1$ and when $\nu \geqslant 2$.

As a first application of (25), we determine an upper estimate for the expression

$$A_0 = (m + 1)^{\nu/n}\,(n + 1)^{2\nu/n}.$$

Since $n \geqslant 60\nu \geqq 60$,

$$n + 1 \leqslant \frac{61}{60}\,n,\ \left(\frac{61}{60}\right)^{2\nu/n} \leqslant \left(\frac{61}{60}\right)^{1/30},\ A_0 \leqslant \left(\frac{61}{60}\right)^{1/30} (m + 1)^{\nu/n}\, n^{2\nu/n},\ = B_0 \text{ say}.$$

Next

$$\frac{\partial \log B_0}{\partial n} = -\frac{\nu}{n^2} \log (m + 1) - \frac{2\nu}{n^2} (\log n - 1)$$

is negative because $\log n \geqslant \log 60 > 1$. Therefore B_0 is not decreased on replacing n by $(m + 1) \log (m + 1)$, and we find that

$$A_0 \leqslant \left(\frac{61}{60}\right)^{1/30} \exp\left\{\frac{\nu \log (m+1) + 2\nu\,(\log (m+1) + \log\log (m+1))}{(m+1) \log (m+1)}\right\}$$

or

$$A_0 \leqslant \left(\frac{61}{60}\right)^{1/30} \exp\left\{\frac{3\nu}{m+1} + \frac{2\nu}{m+1}\,\frac{\log\log (m+1)}{\log (m+1)}\right\}.$$

Here $\frac{\log\log (m+1)}{\log (m+1)}$ decreases with increasing m because $\log (m + 1) \geqslant$ $\geqslant \log 20 > e$; hence

$$\frac{\log\log (m+1)}{\log (m+1)} \leqslant \frac{\log\log 20}{\log 20} < \frac{1}{2},$$

whence finally,

$$A_0 \leqslant \left(\frac{61}{60}\right)^{1/30} \exp\left(\frac{3\nu+\nu}{20\nu}\right) = \left(\frac{61}{60}\right)^{1/30} e^{1/5} < \frac{5}{4}.$$

We next discuss certain factors that occur on the right-hand sides of (23) and (24).

In

$$A_1 = \frac{4\, e^{\pi/2}\,(m+1)^{(\nu/2)-1}\,(e/4)^{\nu}}{(n+1)^{(m+1)/2}},$$

evidently

$\log (m+1) > e$, $n+1 > (m+1)\ \log (m+1) > e(m+1)$, $m+1 \geqslant 20\nu$, $(e/4)^{\nu} < 1$,

whence

$$A_1 < \frac{4\, e^{\pi/2}\,(m+1)^{(\nu/2)-1} \cdot 1}{\{e(m+1)\}^{10\nu}} < 4\, e^{(\pi/2)-10}\,(m + 1)^{-9\nu} < 1.$$

Next let

$$A_2 = \frac{(e/2)^{m+1}\,(4/e)^{(m+1)\nu}}{(n+1)^{m+1}}.$$

Then by the last inequalities and by (25),

$$A_2 < \left\{\frac{(e/2)^1 (4/e)^\nu}{e(m+1)}\right\}^{m+1} \leqslant \left\{\frac{(e/2)^\nu (4/e)^\nu}{e\cdot 20\cdot 2^{5(\nu-1)/2}}\right\}^{(m+1)} = \left(\frac{2^{5/2}}{20\,e\cdot 2^{3\nu/2}}\right)^{m+1} < 1.$$

Let further

$$A_3 = (e^\pi \cdot 2^{-(3\nu/2)})^{1/(m+1)}.$$

Since $\nu \geqslant 1$ and $m + 1 \geqslant 20$,

$$A_3 \leqslant (e^\pi \cdot 2^{-(3/2)})^{1/20} < \tfrac{6}{5}.$$

Consider finally the expression

$$A_4 = \left(\frac{e}{4}\right)^\nu (m+1)^{\nu/2} \left(\frac{4}{e}\right)^{(m+1)\nu} 2^{(m+1)-(3n\nu/2)-1}.$$

Here

$$\nu \geqslant 1,\ \left(\frac{e}{4}\right)^\nu 2^{-1} < 1,\ m+1 < e^{m+1},\ n \geqslant (m+1)\log(m+1),$$

so that

$$A_4 < e^{(m+1)\nu/2}\left(\frac{4}{e}\right)^{(m+1)\nu} 2^{(m+1)\nu-(3/2)(m+1)\nu\log(m+1)} = \left(\frac{8\,e^{-1/2}}{(m+1)^{(3/2)\log 2}}\right)^{(m+1)\nu}.$$

Since now $\frac{3}{2}\log 2 > 1$ and $m + 1 \geqslant 20$, we find that

$$A_4 < \left(\frac{2\,e^{-1/2}}{5}\right)^{(m+1)\nu} < 1.$$

14. The inequalities for the A's lead easily to a great simplification of the result in 12.).

The right-hand side of (23) can be written as

$$A_1 A_2 A_3^{(m+1)n} A_0^{(m+1)n} a^{m\nu}$$

and so, by what has just been proved, is less than

$$1\cdot 1\cdot\left(\frac{6}{5}\right)^{(m+1)n}\left(\frac{5}{4}\right)^{(m+1)n} a^{(m+1)\nu} = \left(\frac{3}{2}\right)^{(m+1)n} a^{(m+1)\nu}.$$

Similarly the right-hand side of (24) has the value

$$A_4 A_0^{(m+1)n}\, 2^{(5/2)(m+1)n\nu} a^{m\nu}$$

and is therefore smaller than

$$\left(\frac{5}{4}\cdot 2^{(5/2)\nu}\right)^{(m+1)n} a^{(m+1)\nu}.$$

We have therefore the following result:

"*Let m and n satisfy the inequalities* (25) *and let further*

(26) $$\left(\frac{4(m+1)}{2^{5\nu/2}\pi}\right)^n \geqslant \left(\frac{3}{2}\right)^n a^\nu.$$

Then

(27) $$|\pi - \omega| > \left\{\left(\frac{5}{4}\cdot 2^{(5/2)\nu}\right)^n a^\nu\right\}^{-(m+1)}."$$

The proof assumed that $|\omega| < 4$, but we may now dispense with this condition. For if $|\omega| \geqslant 4$, then trivially,

$$|\pi - \omega| \geqslant 4 - \pi > \frac{1}{5} > \left\{\left(\frac{5}{4}\cdot 2^{(5/2)\nu}\right)^n a^\nu\right\}^{-(m+1)}.$$

15. The first inequality (25) is satisfied if

(28) $$m = [20 \cdot 2^{(5/2)(\nu-1)}],$$

for then

$$20 \times 2^{(5/2)(\nu-1)} < m + 1 \leqslant 20 \cdot 2^{(5/2)(\nu-1)} + 1.$$

This choice of m means that

$$\frac{2}{3} \times \frac{4(m+1)}{2^{5\nu/2}\pi} \geqslant \frac{2}{3} \times \frac{4 \times 20}{2^{5/2}\pi} = \frac{20\sqrt{2}}{3\pi} > e.$$

The condition (26) is therefore certainly fulfilled if

$$e^n \geqslant a^\nu, \text{ i.e., } n \geqslant \nu \log a.$$

Let then from now on n be defined by the formula,

(29) $$n = [\max((m+1)\log(m+1), \nu \log a)] + 1,$$

so that both inequalities (25) and (26) hold, hence also the inquality (27) for $|\pi - \omega|$.

It is now convenient to distinguish two cases.

If, firstly,

$$a < (m+1)^{(m+1)/\nu},$$

then

$$(m+1)\log(m+1) > \nu \log a,$$

and therefore, by (29),

$$n = [(m+1)\log(m+1)] + 1 \leqslant (m+1)\log(m+1) + 1.$$

Further

$$\frac{5}{2}\, 2^{(5/2)\nu} = \frac{1}{\sqrt{8}}\, 20 \cdot 2^{(5/2)(\nu-1)} < \frac{m+1}{\sqrt{8}} < \frac{m+1}{e},$$

whence

$$\left(\frac{5}{4}\, 2^{(5/2)\nu}\right)^n a^\nu < \left(\frac{m+1}{e}\right)^{(m+1)\log(m+1)+1} (m+1)^{m+1} = \frac{m+1}{e}\, e^{(m+1).\{\log(m+1)\}^2}.$$

Let, secondly,

$$a \geqslant (m+1)^{(m+1)/\nu},$$

so that

$$(m+1)\log(m+1) \leqslant \nu \log a.$$

Now

$$n = [\nu \log a] + 1 \leqslant \nu \log a + 1,$$

hence

$$\left(\frac{5}{4}\, 2^{(5/2)\nu}\right)^n a^\nu < \left(\frac{m+1}{e}\right)^{\nu \log a + 1} a^\nu = \frac{m+1}{e}\, a^{\nu \log(m+1)}.$$

The following result has therefore been obtained:

Theorem 2: *Let ω be a real or complex algebraic number. Denote by R the rational field K if ω is real, and the Gaussian imaginary field $K(i)$ if ω is non-real. Further denote by ν the degree of ω over R, by*

$$a_0 z^\nu + a_1 z^{\nu-1} + \ldots + a_\nu = 0 \qquad (a_0 \neq 0)$$

an equation for ω with integral coefficients in R which is irreducible over this field, and by

$$a = \max(|a_0|, |a_1|, \ldots, |a_\nu|)$$

the height of this equation. Put

$$m = [20 \cdot 2^{(5/2)(\nu-1)}], \qquad \bar{a} = \max(a, (m+1)^{(m+1)/\nu}).$$

Then

(30) $$|\pi - \omega| > \left(\frac{m+1}{e}\right)^{-(m+1)} \bar{a}^{-(m+1)\nu \log(m+1)}.$$

Remarks: 1) We note that the theorem remains true if $\bar{a}$ is replaced by any larger number.

2) When

$$a < (m+1)^{(m+1)/\nu},$$

the estimate (30) is not as good as that by N. I. Fel'dman (Izvestiya Akad. Nauk SSSR, ser. mat. 15, 1951, 53–74), viz.

$$|\pi - \omega| > \exp\{-\gamma_1 \nu (1 + \nu \log \nu + \log a) \log (2 + \nu \log \nu + \log a)\},$$

where γ_1, just as γ_2 in the next line, is a positive absolute constant. Fel'dman's inequality implies that

$$\pi^n - [\pi^n] > \exp\{-\gamma_2 n^2 (\log n)^2\}$$

for all sufficiently large positive integers n, while my result yields a much less good lower estimate.

If, however,

$$a \geqslant (m+1)^{(m+1)/\nu},$$

then Theorem 2 is much stronger, and it furthermore gives a lower bound for $|\pi - \omega|$ free of unknown constants. The exponent of $1/a$,

$$(m+1)\,\nu \log(m+1),$$

is not greater than

$$(20 \cdot 2^{(5/2)(\nu-1)} + 1)\,\nu \,\log\,(20 \cdot 2^{(5/2)(\nu-1)} + 1)$$

and therefore, for large n, is of the order

$$O\,(2^{(5/2)\nu}\,\nu^2)$$

16. As an application of Theorem 2, let us determine a lower bound for $|\sin u\alpha|$ when α is a fixed positive algebraic number and u is a positive integral variable such that $u \geqslant \pi/\alpha$.

Define a second positive integer v by

$$-\frac{\pi}{2} < u\alpha - v\pi \leqslant \frac{\pi}{2}.$$

Then

$$\frac{\alpha}{2\pi} u \leqslant \frac{\alpha}{\pi} u - \tfrac{1}{2} \leqslant v < \frac{\alpha}{\pi} u + \tfrac{1}{2} < \frac{2\alpha}{\pi} u,$$

and therefore

$$\max(u, v) \leqslant \max\left(u, \frac{2\alpha}{\pi} u\right) < \left(\frac{2\alpha}{\pi} + 1\right) u.$$

Let, say, α have the degree ν over the rational field, and let it satisfy the irreducible equation

$$A_0 z^\nu + A_1 z^{\nu-1} + \ldots + A_\nu = 0 \qquad (A_0 \neq 0)$$

with rational integral coefficients of height

$$A = \max(|A_0|, |A_1|, \ldots, |A_\nu|) \geqslant 1.$$

Then the rational multiple of a,

$$\omega = \frac{u}{v} a,$$

is a root of the equation

$$A_0 v^\nu z^\nu + A_1 u v^{\nu-1} z^{\nu-1} + \ldots + A_\nu u^\nu = 0$$

of height

$$a = \max(|A_0 v^\nu|, |A_1 u v^{\nu-1}|, \ldots, |A_\nu u^\nu|) \leqslant A\,(\max(u,v))^\nu < \left(\frac{2a}{\pi}+1\right)^\nu A\, u^\nu.$$

Let again

$$m = [20 \cdot 2^{(5/2)(\nu-1)}], \qquad \bar{a} = \max(a, (m+1)^{(m+1)/\nu}),$$

so that

$$\bar{a} \leqslant \max\left(\left(\frac{2a}{\pi}+1\right)^\nu A\, u^\nu,\ (m+1)^{(m+1)/\nu}\right), = a^* \text{ say,}$$

whence, by Theorem 2,

$$|\pi - \omega| > \left(\frac{m+1}{e}\right)^{-(m+1)} a^{*-(m+1)\nu \log(m+1)}.$$

On the other hand,

$$|\sin t| \geqslant \frac{2}{\pi}|t| \quad \text{if } |t| \leqslant \frac{\pi}{2},$$

hence

$$|\sin u a| = |\sin(u a - v\pi)| \geqslant \frac{2}{\pi} v |\pi - \omega|,$$

and we find, finally, that

$$|\sin u a| > \frac{a}{\pi^2} u \left(\frac{m+1}{e}\right)^{-(m+1)} a^{*-(m+1)\nu \log(m+1)}.$$

In the special case when $a = 1$, Theorem 1 gives a stronger result, viz.

$$|\sin u| > \frac{1}{\pi^2} u^{-41}.$$

This inequality has been proved for $u \geqslant \pi$, i.e. for $u \geqslant 4$, but it is easily verified that it holds also for $1 \leqslant u \leqslant 3$.

By way of example, the power series

$$\sum_{u=1}^{\infty} \frac{z^u}{\sin u a}$$

has the radius of convergence 1, and the Dirichlet series

$$\sum_{u=1}^{\infty} \frac{u^{-s}}{\sin u a}$$

converges when the real part of s is greater than $(m+1)\nu \log(m+1)$.

I wish to thank Mr C. G. LEKKERKERKER for his careful checking of the numerical work of this paper, and for pointing out a minor error.

Mathematics Department,
Manchester University.

October 20, 1952.

Edited by Howard Eves, University of Maine, Orono, Maine

The evolution of extended decimal approximations to π

by J. W. Wrench, Jr., Applied Mathematics Laboratory, David Taylor Model Basin, Washington, D.C.

In his historical survey of the classic problem of "squaring the circle," Professor E. W. Hobson [1]* distinguished three distinct periods, characterized by fundamental differences in method, immediate aims, and available mathematical tools.

The first period—the so-called geometrical period—extended from the earliest empirical determinations of the ratio of the circumference of a circle to its diameter to the invention of the calculus about the middle of the seventeenth century. The main effort was directed toward the approximation of this ratio by the calculation of perimeters or areas of regular inscribed and circumscribed polygons.

The second period began in the middle of the seventeenth century and lasted for more than a hundred years. During this period the methods of the calculus were employed in the development of analytical expressions for π in the form of infinite series, products, and continued fractions.

The third period, which extended from the middle of the eighteenth century to nearly the end of the nineteenth century, was devoted to studies of the nature of the number π. J. H. Lambert [2] proved the irrationality of π in 1761, and F. Lindemann [3] first established its transcendence in 1882.

* Numbers in brackets refer to the references listed at the end of the article.

This article is concerned with the second period and its sequel, which extends to the present day.

According to Hobson [1], the first analytical expression discovered in this period is the infinite product

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots,$$

which was published by John Wallis [4] in 1655.

Lord Brouncker, the first president of the Royal Society, about 1658 found the infinite continued fraction

$$\frac{\pi}{4}=\frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots,$$

which was shown subsequently by Euler to be equivalent to the alternating series

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots,$$

known to G. W. Leibniz in 1674.

The great majority of calculations of π to many decimal places have been based upon the power series

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots,\quad -1\leqq x\leqq 1,$$

which was discovered in 1671 by James Gregory [5]. He failed, however, to note explicitly the special case corresponding to $x=1$, which is ascribed to Leibniz.

Sir Isaac Newton [6] in 1676 discovered the power series

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots, \qquad -1 \leqq x \leqq 1,$$

which has been used by a few computers of π.

In 1755 Leonhard Euler [7] obtained the following useful series:

$$\arctan x = \frac{x}{1+x^2}\left\{1 + \frac{2}{3}\left(\frac{x^2}{1+x^2}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2 + \cdots\right\}.$$

It was by means of Gregory's series, taking $x = 1/\sqrt{3}$, that Abraham Sharp [8], at the suggestion of the English astronomer Edmund Halley, computed π to 72 decimal places in 1699, thereby nearly doubling the greatest accuracy (39 decimal places) attained by earlier computers, who had used geometrical methods. Sharp's calculation was extended by Fautet de Lagny [9] in 1719 to 127 decimals (the 113th place has a unit error).

Newton set $x = \frac{1}{2}$ in his series, and thereby computed π to 14 places. A Japanese computer, Matsunaga Ryohitsu [10], used the same procedure to evaluate π correct to 49 decimal places in 1739. About 1800 a Chinese, Chu Hung, calculated π to 40 places (25 correct) by this series [10].

Most computers of π in modern times have used Gregory's series in conjunction with certain arctangent relations. Only nine of these relations have been employed to any extent in such computations. We shall now consider these formulas, arranged according to the increasing precision of the approximations computed by their use.

$$\text{I.}\quad \frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}$$

Euler [7] in 1755 used this relation in conjunction with his series for arctan x to compute π correct to 20 decimal places in one hour. Baron Georg von Vega [11] in 1794 employed Gregory's series and the preceding relation to evaluate π to 140 decimal places, of which the first 136 were correct. This precision was exceeded by that attained by an unknown calculator whose manuscript, containing an approximation correct to 152 places, was seen in the Radcliffe Library at Oxford toward the close of the eighteenth century.

$$\text{II.}\quad \frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{70} + \arctan\frac{1}{99}$$

Euler published this relation in 1764. It was used by William Rutherford [12] in 1841 to compute π to 208 places (152 correct).

$$\text{III.}\quad \frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}$$

This formula was supplied the calculating prodigy Zacharias Dahse [13] by L. K. Schulz von Strassnitzky of Vienna. Within a period of two months in 1844, Dahse thereby evaluated π correct to 200 places.

$$\text{IV.}\quad \frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

First published by Charles Hutton [14] in 1776, this relation was used by W. Lehmann [15] of Potsdam to compute π to 261 decimals in 1853. Tseng Chi-hung [16] in 1877 used the same formula to evaluate π to 100 decimals in a little more than a month.

$$\text{V.}\quad \frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7}$$

The relation was also published by Hutton [14] in 1776, and independently by Euler in 1779. Vega [17] used it in 1789 to compute 143 decimals (126 correct). In order to remove the uncertainty caused by the discrepant approximations of Rutherford and Dahse, Thomas Clausen [18] extended the calculation to 248 correct decimals in 1847, and Lehman [15] reached 261 decimals in 1853 by this formula, confirming his independent calculation of π to the same extent by relation IV. Edgar Frisby [19] in Washington, D. C. used relation V in conjunction with Euler's series to compute π to 30 places in 1872.

$$\text{VI.}\quad \frac{\pi}{4}=3\arctan\frac{1}{4}+\arctan\frac{1}{20}+\arctan\frac{1}{1985}$$

This formula was published by S. L. Loney [20] in 1893, by Carl Störmer [21] in 1896, and was rediscovered by R. W. Morris [22] in 1944. By means of this formula D. F. Ferguson, then of the Royal Naval College, Eaton, Chester, England, performed a longhand calculation of π to 530 decimal places between May 1944 and May 1945. At that time he discovered a discrepancy between his approximation and the final result of William Shanks—discussed under formula IX—beginning with the 528th place. The first notice of an error in Shanks's well-known approximation appeared in a note [22] published by Ferguson in March 1946. He continued his calculation of π and in July 1946 published [23] a correction to Shanks's value through the 620th decimal place. Subsequently, Ferguson used a desk calculator to reach 710 decimals [24] by January 1947, and finally 808 decimals [25] by September 1947.

$$\text{VII.}\quad \frac{\pi}{4}=8\arctan\frac{1}{10}-\arctan\frac{1}{239}-4\arctan\frac{1}{515}$$

S. Klingenstierna discovered this relation in 1730; it was rediscovered more than a century later by Schellbach [26]. It was used by C. C. Camp [27] in 1926 to evaluate $\pi/4$ to 56 places. D. H. Lehmer [28] recommended it in conjunction with the next formula for the calculation of π to many figures. G. E. Felton on March 31, 1957 completed a calculation of π to 10021 places on a Pegasus computer at the Ferranti Computer Centre in London. This required 33 hours of computer time. The result was published to 10000 places [29]. A check calculation using formula VIII revealed that, because of a machine error, this result was incorrect after 7480 decimal places.

Gauss [30] investigated the derivation of arctangent relations and reduced it to a problem in Diophantine analysis. Relation VIII is one of several formulas he developed. J. P. Ballantine [31] substantiated Lehmer's claim that this formula is especially effective for extensive calculation, by discussing its use in conjunction with Euler's series for the arctangent.

$$\text{VIII.}\quad \frac{\pi}{4}=12\arctan\frac{1}{18}+8\arctan\frac{1}{57}-5\arctan\frac{1}{239}$$

Felton carried out a second calculation to 10021 places, and by March 1, 1958 had removed all discrepancies from his results, so that the approximations computed from formulas VII and VIII agreed to within 3 units in the 10021st decimal place. The corrected result remains unpublished.

$$\text{IX.}\quad \frac{\pi}{4}=4\arctan\frac{1}{5}-\arctan\frac{1}{239}$$

This is the most celebrated of all the relations of this kind. John Machin, its discoverer, computed π correct to 100 decimals by means of it in conjunction with Gregory's series, and the result [32] appeared in 1706. Clausen [18] in 1847 used this relation in addition to Hutton's formula V to compute π correct to 248 decimal places, as has already been noted.

Rutherford resumed his calculation of π in 1852, using Machin's formula this time, as did his former pupil William Shanks. Shanks's first published approximation to π contained 530 decimal places, and was incorporated in Rutherford's note [33], published in 1853, which set forth his approximation to 441 decimals.

Later that year Shanks published his book [34] containing an approximation to 607 places and giving all details of the calculation to 530 places. It is now known that Shanks's value was incorrectly calculated beyond 527 decimal places. The accuracy of that value was further vitiated by a blunder committed by Shanks in correcting his copy prior to publication, with the result that similar errors appear in decimal places 460–462 and 513–515. These errors persist in Shanks's first paper of 1873 [35] containing the extension to 707 decimals of his earlier approximation. His second paper of that year [36], which contained his final approximation to π, gives corrections of these errors; however, there appears an inadvertent typographical error in the 326th decimal place of his final value. In retrospect, we now realize that Shanks's first value published in 1853 was the most accurate he ever published.

The accuracy of Shanks's approximation to at least 500 decimals was confirmed by the independent calculations of Professor Richter [37] of Elbing, Germany, who in 1853–1854 computed successive approximations to 330, 400, and 500 places. Richter's communications do not reveal the formula that he used.

Machin's formula was used by H. S. Uhler in an unpublished computation correct to 282 places, which was completed in August 1900.

F. J. Duarte computed π correct to 200 places by this method in 1902. The result was published [38] six years later.

As a by-product of his calculation of the natural logarithms of small primes, Uhler in 1940 noted [39] confirmation to 333 decimal places of Shanks's approximation.

In December 1945, Professor R. C. Archibald suggested that the writer undertake the computation of π by Machin's formula in order to provide an independent check of the accuracy of Ferguson's calculations. With the collaboration of Levi B. Smith, who evaluated arctan 1/239 to 820 decimal places, the writer computed π to 818 places by February 1947, using a desk calculator. The result was published [24] to 808 places in April 1947, and was verified to 710 places by Ferguson in a note published concurrently [24]. The limit of 808 decimals in the published value was chosen to provide precision comparable to that obtained by P. Pedersen [40] in his approximation to e.

Collation of this 808-place approximation with results obtained by Ferguson later that year revealed several erroneous figures beyond the 723rd place in the writer's approximation to arctan $\frac{1}{5}$. These errors vitiated the corresponding figures in the approximation to π. Corrections of these errors and extensions of Ferguson's results appeared in a joint paper [25] by Ferguson and the writer in January 1948, which concluded with an 808-place approximation to π of guaranteed accuracy.

Subsequently, Smith and the writer resumed their calculations and by June 1949 had obtained an approximation to about 1120 decimal places [41]. Before final checking of this extension could be completed, the ENIAC (Electronic Numerical Integrator and Computer) at the Ballistic Research Laboratories, Aberdeen Proving Ground, was employed by George W. Reitwiesner and his associates in September 1949 to evaluate π to about 2037 places (2040 working decimals) in a total time (including card handling) of 70 hours [42]. Machin's formula was also used in this computation.

In November 1954, Smith and the writer extended their calculation to 1150 places, and in January 1956 reverted to this work once more to attain their final result, which was terminated at 1160 places, of which the first 1157 agree with those obtained on the ENIAC.

A calculation of π was performed in duplicate on the NORC (Naval Ordnance Research Calculator) in November 1954 and in January 1955 as a demonstration problem, prior to the delivery of that computer to the U. S. Naval Proving

Grounds at Dahlgren, Virginia. Again, Machin's formula was selected, and the calculation was completed to 3093 decimal places in 13 minutes running time. A report of this work, in which the value of π was presented unrounded to 3089 decimal places, was published by S. C. Nicholson and J. Jeenel [43] of the Watson Scientific Computing Laboratory, in New York.

In January 1958, François Genuys [44] programmed and carried out the evaluation of π correct to 10000 decimal places on an IBM 704 Electronic Data Processing System at the Paris Data Processing Center. Machin's formula in conjunction with Gregory's series was used. Only 40 seconds were required to attain the 707 decimal-place precision reached by Shanks, and one hour and forty minutes was required to reach the 10000 places of the final result.

On July 20, 1959, the program of Genuys was used on an IBM 704 system at the Commissariat à l'Energie Atomique in Paris to compute π to 16167 decimal places. This latest approximation is unpublished at present.

The motivation of modern calculations of π to many decimal places was conjectured by Professor P. S. Jones [45] in 1950 as being attributable to "intellectual curiosity and the challenge of an unchecked and long untouched computation." This reason for undertaking such work should be supplemented by reference to the recurrent interest in determining a statistical measure of the randomness of distribution of the digits in the decimal representation of π.

Augustus De Morgan [46] drew attention to the deficiency in the number of appearances of the digit 7 in Shanks's 607-place approximation to π. In 1897 E. B. Escott [47] raised the question whether the deficiency of 7's noted in Shanks's final approximation could be explained.

In June 1949, the late Professor John von Neumann expressed an interest in utilizing the ENIAC to determine the value of π and e to many places as the basis for a statistical study of the distribution of their decimal digits. A statistical treatment of the first 2000 decimal digits of both π and e was published by N. C. Metropolis, G. Reitwiesner, and J. von Neumann [48]. Further analysis of these data was performed by R. E. Greenwood [49], using the coupon collector's test. A

TABLE 1

CUMULATIVE DISTRIBUTION OF THE FIRST 16000 DECIMAL DIGITS OF π

THOUSAND	DIGIT									
	0	1	2	3	4	5	6	7	8	9
1	93	116	103	102	93	97	94	95	101	106
2	182	212	207	188	195	205	200	197	202	212
3	259	309	303	265	318	315	302	287	310	332
4	362	429	408	368	405	417	398	377	405	431
5	466	532	496	459	508	525	513	488	492	512
6	557	626	594	572	613	622	619	606	582	609
7	657	733	692	686	702	730	708	694	680	718
8	754	833	811	781	809	834	816	786	764	812
9	855	936	911	884	910	933	914	883	854	920
10	968	1026	1021	974	1012	1046	1021	970	948	1014
11	1070	1099	1111	1080	1133	1150	1129	1070	1031	1127
12	1162	1193	1214	1176	1233	1262	1227	1166	1144	1223
13	1266	1314	1316	1272	1343	1358	1324	1260	1246	1301
14	1365	1416	1419	1383	1440	1455	1426	1344	1339	1413
15	1456	1513	1511	1491	1553	1549	1520	1441	1458	1508
16	1556	1601	1593	1602	1670	1659	1615	1548	1546	1610

count of each of the decimal digits appearing in the NORC approximation appears in the paper of Nicholson and Jeenel [43]. A number of recent investigators have discussed the distribution of digits in Shanks's approximation and in the corrected value of π. These investigators include F. Bukovszky [50], W. Hope-Jones [51], E. H. Neville [52], and B. C. Brookes [53].

The writer has recently completed a count by centuries of the 16167 decimal digits constituting the fractional part of the latest approximation to π. An abridgment of this information is presented in the accompanying table.

The standard χ^2 test for goodness of fit reveals no abnormal behavior in the distribution of digits in this sample; in particular, there appears to be no basis for supposing that π is not simply normal [54] in the decimal scale of notation. It has been pointed out recently by Ivan Niven [55] that the normality of such numbers as π, e, and $\sqrt{2}$ has yet to be proved.

Numerical studies directed toward the empirical investigation of the normality of π clearly require increasingly higher decimal approximations, which can best be obtained by use of ultra-high-speed electronic computers now under design and development.

References

1. E. W. Hobson, *"Squaring the Circle," a History of the Problem* (Cambridge, 1913; reprinted by Chelsea Publishing Company, New York, 1953).
2. J. H. Lambert, "Mémoire sur quelques propriétés rémarquables des quantités transcendentes circulaires et logarithmiques," *Histoire de l'Académie de Berlin, 1761* (1768).
3. F. Lindemann, "Ueber die Zahl π," *Mathematische Annalen*, 20 (1882), 213–225.
4. J. Wallis, *Arithmetica Infinitorum* (1655).
5. R. C. Archibald, *Outline of the History of Mathematics* (6th ed.; Buffalo, N. Y.: The Mathematical Association of America, 1949), p. 40.
6. Letter from Newton to Oldenburg dated October 24, 1676.
7. E. Beutel, *Die Quadratur des Kreises* (Leipzig, 1920), p. 40. See also E. W. Hobson,[1] p. 39.
8. H. Sherwin, *Mathematical Tables* (London, 1705), p. 59.
9. F. de Lagny, "Mémoire Sur la Quadrature du Circle, & sur la mesure de tout Arc, tout Secteur, & tout Segment donné," *Histoire de l'Académie Royale des Sciences, 1719* (Paris, 1721), pp. 135–145.
10. Y. Mikami, *The Development of Mathematics in China and Japan* (Leipzig, 1913), p. 202 and p. 141.
11. G. Vega, *Thesaurus Logarithmorum Completus* (Leipzig, 1794; reprinted by G. E. Stechert & Co., New York, 1946), p. 633.
12. W. Rutherford, "Computation of the Ratio of the Diameter of a Circle to its Circumference to 208 places of figures," *Philosophical Transactions of the Royal Society of London*, 131 (1841), 281–283.
13. Z. Dahse, "Der Kreis-Umfang für den Durchmesser 1 auf 200 Decimalstellen berechnet," *Journal für die reine und angewandte Mathematik*, 27 (1844), 198.
14. *Philosophical Transactions of the Royal Society of London*, 46 (1776), 476–492.
15. W. Lehmann, "Beitrag zur Berechnung der Zahl π, welche das Verhältniss des Kreis-Durchmessers zum Umfang ausdrückt," *Archiv der Mathematik und Physik*, 21 (1853), 121–174.
16. Y. Mikami,[10] pp. 141–142.
17. *Nova Acta Academiae Scientiarum Imperialis Petropolitanae*, 9 (1790), 41.
18. *Astronomische Nachrichten*, 25 (1847), col. 207–210.
19. E. Frisby, "On the calculation of π," *Messenger of Mathematics*, 2 (1873), 114–118.
20. S. L. Loney, *Plane Trigonometry* (Cambridge, 1893), p. 277.
21. C. Störmer, "Sur l'application de la théorie des nombres entiers complexes à la solution en nombres rationnels $x_1, x_2, \cdots, x_n, c_1, c_2, \cdots, c_n, k$ de l'équation $c_1 \operatorname{arctg} x_1 + c_2 \operatorname{arctg} x_2 + \cdots + c_n \operatorname{arctg} x_n = k\pi/4$," *Archiv for Mathematik og Naturvidenskab*, 19 (1896), 70.
22. D. F. Ferguson, "Value of π," *Nature*, 157 (1946), 342. See also D. F. Ferguson, "Evaluation of π. Are Shanks' figures correct?" *Mathematical Gazette*, 30 (1946), 89–90.
23. R. C. Archibald, "Approximations to π," *Mathematical Tables and other Aids to Computation*, 2 (1946-1947), 143–145.
24. L. B. Smith, J. W. Wrench, Jr., and D. F. Ferguson, "A New Approximation to π," *ibid.*, 2 (1946–1947), 245–248.
25. D. F. Ferguson and J. W. Wrench, Jr., "A New Approximation to π (conclusion)," *ibid.*, 3 (1948-1949), 18–19. See also R. Liénard, "Constantes mathématiques et système binaire," *Intermédiaire des Recherches Mathématiques*, 5 (1948), 75.

26. K. H. Schellbach, "Über den Ausdruck $\pi = (2/i) \log i$," *Journal für die reine und angewandte Mathematik*, 9 (1832), 404–406.
27. C. C. Camp, "A New Calculation of π," *American Mathematical Monthly*, 33 (1926), 474.
28. D. H. Lehmer, "On Arccotangent Relations for π," *ibid.*, 45 (1938), 657–664.
29. G. E. Felton, "Electronic Computers and Mathematicians," *Abbreviated Proceedings of the Oxford Mathematical Conference for Schoolteachers and Industrialists at Trinity College, Oxford, April 8–18, 1957*, p. 12–17; footnote, p. 12–53.
30. C. F. Gauss, *Werke* (Göttingen, 1863; 2nd ed., 1876), Vol. 2, p. 499–502.
31. J. P. Ballantine, "The Best (?) Formula for Computing π to a Thousand Places," *American Mathematical Monthly*, 46 (1939), 499–501.
32. W. Jones, *Synopsis palmiorum matheseos* (London, 1706), p. 263.
33. W. Rutherford, "On the Extension of the value of the ratio of the Circumference of a circle to its Diameter," *Proceedings of the Royal Society of London*, 6 (1850–1854), 273–275. See also *Nouvelles Annales des Mathématiques*, 14 (1855), 209–210.
34. W. Shanks, *Contributions to Mathematics, comprising chiefly the Rectification of the Circle to 607 Places of Decimals* (London, 1853).
35. W. Shanks, "On the Extension of the Numerical Value of π," *Proceedings of the Royal Society of London*, 21 (1873), 318.
36. W. Shanks, "On certain Discrepancies in the published numerical value of π," *ibid.*, 22 (1873), 45–46.
37. *Archiv der Mathematik und Physik*, 21 (1853), 119; 22 (1854), 473; 23 (1854), 475–476; 25 (1855), 471–472 (posthumous). See also *Nouvelles Annales des Mathématiques*, 13 (1854), 418–423.
38. *Comptes Rendus de l'Académie des Sciences de Paris*, Vol. 146, 1908. See also *L'Intermédiaire des Mathématiciens*, 27 (1920), 108–109, and F. J. Duarte, *Monografía sobre los Números π y e* (Caracas, 1949).
39. H. S. Uhler, "Recalculation and Extension of the Modulus and of the Logarithms of 2, 3, 5, 7, and 17," *Proceedings of the National Academy of Sciences*, 26 (1940), 205–212.
40. D. H. Lehmer, Review 275, *Mathematical Tables and other Aids to Computation*, 2 (1946–1947), 68–69.
41. J. W. Wrench, Jr., and L. B. Smith, "Values of the terms of the Gregory series for arccot 5 and arccot 239 to 1150 and 1120 decimal places, respectively," *ibid.*, 4 (1950), 160–161.
42. G. Reitwiesner, "An ENIAC Determination of π and e to more than 2000 Decimal Places," *ibid.*, 4 (1950), 11–15.
43. S. C. Nicholson and J. Jeenel, "Some Comments on a NORC Computation of π," *ibid.*, 9 (1955), 162–164.
44. F. Genuys, "Dix milles décimales de π," *Chiffres*, 1 (1958), 17–22.
45. P. S. Jones, "What's New About π?," The Mathematics Teacher, 43 (1950), 120–122.
46. A. De Morgan, *A Budget of Paradoxes* (1st ed., 1872; 2nd ed., Chicago: The Open Court Publishing Company, 1915), Vol. 2, p. 65. See also James R. Newman, *The World of Mathematics* (New York: Simon and Schuster, 1956), Vol. 4, pp. 2379–2380.
47. E. B. Escott, Question 1154, *L'Intermédiaire des Mathématiciens*, 4 (1897), 221.
48. N. C. Metropolis, G. Reitwiesner, and J. von Neumann, "Statistical Treatment of the Values of First 2000 Decimal Digits of e and π Calculated on the ENIAC," *Mathematical Tables and other Aids to Computation*, 4 (1950), 109–111.
49. R. E. Greenwood, "Coupon Collector's Test for Random Digits," *ibid.*, 9 (1955), 1–5.
50. F. Bukovsky, "The Digits in the Decimal Form of π," *The Mathematical Gazette*, 33 (1949), 291.
51. W. Hope-Jones, "Surprising," *ibid.*, Vol. 35, 1951.
52. E. H. Neville, "The Digits in the Decimal Form of π," *ibid.*, 35 (1951), 44–45.
53. B. C. Brookes, "On the Decimal for π," *ibid.*, 36 (1952), 47–48.
54. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford, 1938), pp. 123–127.
55. I. Niven, *Irrational Numbers*, Carus Monograph No. 11 (Buffalo, N. Y.: The Mathematical Association of America, 1956), p. 112.

"Well over half of all high schools enrolling more than 300 pupils are grouping pupils according to their ability and achievement records, thus making possible enriched and advanced learning."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

Calculation of π to 100,000 Decimals

By Daniel Shanks and John W. Wrench, Jr.

1. Introduction. The following comparison of the previous calculations of π performed on electronic computers shows the rapid increase in computational speeds which has taken place.

Author		Machine	Date	Precision	Time
Reitwiesner	[1]	ENIAC	1949	2037D	70 hours
Nicholson & Jeenel	[2]	NORC	1954	3089D	13 min.
Felton	[3]	Pegasus	1958	10000D	33 hours
Genuys	[4]	IBM 704	1958	10000D	100 min.
Unpublished	[5]	IBM 704	1959	16167D	4.3 hours

All these computations, except Felton's, used Machin's formula:

$$\pi = 16 \tan^{-1} \tfrac{1}{5} - 4 \tan^{-1} \tfrac{1}{239}. \tag{1}$$

Other things being equal, that is, assuming the use of the same machine and the same program, an increase in precision by a factor f requires f times as much memory, and f^2 times as much machine time. For example, a hypothetical computation of π to 100,000D using Genuys' program would require 167 hours on an IBM 704 system and more than 38,000 words of core memory. However, since the latter is not available, the program would require modification, and this would extend the running time. Further, since the probability of a machine error would be more than 100 times that during Genuys' computation, prudence would require still other program modifications, and, therefore, still more machine time.

2. A New Program. We discuss here a computation of π to more than 100,000D, which required 8 hours 43 minutes on an IBM 7090 system. This increase in speed by a factor of about 20 is largely due to the increased speed of the 7090 (it is about 7 times as fast as a 704), but substantial gains were also obtained by programming changes.

The formula we used, namely,

$$\pi = 24 \tan^{-1} \tfrac{1}{8} + 8 \tan^{-1} \tfrac{1}{57} + 4 \tan^{-1} \tfrac{1}{239}, \tag{2}$$

is due to Störmer [6], and has not been previously used in high-precision computation. The computation time breaks down as follows:

$8 \tan^{-1} \frac{1}{57}$: 3 hours 7 minutes

$4 \tan^{-1} \frac{1}{239}$: 2 hours 20 minutes

$24 \tan^{-1} \frac{1}{8}$: 2 hours 34 minutes

Received September 7, 1961.

Conversion (binary-to-decimal): 42 minutes.

To obtain these favorable times, two devices were used.

a) Instead of evaluating the Taylor series

$$A \tan^{-1} \frac{1}{m} = \sum_{k=0}^{\infty} \frac{(-1)^k A m}{(2k+1)m^{2(k+1)}} \tag{3}$$

term by term, one may compute two terms at a time by using

$$A \tan^{-1} \frac{1}{m} = \sum_{k=0}^{\infty} \frac{Am[(4k+3)m^2 - (4k+1)]}{(16k^2+16k+3)m^{4(k+1)}}. \tag{4}$$

This substitutes 2 (multi-precision) divisions, 1 multiplication, and 1 addition for 4 divisions, 1 subtraction, and 1 addition, and this eliminates 27% of the computation time. Further improvement in this direction, i.e., three terms at a time, is not possible here, since the divisors in (4), m^4 and $(16k^2 + 16k + 3)$, already tend to fill a 7090 word, whose maximum numerical size is $2^{35} - 1$.

b) Since the 7090 is a binary machine, the (multi-precision) division by m^4 and the multiplication by $[(4k+3)m^2 - (4k+1)]$ may be replaced by a simple shifting operation for the case $m = 8$ in $24 \tan^{-1} \frac{1}{8}$. Had this not been done the 2 hours 34 minutes listed above would be, instead, 6 hours 7 minutes with the double-term formula, (4), or 8 hours 21 minutes with (3).

The program using (4) requires three blocks of storage, one for the current $Am/m^{4(k+1)}$, one for the current term, and one for the partial sum. Since a 7090 word is equivalent to 10.536 decimal digits, the working storage for an N-place π is $3N/10.536$. Therefore, a core memory of 32,768 easily suffices for a computation to more than 100,000D.

3. The Check. Inasmuch as such a computation requires *billions* of (arithmetical) operations, it is clear that a check is necessary. This was obtained with Gauss's formula [7]:

$$\pi = 48 \tan^{-1} \tfrac{1}{18} + 32 \tan^{-1} \tfrac{1}{57} - 20 \tan^{-1} \tfrac{1}{239}. \tag{5}$$

During the computation, using (2), the numbers

$$A = 8 \tan^{-1} \tfrac{1}{57}$$

and

$$B = 8 \tan^{-1} \tfrac{1}{57} + 4 \tan^{-1} \tfrac{1}{239}$$

were written on tape. At the start of the check A and B were read into memory and $9A - 5B = 32 \tan^{-1} \frac{1}{57} - 20 \tan^{-1} \frac{1}{239}$ was computed (in 1 second). To this difference was added the new number $48 \tan^{-1} \frac{1}{18}$, which was computed by (4) in 4 hours 22 minutes. The check, therefore, takes less time than the original run (8 hours 1 minute), and is perfectly valid, since an error in any of the four arctangents will lead to a discrepancy between the results of (2) and (5).

4. The Result. The run and the check were both made on July 29, 1961, and such a discrepancy did in fact occur. The two values of π, in binary form, were compared by the machine in $\frac{1}{5}$ second, and were found to agree to only 234,848 bits. This is equivalent to 70,695 decimal places. Subsequently the error was isolated. It was found to have occurred in the computation of $24 \tan^{-1} \frac{1}{8}$. The second value of π, computed using (5), was therefore correct throughout. When $24 \tan^{-1} \frac{1}{8}$ was recomputed, the two values of π agreed up to the last word. This comprised 333,075 bits, or 100,265 decimal places. The first 100,000 of these are given here.

The computation was of such a character that it was known *a priori* that the term involving $\tan^{-1} \frac{1}{239}$ contains most of the round-off and truncation error, and consequently that we have the inequalities:

$$\pi \text{ computed by } (2) < \pi < \pi \text{ computed by } (5).$$

Thus the check and the computation give upper and lower bounds respectively, and, to the extent that they agree, π is determined absolutely.

Care has been taken in the output routines—in writing and printing—and, since the reproduction here is photographic, we believe that this value is entirely free from error.

5. A Million Decimals? Can π be computed to 1,000,000 decimals with the computers of today? From the remarks in the first section we see that the program which we have described would require times of the order of *months*. But since the memory of a 7090 is too small, by a factor of ten, a modified program, which writes and reads partial results, would take longer still. One would really want a computer 100 times as fast, 100 times as reliable, and with a memory 10 times as large. No such machine now exists.

There are, of course, many other formulas similar to (1), (2), and (5), and other programming devices are also possible, but it seems unlikely that any such modification can lead to more than a rather small improvement.

Are there *entirely different* procedures? This is, of course, possible. We cite the following: compute $1/\pi$ and then take its reciprocal. This sounds fantastic, but, in fact, it can be faster than the use of equation (2). One can compute $1/\pi$ by Ramanujan's formula [8]:

$$(6) \qquad \frac{1}{\pi} = \frac{1}{4}\left(\frac{1123}{882} - \frac{22583}{882^3}\frac{1}{2} \cdot \frac{1\cdot 3}{4^2} + \frac{44043}{882^5}\frac{1\cdot 3}{2\cdot 4} \cdot \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots\right).$$

The first factors here are given by $(-1)^k(1123 + 21460k)$. A binary value of $1/\pi$ equivalent to 100,000D, can be computed on a 7090 using equation (6) in 6 hours instead of the 8 hours required for the application of equation (2).* To reciprocate this value of $1/\pi$ would take about 1 hour. Thus, we can reduce the time required by (2) by an hour. But unfortunately we lose our overlapping check, and, in any case, this small gain is quite inadequate for the present question.

One could hope for a theoretical approach to this question of optimization—a theory of the "*depth*" of numbers—but no such theory now exists. One can guess that e is not as "deep" as π,† but try to prove it!

Such a theory would, of course, take years to develop. In the meantime—say, in 5 to 7 years—such a computer as we suggested above (100 times as fast, 100 times as reliable, and with 10 times the memory) will, no doubt, become a reality. At that time a computation of π to 1,000,000D will not be difficult.

* We have computed $1/\pi$ by (6) to over 5000D in less than a minute.

† We have computed e on a 7090 to 100,265D by the obvious program. This takes 2.5 hours instead of the 8-hour run for π by (2).

6. Acknowledgement. We wish to thank the IBM Corporation for its generosity in allowing us to use, free of charge, the 7090 in the IBM Data Processing Center, New York, for the computation described above.

Applied Mathematics Laboratory
David Taylor Model Basin
Washington 7, D. C.

1. G. Reitwiesner, "An ENIAC determination of π and e to more than 2000 decimal places," *MTAC*, v. 4, 1950, p. 11–15.
2. S. C. Nicholson & J. Jeenel, "Some comments on a NORC computation of π," *MTAC*, v. 9, 1955, p. 162–164.
3. G. E. Felton, "Electronic computers and mathematicians," *Abbreviated Proceedings of the Oxford Mathematical Conference for Schoolteachers and Industrialists at Trinity College, Oxford, April 8–18, 1957*, p. 12–17, footnote p. 12–53. This published result is correct to only 7480D, as was established by Felton in a second calculation, using formula (5), completed in 1958 but apparently unpublished. For a detailed account of calculations of π see J. W. Wrench, Jr., "The evolution of extended decimal approximations to π," *The Mathematics Teacher*, v. 53, 1960, p. 644–650.
4. F. Genuys, "Dix milles decimales de π," *Chiffres*, v. 1, 1958, p. 17–22.
5. This unpublished value of π to 16167D was computed on an IBM 704 system at the Commissariat à l'Energie Atomique in Paris, by means of the program of Genuys.
6. C. Störmer, "Sur l'application de la théorie des nombres entiers complexes à la solution en nombres rationnels, $x_1, x_2, \cdots, x_n, c_1, c_2, \cdots, c_n, k$ de l'équation $c_1 \operatorname{arctg} x_1 + c_2 \operatorname{arctg} x_2 + \cdots + c_n \operatorname{arctg} x_n = k\pi/4$," *Archiv for Mathematik og Naturvidenskab*, v. 19, 1896, p. 69.
7. C. F. Gauss, *Werke*, Göttingen, 1863; 2nd ed., 1876, v. 2, p. 499–502.
8. S. Ramanujan, "Modular equations and approximations to π," *Quart. J. Pure Appl. Math.*, v. 45, 1914, p. 350–372; *Collected papers of Srinivasa Ramanujan*, Cambridge, 1927, p. 23–39.

Note added in proof, December 1, 1961. J. M. Gerard of IBM United Kingdom Limited, who was then unaware of the computation described above, computed π to 20,000 D on the 7090 in the London Data Centre on July 31, 1961. His program used Machin's formula, (1), and required 39 minutes running time. His result agrees with ours to that number of decimals.

PI = 3.+

1415926535	8979323846	2643383279	5028841971	6939937510	5820974944	5923078164	0628620899	8628034825	3421170679
8214808651	3282306647	0938446095	5058223172	5359408128	4811174502	8410270193	8521105559	6446229489	5493038196
4428810975	6659334461	2847564823	3786783165	2712019091	4564856692	3460348610	4543266482	1339360726	0249141273
7245870066	0631558817	4881520920	9628292540	9171536436	7892590360	0113305305	4882046652	1384146951	9415116094
3305727036	5759591953	0921861173	8193261179	3105118548	0744623799	6274956735	1885752724	8912279381	8301194912
9833673362	4406566430	8602139494	6395224737	1907021798	6094370277	0539217176	2931767523	8467481846	7669405132
0005681271	4526356082	7785771342	7577896091	7363717872	1468440901	2249534301	4654958537	1050792279	6892589235
4201995611	2129021960	8640344181	5981362977	4771309960	5187072113	4999999837	2978049951	0597317328	1609631859
5024459455	3469083026	4252230825	3344685035	2619311881	7101000313	7838752886	5875332083	8142061717	7669147303
5982534904	2875546873	1159562863	8823537875	9375195778	1857780532	1712268066	1300192787	6611195909	2164201989

3809525720	1065485863	2788659361	5338182796	8230301952	0353018529	6899577362	2599413891	2497217752	8347913151
5574857242	4541506959	5082953311	6861727855	8890750983	8175463746	4939319255	0604009277	0167113900	9848824012
8583616035	6370766010	4710181942	9555961989	4676783744	9448255379	7747268471	0404753464	6208046684	2590694912
9331367702	8989152104	7521620569	6602405803	8150193511	2533824300	3558764024	7496473263	9141992726	0426992279
6782354781	6360093417	2164121992	4586315030	2861829745	5570674983	8505494588	5869269956	9092721079	7509302955
3211653449	8720275596	0236480665	4991198818	3479775356	6369807426	5425278625	5181841757	4672890977	7727938000
8164706001	6145249192	1732172147	7235014144	1973568548	1613611573	5255213347	5741849468	4385233239	0739414333
4547762416	8625189835	6948556209	9219222184	2725502542	5688767179	0494601653	4668049886	2723279178	6085784383
8279679766	8145410095	3883786360	9506800642	2512520511	7392984896	0841284886	2694560424	1965285022	2106611863
0674427862	2039194945	0471237137	8696095636	4371917287	4677646575	7396241389	0865832645	9958133904	7802759009

9465764078	9512694683	9835259570	9825822620	5224894077	2671947826	8482601476	9909026401	3639443745	5305068203
4962524517	4939965143	1429809190	6592509372	2169646151	5709858387	4105978859	5977297549	8930161753	9284681382
6868386894	2774155991	8559252459	5395943104	9972524680	8459872736	4469584865	3836736222	6260991246	0805124388
4390451244	1365497627	8079771569	1435997700	1296160894	4169486855	5848406353	4220722258	2848864815	8456028506
0168427394	5226746767	8895252138	5225499546	6672782398	6456596116	3548862305	7745649803	5593634568	1743241125
1507606947	9451096596	0940252288	7971089314	5669136867	2287489405	6010150330	8617928680	9208747609	1782493858
9009714909	6759852613	6554978189	3129784821	6829989487	2265880485	7564014270	4775551323	7964145152	3746234364
5428584447	9526586782	1051141354	7357395231	1342716610	2135969536	2314429524	8493718711	0145765403	5902799344
0374200731	0578539062	1983874478	0847848968	3321445713	8687519435	0643021845	3191048481	0053706146	8067491927
8191197939	9520614196	6342875444	0643745123	7181921799	9839101591	9561814675	1426912397	4894090718	6494231961

5679452080	9514655022	5231603881	9301420937	6213785595	6638937787	0830390697	9207734672	2182562599	6615014215
0306803844	7734549202	6054146659	2520149744	2850732518	6660021324	3408819071	0486331734	6496514539	0579626856
1005508106	6587969981	6357473638	4052571459	1028970641	4011097120	6280439039	7595156771	5770042033	7869936007
2305587631	7635942187	3125147120	5329281918	2618612586	7321579198	4148488291	6447060957	5270695722	0917567116
7229109816	9091528017	3506712748	5832228718	3520935396	5725121083	5791513698	8209144421	0067510334	6711031412
6711136990	8658516398	3150197016	5151168517	1437657618	3515565088	4909989859	9823873455	2833163550	7647918535
8932261854	8963213293	3089857064	2046752590	7091548141	6549859461	6371802709	8199430992	4488957571	2828905923
2332609729	9712084433	5732654893	8239119325	9746366730	5836041428	1388303203	8249037589	8524374417	0291327656
1809377344	4030707469	2112019130	2033038019	7621101100	4492932151	6084244485	9637669838	9522868478	3123552658
2131449576	8572624334	4189303968	6426243410	7732269780	2807318915	4411010446	8232527162	0105265227	2111660396

6655730925	4711055785	3763466820	6531098965	2691862056	4769312570	5863566201	8558100729	3606598764	8611791045
3348850346	1136576867	5324944166	8039626579	7877185560	8455296541	2665408530	6143444318	5867697514	5661406800
7002378776	5913440171	2749470420	5622305389	9456131407	1127000407	8547332699	3908145466	4645880797	2708266830
6343285878	5698305235	8089330657	5740679545	7163775254	2021149557	6158140025	0126228594	1302164715	5097925923
0990796547	3761255176	5675135751	7829666454	7791745011	2996148903	0463994713	2962107340	4375189573	5961458901
9389713111	7904297828	5647503203	1986915140	2870808599	0480109412	1472213179	4764777262	2414254854	5403321571
8530614228	8137585043	0633217518	2979866223	7172159160	7716692547	4873898665	4949450114	6540628433	6639379003
9769265672	1463853067	3609657120	9180763832	7166416274	8888007869	2560290228	4721040317	2118608204	1900042296
6171196377	9213375751	1495950156	6049631862	9472654736	4252308177	0367515906	7350235072	8354056704	0386743513
6222247715	8915049530	9844489333	0963408780	7693259939	7805419341	4473774418	4263129860	8099888687	4132604721

FROM 5001D TO 10000D

5695162396	5864573021	6315981931	9516735381	2974167729	4786724229	2465436680	0980676928	2382806899	6400482435
4037014163	1496589794	0924323789	6907069779	4223625082	2168895738	3798623001	5937764716	5122893578	6015881617
5578297352	3344604281	5126272037	3431465319	7777416031	9906655418	7639792933	4419521541	3418994854	4473456738
3162499341	9131814809	2777710386	3877343177	2075456545	3220777092	1201905166	0962804909	2636019759	8828161332
3166636528	6193266863	3606273567	6303544776	2803504507	7723554710	5859548702	7908143562	4014517180	6246436267
9456127531	8134078330	3362542327	8394497538	2437205835	3114771199	2606381334	6776879695	9703098339	1307710987
0408591337	4641442822	7726346594	7047458784	7787201927	7152807317	6790770715	7213444730	6057007334	9243693113
8350493163	1284042512	1925651798	0694113528	0131470130	4781643788	5185290928	5452011658	3934196562	1349143415
9562586586	5570552690	4965209858	0338507224	2648293972	8584783163	0577775606	8887644624	8246857926	0395352773
4803048029	0058760758	2510474709	1643961362	6760449256	2742042083	2085661190	6254543372	1315359584	5068772460

2901618766	7952406163	4252257719	5429162991	9306455377	9914037340	4328752628	8896399587	9475729174	6426357455
2540790914	5135711136	9410911939	3251910760	2082520261	8798531887	7058429725	9167781314	9699009019	2116971737
2784768472	6860849003	3770242429	1651300500	5168323364	3503895170	2989392233	4517220138	1280696501	1784408745
1960121228	5993716231	3017114448	4640903890	6449544400	6198690754	8516026327	5052983491	8740786680	8818338510
2283345085	0486082503	9302133219	7155184306	3545500766	8282949304	1377655279	3975175461	3953984683	3936383047
4611996653	8581538420	5685338621	8672523340	2830871123	2827892125	0771262946	3229563989	8989358211	6745627010
2183564622	0134967151	8819097303	8119800497	3407239610	3685406643	1939509790	1906996395	5245300545	0580685501
9567302292	1913933918	5680344903	9820595510	0226353536	1920419947	4553859381	0234395544	9597783779	0237421617
2711172364	3435439478	2218185286	2408514006	6604433258	8856986705	4315470696	5747458550	3323233421	0730154594
0516553790	6866273337	9958511562	5784322988	2737231989	8757141595	7811196358	3300594087	3068121602	8764962867

4460477464	9159950549	7374256269	0104903778	1986835938	1465741268	0492564879	8556145372	3478673303	9046883834
3634655379	4986419270	5638729317	4872332083	7601123029	9113679386	2708943879	9362016295	1541337142	4892830722
0126901475	4668476535	7616477379	4675200490	7571555278	1965362132	3926406160	1363581559	0742202020	3187277605
2772190055	6148425551	8792530343	5139844253	2234157623	3610642506	3904975008	6562710953	5919465897	5141310348
2276930624	7435363256	9160781547	8181152843	6679570611	0861533150	4452127473	9245449454	2368288606	1340841486
3776700961	2071512491	4043027253	8607648236	3414334623	5189757664	5216413767	9690314950	1910857598	4423919862
9164219399	4907236234	6468441173	9403265918	4044378051	3338945257	4239950829	6591228508	5558215725	0310712570
1266830240	2929525220	1187267675	6220415420	5161841634	8475651699	9811614101	0029960783	8690929160	3028840026
9104140792	8862150784	2451670908	7000699282	1206604183	7180653556	7252532567	5328612910	4248776182	5829765157
9598470356	2226293486	0034158722	9805349896	5022629174	8788202734	2092222453	3985626476	6914905562	8425039127

5771028402	7998066365	8254889264	8802545661	0172967026	6407655904	2909945681	5065265305	3718294127	0336931378
5178609040	7086671149	6558343434	7693385781	7113864558	7367812301	4587687126	6034891390	9562009939	3610310291
6161528813	8437909904	2317473363	9480457593	1493140529	7634757481	1935670911	0137751721	0080315590	2485309066
9203767192	2033229094	3346768514	2214477379	3937517034	4366199104	0337511173	5471918550	4644902636	5512816228
8244625759	1633303910	7225383742	1821408835	0865739177	1509682887	4782656995	9957449066	1758344137	5223970968
3408005355	9849175417	3818839994	4697486762	6551658276	5848358845	3142775687	9002909517	0283529716	3445621296
4043523117	6006651012	4120065975	5851276178	5838292041	9748442360	8007193045	7618932349	2292796501	9875187212
7267507981	2554709589	0455635792	1221033346	6974992356	3025494780	2490114195	2123828153	0911407907	3860251522
7429958180	7247162591	6685451333	1239480494	7079119153	2673430282	4418604142	6363954800	0448002670	4962482017
9289647669	7583183271	3142517029	6923488962	7668440323	2609275249	6035799646	9256504936	8183609003	2380929345

9588970695	3653494060	3402166544	3755890045	6328822505	4525564056	4482465151	8754711962	1844396582	5337543885
6909411303	1509526179	3780029741	2076651479	3942590298	9695946995	5657612186	5619673378	6236256125	2163208628
6922210327	4889218654	3648022967	8070576561	5144632046	9279068212	0738837781	4233562823	6089632080	6822246801
2248261177	1858963814	0918390367	3672220888	3215137556	0037279839	4004152970	0287830766	7094447456	0134556417
2543709069	7939612257	1429894671	5435784687	8861444581	2314593571	9849225284	7160504922	1242470141	2147805734
5510500801	9086996033	0276347870	8108175450	1193071412	2339086639	3833952942	5786905076	4310063835	1983438934
1596131854	3475464955	6978103829	3097164651	4384070070	7360411237	3599843452	2516105070	2705623526	6012764848
3084076118	3013052793	2054274628	6540360367	4532865105	7065874882	2569815793	6789766974	2205750596	8344086973
5020141020	6723585020	0724522563	2651341055	9240190274	2162484391	4035998953	5394590944	0704691209	1409387001
2645600162	3742880210	9276457931	0657922955	2498872758	4610126483	6999892256	9596881592	0560010165	5256375678

FROM 10001D TO 15000D

5667227966	1988578279	4848855834	3975187445	4551296563	4434803966	4205579829	3680435220	2770984294	2325330225
7634180703	9476994159	7915945300	6975214829	3366555661	5678736400	5366656416	5473217043	9035213295	4352916941
4599041608	7532018683	7937023488	8689479151	0716378529	0234529244	0773659495	6305100742	1087142613	4974595615
1384987137	5704710178	7957310422	9690666702	1449863746	4595280824	3694457897	7233004876	4765241339	0759204340
1963403911	4732023380	7150952220	1068256342	7471646024	3354400515	2126693249	3419673977	0415956837	5355516673
0273900749	7297363549	6453328886	9844061196	4961627734	4951827369	5588220757	3551766515	8985519098	6665393549
4810688732	0685990754	0792342402	3009259007	0173196036	2254756478	9406475483	4664776041	1463233905	6513433068
4495397907	0903023460	4614709616	9688688501	4083470405	4607429586	9913829668	2468185710	3188790652	8703665083
2431974404	7718556789	3482308943	1068287027	2280973624	8093996270	6074726455	3992539944	2808113736	9433887294
0630792615	9599546262	4629707062	5948455690	3471197299	6409089418	0595343932	5123623550	8134949004	3642785271

3831591256	8989295196	4272875739	4691427253	4366941532	3610045373	0488198551	7065941217	3524625895	4873016760
0298865925	7866285612	4966552353	3829428785	4253404830	8330701653	7228563559	1525347844	5981831341	1290019992
0598135220	5117336585	6407826484	9427644113	7639386692	4803118364	4536985891	7544264739	9882284621	8449008777
6977631279	5722672655	5625962825	4276531830	0134070922	3343657791	6012809317	9401718598	5999338492	3549564005
7099558561	1349802524	9906698423	3017350358	0440811685	5265311709	9570899427	3287092584	8789443646	0050410892
2669178352	5870785951	2983441729	5351953788	5534573742	6085902908	1765155780	3905946408	7350612322	6112009373
1080485485	2635722825	7682034160	5048466277	5045003126	2008007998	0492548534	6941469775	1649327095	0493463938
2432227188	5159740547	0214828971	1177792376	1225788734	7718819682	5462981268	6858170507	4027255026	3329044976
2778944236	2167411918	6269439650	6715157795	8675648239	9391760426	0176338704	5499017614	3641204692	1823707648
8783419689	6861181558	1587360629	3860381017	1215855272	6683008238	3404656475	8804051380	8016336388	7421637140

6435495561	8689641122	8214075330	2655100424	1048967835	2858829024	3670904887	1181909094	9453314421	8287661810
3100735477	0549815968	0772009474	6961343609	2861484941	7850171807	7930681085	4690009445	8995279424	3981392135
0558642219	6483491512	6390128038	3200109773	8680662877	9239718014	6134324457	2640097374	2570073592	1003154150
8936793008	1699805365	2027600727	7496745840	0283624053	4603726341	6554259027	6018348403	0681138185	5105979705
6640075094	2608788573	5796037324	5141467867	0368809880	6097164258	4975951380	6930944940	1515422221	9432913021
7391253835	5915031003	3303251117	4915696917	4502714943	3151558854	0392216409	7229101129	0355218157	6282328318
2342548326	1119128009	2825256190	2052630163	9114772473	3148573910	7775874425	3876117465	7867116941	4776421441
1112635835	5387136101	1023267987	7564102468	2403226483	4641766369	8066378576	8134920453	0224081972	7856471983
9630878154	3221166912	2464159117	7673225326	4335686146	1865452226	8126887268	4459684424	1610785401	6768142080
8850280054	1436131462	3082102594	1737562389	9420757136	2751674573	1891894562	8352570441	3354375857	5342698699

4725470316	5661399199	9682628247	2706413362	2217892390	3176085428	9437339356	1889165125	0424404008	9527198378
7386480584	7268954624	3882343751	7885201439	5600571048	1194988423	9060613695	7342315590	7967034614	9143447886
3604103182	3507365027	7859089757	8272731305	0488939890	0992391350	3373250855	9826558670	8924261242	9473670193
9077271307	0686917092	6462548423	2407485503	6608013604	6689511840	0936686095	4632500214	5852930950	0009071510
5823626729	3264537382	1049387249	9669933942	4685516483	2611341461	1068026744	6637334375	3407642940	2668297386
5220935701	6263846485	2851490362	9320199199	6882851718	3953669134	5222444708	0459239660	2817156551	5656661113
5982311225	0628905854	9145097157	5539002439	3153519090	2107119457	3002438801	7661503527	0862602537	8817975194
7806101371	5004489917	2100222013	3501310601	6391541589	5780371177	9277522597	8742891917	9155224171	8958536168
0594741234	1933984202	1874564925	6443462392	5319531351	0331147639	4911995072	8584306583	6193536932	9699289837
9149419394	0608572486	3968836903	2655643642	1664425760	7914710869	9843157337	4964883529	2769328220	7629472823

8153740996	1545598798	2598910937	1712621828	3025848112	3890119682	2142945766	7580718653	8065064870	2613389282
2994972574	5303328389	6381843944	7707794022	8435988341	0035838542	3897354243	9564755568	4095224844	5541392394
1000162076	9363684677	6413017819	6593799715	5746854194	6334893748	4391297423	9143365936	0410035234	3777065888
6778113949	8616478747	1407932638	5873862473	2889645643	5987746676	3847946650	4074111825	6583788784	5485814896
2961273998	4134427260	8606187245	5452360643	1537101127	4680977870	4464094758	2803487697	5894832824	1239292960
5829486191	9667091895	8089833201	2103184303	4012849511	6203534280	1441276172	8583024355	9830032042	0245120728
7253558119	5840149180	9692533950	7577840006	7465526031	4461670508	2768277222	3534191102	6341631571	4740612385
0425845988	4199076112	8725805911	3935689601	4316682831	7632356732	5417073420	8173322304	6298799280	4908514094
7903688786	8789493054	6955703072	6190095020	7643349335	9106024545	0864536289	3545686295	8531315337	1838682656
1786227363	7169757741	8302398600	6591481616	4049449650	1173213138	9574706208	8474802365	3710311508	9842799275

FROM 15001D TO 20000D

4426853277	9743113951	4357417221	9759799359	6852522857	4526379628	9612691572	3579866205	7340837576	6873884266
4059909935	0500081337	5432454635	9675048442	3528487470	1443545419	5762584735	6421619813	4073468541	1176688311
8654489377	6979566517	2796623267	1481033864	3913751865	9467300244	3450054499	5399742372	3287124948	3470604406
3471606325	8306498297	9551010954	1836235030	3094530973	3583446283	9476304775	6450150085	0757894954	8931393944
8992161255	2559770143	6858943585	8775263796	2559708167	7643800125	4365023714	1278346792	6101995585	2247172201
7772370041	7808419423	9487254068	0155603599	8390548985	7235467456	4239058585	0216719031	3952629445	5439131663
1345308939	0620467843	8778505423	9390524731	3620129476	9187497519	1011472315	2893267725	3391814660	7300089027
7689631148	1090220972	4520759167	2970078505	8071718638	1054967973	1001678708	5069420709	2232908070	3832634534
5203802786	0990556900	1341371823	6837099194	9516489600	7550493412	6787643674	6384902063	9640197666	8559233565
4639138363	1857456981	4719621084	1080961884	6054560390	3845534372	9141446513	4749407848	8442377217	5154334260
3066988317	6833100113	3108690421	9390310801	4378433415	1370924353	0136776310	8491351615	6422698475	0743032971
6746964066	6531527035	3254671126	6752246055	1199581831	9637637076	1799191920	3579582007	5956053023	4626775794
3936307463	0569010801	1494271410	0939136913	8107258137	8135789400	5599500183	5425118417	2136055727	5221035268
0373572652	7922417373	6057511278	8721819084	4900617801	3889710770	8229310027	9766593583	8758909395	6881485602
6322439372	6562472776	0378908144	5883785501	9702843779	3624078250	5270487581	6470324581	2908783952	3245323789
6029841669	2254896497	1560698119	2186584926	7704039564	8127810217	9913217416	3058105545	9880130048	4562997651
1212415363	7451500563	5070127815	9267142413	4210330156	6165356024	7338078430	2865525722	2753049998	8370153487
9300806260	1809623815	1613669033	4111138653	8510919367	3938352293	4588832255	0887064507	5394739520	4396807906
7086806445	0969865488	0168287434	3786126453	8158342807	5306184548	5903798217	9945996811	5441974253	6344399602
9025100158	8827216474	5006820704	1937615845	4712318346	0072629339	5505482395	5713725684	0232268213	0124767945
2264482091	0235647752	7230820810	6351889915	2692889108	4555711266	0396503439	7896278250	0161101532	3516051965
5904211844	9499077899	9200732947	6905868577	8787209829	0135295661	3978884860	5097860859	5701773129	8155314951
6814671769	5976099421	0036183559	1387778176	9845875810	4466283998	8060061622	9848616935	3373865787	7359833616
1338413385	3684211978	9389001852	9569196780	4554482858	4837011709	6721253533	8758621582	3101331038	7766827211
5726949518	1795897546	9399264219	7915523385	7662316762	7547570354	6994148929	0413018638	6119439196	2838870543
6777432242	7680913236	5449485366	7680000010	6526248547	3055861598	9991401707	6983854831	8875014293	8908995068
5453076511	6803337322	2651756622	0752695179	1442252808	1651716677	6672793035	4851542040	2381746089	2328391703
2754257508	6765511785	9395002793	3895920576	6827896776	4453184040	4185540104	3513483895	3120132637	8369283580
8271937831	2654961745	9970567450	7183320650	3455664403	4490453627	5600112501	8433560736	1222765949	2783937064
7842645676	3388188075	6561216896	0504161139	0390639601	6202215368	4941092605	3876887148	3798955999	9112099164
6464411918	5682770045	7424343402	1672276445	5893301277	8158686952	5069499364	6101756850	6016714535	4315814801
0545886056	4550133203	7586454858	4032402987	1709348091	0556211671	5468484778	0394475697	9804263180	9917564228
0987399876	6973237695	7370158080	6822904599	2123661689	0259627304	3067931653	1149401764	7376938735	1409336183
3216142802	1497633991	8983548487	5625298752	4238730775	5955595546	5196394401	8218409984	1248982623	6737714672
2606163364	3296406335	7281070788	7581640438	1485018841	1431885988	2769449011	9321296827	1588841338	6943468285
9006664080	6314077757	7257056307	2940049294	0302420498	4165654797	3670548558	0445865720	2276378404	6682337985
2827105784	3197535417	9501134727	3625774080	2134768260	4502285157	9795797647	4670228409	9956160156	9108903845
8245026792	6594205550	3958792298	1852648007	0683765041	8365620945	5543461351	3415257006	5974881916	3413595567
1964965403	2187271602	6485930490	3978748958	9066127250	7948282769	3895352175	3621850796	2977851461	8843271922
3223810158	7444505286	6523802253	2843891375	2738458923	8442253547	2653098171	5784478342	1582232702	0690287232
3300538621	6347988509	4695472004	7952311201	5043293226	6282727632	1779088400	8786148022	1475376578	1058197022
2630971749	5072127248	4794781695	7296142365	8595782090	8307332335	6034846531	8730293026	6596450137	1837542889
7557971449	9246540386	8179921389	3469244741	9850973346	2679332107	2686870768	0626399193	6196504409	9542167627
8409146698	5692571507	4315740793	8053239252	3947755744	1591845821	5625181921	5523370960	7483329234	9210345146
2643744980	5596103307	9941453477	8457469999	2128599999	3996122816	1521931488	8769388022	2810830019	8601654941
6542616968	5867883726	0958774567	6182507275	9929508931	8052187292	4610867639	9589161458	5505839727	4209809097
8172932393	0106766386	8240401113	0402470073	5085782872	4627134946	3685318154	6969046696	8693925472	5194139929
1465242385	7762550047	4852954768	1479546700	7050347999	5888676950	1612497228	2040303995	4632788306	9597624936
1510102436	5553522306	9061294938	8599015734	6610237122	3547891129	2547696176	0050479749	2806072126	8039226911
0277722610	2544149221	5765045081	2067717357	1202718024	2968106203	7765788371	6690910941	8074487814	0490755178

FROM 20001D TO 25000D

2038565390	9910477594	1413215432	8440625030	1802757169	6508209642	7348414695	7263978842	5600845312	1406593580
9041271135	9200419759	8513625479	6160632288	7361813673	7324450607	9244117639	9759746193	8358457491	5988097667
4470930065	4634242346	0634237474	6660804317	0126005205	5928493695	9414340814	6852981505	3947178900	4518357551
5412522359	0590687264	8786357525	4191128887	7371766374	8602766063	4960353679	4702692322	9718683277	1739323619
2007774522	1262475186	9833495151	0198642698	8784717193	9664976907	0825217423	3656627259	2844062043	0214113719
9227852699	8469884770	2323823840	0556555178	8908766136	0130477098	4386116870	5231055314	9162517283	7327286760
0724817298	7637569816	3354150746	0883866364	0693470437	2066886512	7568826614	9730788657	0156850169	1864748854
1679154596	5072342877	3069985371	3904300266	5307839877	6385032381	8215535597	3235306860	4301067576	0838908627
0498418885	9513809103	0423595782	4951439885	9011318583	5840667472	3702971497	8508414585	3085781339	1562707603
5639076394	7311455495	8322669457	0249413983	1634332378	9759556808	5683629725	3867913275	0555425244	9194358912

8405045226	9538121791	3191451350	0993846311	7740179715	1228378546	0116035955	4028644059	0249646693	0707769055
4810288502	0808580087	8115773817	1917417760	1733073855	4758006056	0143377432	9901272867	7253043182	5197579167
9296996504	1460706645	7125888346	9797964293	1622965520	1687973000	3564630457	9308840327	4807718115	5533090988
7025505207	6804630346	0865816539	4876951960	0440848206	5967379473	1680864156	4565053004	9881616490	5788311543
4548505266	0069823093	1577765003	7807046612	6470602145	7505793270	9620478256	1524714591	8965223608	3966456241
0519551052	2357239739	5128818164	0597859142	7914816542	6328920042	8160913693	7773722299	9833270820	8296995573
7727375667	6155271139	2258805520	1898876201	1416800546	8736558063	3471603734	2917039079	8639652296	1312801782
6797172898	2293607028	8069087768	6605932527	4637840539	7691848082	0410219447	1971386925	6084162451	1239806201
1318454124	4782050110	7987607171	5568315407	8865439041	2108730324	0201068534	1947230476	6667217498	6986854707
6781205124	7367924791	9315085644	4775379853	7997322344	5612278584	3296846647	5133365736	9238720146	4723679427

8700425032	5558992688	4349592876	1240075587	5694641370	5625140011	7971331662	0715371543	6006876477	3186755871
4878398908	1074295309	4106059694	4315847753	9700943988	3949144323	5366853920	9946879645	0665339857	3888786614
7629443414	0104988899	3160051207	6781035886	1166020296	1193639682	1349607501	1164983278	5635316145	1684576956
8710900299	9769841263	2665023477	1672865737	8579085746	6460772283	4154031144	1529418804	7825438761	7707904300
0156698677	6795760909	9669360755	9496515273	6349811896	4130433116	6277471233	8817406037	3174397054	0670310967
6765748695	3587896700	3192586625	9410510533	5843846560	2339179674	9267844763	7084749783	3365557900	7384191473
1988627135	2595462518	1604342253	7299628632	6749682405	8060296421	1463864368	6422472488	7283434170	4415734824
8183330164	0566959668	8667695634	9141632842	6414974533	3499994800	0266998758	8815935073	5781519588	9900539512
0853510357	2613736403	4367534714	1048360175	4648830040	7846416745	2167371904	8310967671	1344349481	9262681110
7399482506	0739495073	5031690197	3185211955	2635632584	3390998224	9862406703	1076831844	6607291248	7475403161

7969941139	7387765899	8685541703	1884778867	5929026070	0432126661	7919223520	9382278788	8098863359	9116081923
5355570464	6349113208	5918979613	2791319756	4909760001	3996234445	5350143464	2686046449	5862476909	4347048293
2941404111	4654092398	8344435159	1332010773	9441118407	4107684981	0663472410	4823935827	4019449356	6516108846
3125678529	7769734684	3030614624	1803585293	3159734583	0384554103	3701091676	7763742762	1021370135	4854450926
3071901147	3184857492	3318167207	2137279355	6795284439	2548156091	3728128406	3330393735	6242001604	5664557414
5881660521	6660873874	8047243391	2129558777	6390696903	7078828527	7538940524	6075849623	1574369171	1317613478
3882719416	8606625721	0368513215	6647800147	6752310393	5786068961	1125996028	1839309548	7090590738	6135191459
1819510297	3278755710	4972901148	7171897180	0469616977	7001791391	9613791417	1627070189	5846921434	3696762927
4591099400	6008498356	8425201915	5937037010	1104974733	9493877885	9894174330	3178534870	7603221982	9705797511
9144051099	4235883034	5463534923	4982688362	4043327267	4155403016	1950568065	4180939409	9820206099	9414021689

0900708213	3072308966	2119775530	6659188141	1915778362	7292746156	1857103721	7247100952	1423696483	0864102592
8874579993	2237495519	1221951903	4244523075	3513380685	6807354464	9951272031	7448719540	3976107308	0602699062
5807602029	2731455252	0780799141	8429063884	4373499681	4582733720	7266391767	0201183004	6481900024	1308350884
6584152148	9912761065	1374153943	5657211390	3285749187	6909441370	2090517031	4877734616	5287984823	5338297260
1361109845	1484182380	8120540996	1252745808	8109948697	2216128524	8974255555	1607637167	5054896173	0168096138
0381191436	1143992106	3800508321	4098760459	9309324851	0251682944	6726066613	8151745712	5597549535	8023998314
6982203613	3808284993	5670557552	4712902745	3977621404	9318201465	8008021566	5360677655	0878380430	4134310591
8046068008	3459113664	0834887408	0057412725	8670479225	8319127415	7390809143	8313845642	4150940849	1339180968
4025116399	1936853225	5573389669	5374902662	0923261318	8558915808	3245557194	8453875628	7861288590	0410600607
3746501402	6278240273	4696252821	7174941582	3317492396	8353013617	8653673760	6421667781	3773995100	6589528877

FROM 25001D TO 30000D

4276626368	4183068019	0804609849	8094697636	6733566228	2915132352	7888061577	6827815958	8669180238	9403330764
4191240341	2022316368	5778603572	7694154177	8826435238	1319050280	8701857504	7046312933	3537572853	8660588890
4583111450	7739429352	0199432197	1171642235	0056440429	7989208159	4307167019	8574692738	4865383343	6145794634
1759225738	9858800169	8014757420	5429958012	4295810545	6510831046	2972829375	8416116253	2562516572	4980784920
9989799062	0035936509	9347215829	6517413579	8491047111	6607915874	3698654122	2348341887	7229294463	3517865385
6731962559	8520260729	4767407261	6767145573	6498121056	7771689348	4917660771	7052771876	0119990814	4113058645
5779105256	8430481144	0261938402	3224709392	4980293355	0731845890	3553971330	8844617410	7959162511	7148648744
6861124760	5428673436	7090466784	6867027409	1881014249	7111496578	1772427934	7070216688	2956108777	9440504843
7528443375	1088282647	7197854000	6509704033	0218625561	4733211777	1174413350	2816088403	5178145254	1964320309
5760186946	4908868154	5285621346	9883554445	6024955666	8436602922	1951248309	1060537720	1980218310	1032704178

3866544718	1260397190	6884623708	5751808003	5327047185	6594994761	2424811099	9288679158	9690495639	4762460842
4065930948	6215076903	1498702067	3533848349	5508363660	1784877106	0809804269	2471324100	0946401437	3603265645
1845667924	5666955100	1502298330	7984960799	4988249706	1723674493	6122622296	1790814311	4146609412	3415935930
9585407913	9087208322	7335495720	8075716517	1876599449	8569379562	3875551617	5754380917	8052802946	4200447215
3962807463	6021132942	5591600257	0735628126	3873310600	5891065245	7080244749	3754318414	9401482119	9962764531
0680066311	8382376163	9663180931	4446712986	1552759820	1451410275	6006892975	0246304017	3514891945	7636078935
2855505317	3314164570	5049964438	9093630843	8744847839	6168405184	5273288403	2345202470	5685164657	1647713932
3775517294	7951261323	9822960239	4548579754	5865174587	8771331813	8752959809	4121742273	0035229650	8089177705
0682592488	2232215493	8048371454	7816472139	7682096332	0508305647	9204820859	2047549985	7320388876	3916019952
4091893894	5576768749	7308569559	5801065952	6503036266	1597506622	2508406742	8898265907	5106375635	6996821151

0949669744	5805472886	9363102036	7823250182	3237084597	9011154847	2087618212	4778132663	3041207621	6587312970
8112307581	5982124863	9807212407	8688781145	0165582513	6178903070	8608701989	7588980745	6643955157	4153631931
9198107057	5336633738	0382721527	9884935039	7480015890	5194208797	1130805123	3933221903	4662499171	6915094854
1401871060	3546037946	4337900589	0957721180	8044657439	6280618671	7861017156	7409676620	8029576657	7051291209
9079443046	3289294730	6159510430	9022214393	7184956063	4056189342	5130572682	9146578329	3340524635	0289291754
7087256484	2600349629	6116541382	3007731332	7298305001	6025672401	4185152041	8907011542	8857992081	2198449315
6999059182	0118197335	0012618772	8036812481	9958770702	0753240636	1259313438	5955425477	8196114293	5163561223
4966615226	1473539967	4051584998	6035529533	2924575238	8810136202	3476246690	5581643896	7863097627	3655047243
4864307121	8494373485	3006063876	4456627218	6661701238	1277156213	7974614986	1328744117	7145524447	0899714452
2885662942	4402301847	9120547849	8574521634	6964489738	9206240194	3518310088	2834802492	4908540307	7863875165

9113028739	5878709810	0772718271	8745290139	7283661484	2142871705	5317965430	7650453432	4600536361	4726181809
6997693348	6264077435	1999286863	2383508875	6683595097	2655748154	3194019557	6850437248	0010204137	4983187225
9677387154	9583997184	4490727914	1965845930	0839426370	2087563539	8216962055	3248032122	6749891140	2678528599
6734052420	3109179789	9905718821	9493913207	5343170798	0023736590	9853755202	3891164346	7185582906	8537118979
5262623449	2483392496	3424497146	5684659124	8918556629	5893299090	3523923333	3647435203	7077010108	4388003290
7598342170	1855422838	6161721041	7603011645	9187805393	6744747205	9985023582	8918336929	2233732399	9480437108
4196594731	6265482574	8099482509	9918330069	7656936715	9689364493	3488647442	1350084070	0660883597	2350395323
4017958255	7036016936	9909886711	3210979889	7070517280	7558551912	6993067309	9250704070	2455685077	8679069476
6126298082	2516331363	9952117098	4528092630	3759224267	4257559989	2892783704	7444521893	6320348941	5521044597
2618838003	0067761793	1381399162	0580627016	5102445886	9247649246	8919246121	2531027573	1390840470	0071435613

6231699237	1694848132	5542009145	3041037135	4532966206	3921054798	2439212517	2540132314	9027405858	9206321758
9494345489	0684639931	3757091034	6332714153	1622328055	2297297953	8018801628	5907357295	5416278867	6498274186
1642187898	8574107164	9069191851	1628152854	8679417363	8906653885	7642291583	4250067361	2453849160	6741373401
7357277995	6341043326	8835695078	1493137800	7362354180	0706191802	6732855119	1942676091	2210359874	6924117283
7493126163	3950012395	9924050845	4375698507	9570462226	6461900010	3500490183	0341535458	4283376437	8111988556
3187777925	3720116671	8539541835	9844383052	0376281944	0761594106	8207169703	0228515225	0573126093	0468984234
3315273213	1361216582	8080752126	3154773060	4423774753	5059522871	7440266638	9148817173	0864361113	8906942027
9088143119	4487994171	5404210341	2190847094	0802540239	3294294549	3878640230	5129271190	9751353600	0921971105
4120966831	1151632870	5423028470	0731206580	3262641711	6165957613	2723515666	6253667271	8998534199	8952368848
3099930275	7419916463	8414270779	8870887422	9277053891	2271724863	2202889842	5125287217	8260305009	9451082478

FROM 30001D TO 35000D

3572905691	9885554678	8607946280	5371227042	4665431921	4528176074	1482403827	8358297193	0101788834	5674167811
3989547504	4833931468	9630763396	6572267270	4339321674	5421824557	0625247972	1997866854	2798977992	3395790575
8189062252	5473582205	2364248507	8340711014	4980478726	6919901864	3882293230	5382318559	7328697809	2225352959
1017341407	3348847610	0556401824	2392192695	0620831838	1454698392	3664613639	8910121021	7709597670	4908305081
6547041946	6437131229	9692358895	3849301363	5657618610	6062228705	5994233716	3102127845	7446463989	7381885667
4626087948	2018647487	6727272220	6267646533	8099801966	8836809941	5907577685	2639865146	2533363124	5053640261
0569605513	1838131742	6118442018	9088853196	3569869627	9503673842	4313011331	7533053298	0201668881	7481342988
6815855778	1034323175	3064784983	2106297184	2518438553	4427620128	2345707169	8853051832	6179641178	5796088881
5032960229	0705614476	2209150947	3903594664	6916235396	8092013945	7817589108	8931992112	2600739281	4916948161
5273842736	2642980982	3406320024	4024495894	4561291670	4950823581	2487391799	6486411334	8032475777	5219708932

7722623494	8601504665	2681439877	0516153170	2669692970	4928316285	5042128981	4670619533	1970269507	2143782304
7687528028	7354126166	3917082459	2517001071	4180854800	6369232594	6201900227	8087409859	7719218051	5853214739
2653251559	0354102092	8466592529	9914353791	8253145452	9059841581	7637058927	9069098969	1116438118	7809435371
5213322614	4362531449	0127454772	6957393934	8154691631	1624928873	5747188240	7150399500	9446731954	3161938554
8520766573	8825139639	1635767231	5100555603	7263394867	2082078086	5373494244	0115799667	5073607111	5935133195
9197120948	9647175530	2453136477	0942094635	6969822266	7377520994	5168450643	6238242118	5353488798	9395673187
8066061078	8544000550	8276570305	5874485418	0577889171	9207881423	3511386629	2966717964	3468760077	0479995378
8338787034	8718021842	4373421122	7394025571	7690819603	0920182401	8842705704	6092622564	1783752652	6335832424
0661253311	5294234579	6556950250	6810018310	9004112453	7901533296	6156970522	3792103257	0693705109	0830789479
9990049993	9532215362	2748476603	6136776979	7856738658	4670936679	5885837887	9562594646	4891376652	1995882869

3380183601	1932368578	5585581955	5604215625	0883650203	3220245137	6215820461	8106705195	3306530606	0650105488
7167245377	9428313388	7163139559	6905832083	4168984760	6560711834	7136218123	2462272588	4199028614	2087284956
8796393254	6428534307	5301105285	7138296437	0999035694	8885285190	4029560473	4613113826	3878897551	7885604249
9874831638	2804046848	6189381895	9054203988	9872650697	6202019955	4841265000	5394428203	9301274816	3815853039
6439925470	2016727593	2857436666	1644110962	5663373054	0921951967	5148328734	8089574777	7527834422	1091073111
3518280460	3634719818	5655572957	1447476825	5285786334	9342858423	1187494400	0322969069	7758315903	8580393535
2135886007	9600342097	5473922967	3331064939	5601812237	8128545843	1760556173	3861126734	7807458506	7606304822
9409653041	1183066710	8189303110	8871728167	5195796753	4718853722	9309616143	2040063813	2246584111	1157758358
5811350185	6904781536	8938137718	4728147519	9835050478	1297718599	0847076219	7460588742	3256995828	8925350419
3795826061	6211842368	7685114183	1606831586	7994601652	0577405294	2305360178	0313357263	2670547903	3840125730

5912339601	8801378254	2192709476	7337191987	2873852480	5742124892	1183470876	6296672072	7232565056	5129333126
0595057777	2754247124	1648312832	9820723617	5057467387	0128209575	5443059683	9555568686	1188397135	5220844528
5264008125	2027665557	6774959696	2661260456	5245684086	1392382657	6858338469	8499778726	7065551918	5446869846
9478495734	6226062942	1962455708	5371272776	5230989554	5019303773	2166649182	5781546772	9200521266	7143463209
6378918523	2321501897	6126034373	6840671941	9303774688	0999296877	5824410478	7812326625	3181845960	4538535438
3911449677	5312864260	9252115376	7325886672	2604042523	4910870269	5809964759	5805794663	9734190640	1003636190
4042033113	5793365424	2630356145	7009011244	8008900208	0147805660	3710154122	3288914657	2239314507	6071670643
5568274377	4396578906	7972687438	4730763464	5167756210	3098604092	7170909512	8086309029	7385044527	1828927496
8921210667	0081648583	3955377359	1913695015	3162018908	8874842107	9870689911	4804669270	6509407620	4650277252
8650728905	3285485614	3316081269	3005693785	4178610969	6920253886	5034577183	1766868859	2368148847	5276498468

8219497397	2970773718	7188400414	3231276365	0481453112	2850990020	7424092558	5925292610	3021067368	1543470152
5234878635	1643976235	8604191941	2969769040	5264832347	0099111542	4260127343	8022089331	0966863678	9869497799
4001260164	2276092608	2349304118	0643829138	3473546797	2539926233	8791582998	4864592717	3405922562	0749105308
5315371829	1168163721	9395188700	9577881815	8685046450	7699343940	9874335144	3162633031	7247747486	8979182092
3948083314	3970840673	0840795893	5810896656	4775859905	5637695252	3265361442	4780230826	8118310377	3588708924
0613031336	4773710116	2821461466	1679404090	5186152603	6009252194	7218890918	1073358719	6414214447	8654899528
5823439470	5007983038	8538860831	0357193060	0277119455	8021911942	8999227223	5345870756	6246926177	6631788551
4435021828	7026685610	6650035310	5021631820	6017609217	9846849368	6316129372	7951873078	9726373537	1715025637
8733579771	8081848784	5886650433	5824377004	1477104149	3492743845	7587107159	7315594394	2641257027	0965125108
1155482479	3940359768	1188117282	4721582501	0949609662	5393395380	9221955919	1818855267	8062149923	1727631632

FROM 35001D TO 40000D

1833989693 8075616855 9117529984 5013206712 9392404144 5938623988 0938124045 2191484831 6462101473 8918251010
9096773869 0664041589 7361047643 6500068077 1056567184 8628149637 1118832192 4456639458 1449148616 5500495676
9826903089 1118568798 6929470513 5248160917 4324301538 3684707292 8989828460 2223730145 2655679898 6277679680
9146979837 8268764311 5988321090 4371561129 9766521539 6354644208 6919756737 0005738764 9784376862 8768179249
7469438427 4652563163 2300555130 4174227341 6464551278 1278457777 2457520386 5437542828 2567141288 5834544435
1325620544 6424101103 7955464190 5811686230 5964476958 7054072141 9852121067 3433241075 6767575818 4569906930
4604752277 0167005684 5439692340 4171108988 8993416350 5851578873 5343081552 0811772071 8803791040 4698306957
8685473937 6564336319 7978680367 1873079693 9242363214 4845035477 6315670255 3900654231 1792015346 4977929066
2415083288 5839529054 2637687668 9688050333 1722780018 5885069736 2324038947 0047189761 9347344308 4374437599
2503417880 7972235859 1342458131 4404984770 1732361694 7197657153 5319775499 7162785663 1190469126 0918259124

9890367654 1769799036 2375528652 6375733763 5269693443 5440047306 7198868901 9681474287 6779086697 9688522501
6369498567 3021752313 2529265375 8964151714 7955953878 4278499866 4563028788 3196209983 0494519874 3963690706
8276265748 5810439112 2326187940 5994155406 3270131989 8957037611 0532360629 8674803779 1537675115 8304320849
8720920280 9297526498 1256916342 5000522908 8726469252 8466610466 5392171482 0801305022 9805263783 6426959733
7070539227 8915351056 8883938113 2497570713 3102950443 0346715989 4487868471 1643832805 0692507766 2745001220
0352620370 9466023414 6489983902 5258883014 8678162196 7751945831 6771876275 7200505439 7944124599 0077115205
1546199305 0983869825 4284640725 5540927403 1325716326 4079293418 3342147090 4125425335 2324802193 2277075355
5467958716 3835875018 1593387174 2360615511 7101312352 5633485820 3651461418 7004920570 4372018261 7331947157
0086757853 9336078622 7395581857 9758725874 4102542077 1054753612 9404746010 0094095444 9596628814 8691590389
9071865980 5636171376 9222729076 4197755177 7201042764 9694961105 6220592502 4202177042 6962215495 8726453989

2276976603 1052498085 5759471631 0758701332 0886146326 6412591148 6338812202 8444069416 9488261529 5776253250
1987035987 0674380469 8219420563 8125583343 6421949232 2759372212 8905642094 3082352544 0841108645 4536940496
9271494003 3197828613 1818618881 1118408257 8659287574 2638445005 9944229568 5864604810 3301538891 1499486935
4360302218 1094346676 4000022362 5505736312 9462629609 6198760564 2599639461 3869233083 7196265954 7392346241
3459779574 8524647837 9807956931 9865081597 7675350553 9189911513 3525229873 6112779182 7485420086 8953965835
9421963331 5028695611 9201229888 9887006079 9927954111 8826902307 8913107603 6176347794 8943203210 2773359416
9086500719 3280401716 3840644987 8717537567 8118532132 8408216571 1075495282 9497493621 4608215583 2056872321
8557406516 1096274874 3750980922 3021160998 2633033915 4694946444 9100451528 0925089745 0748967603 2409076898
3652940657 9201983152 6541065813 6823791984 0906457124 6894847020 9357761193 1399802468 1340520039 4781949866
2026240089 0215016616 3813538381 5150377350 2296607462 7952910384 0686855690 7015751662 4192987244 4827194293

3100485482 4454580718 8976330032 3252582158 1280327467 9620028147 6243182862 2171054352 8983482082 7345168018
6131719593 3247110746 6222850871 0666117703 4653528395 7762599774 4672185715 8161264111 4327179434 7885990892
8084866949 1413909771 6736900277 7585026866 4654056595 0394867841 1107901161 0400857274 4562938425 4941675946
0548711723 5946429105 8509099502 1495879311 2196135908 3158826206 8233215615 3086833730 8381732793 2819698387
5087083483 8804638847 8441884003 1847126974 5437093732 9836240287 5197920802 3218787448 8287284372 7378017827
0080587824 1074935751 4889978911 7397461293 2035108143 2703251409 0304874622 6294234432 7571260086 6425083331
8768865075 6429271605 5252895449 2153765175 1492196367 1810494353 1785838345 3865255656 6406572513 6357506435
3236508936 7904317025 9787817719 0314867963 8408288102 0946149007 9715137717 0990619549 6964007086 7667102330
0486726314 7551053723 1757114322 3174114116 8062286420 6388906210 1923552235 4671166213 7499693269 3217370431
0598722503 9456574924 6169782609 7025335947 5020913836 6737728944 3869640002 8110344026 0847128990 0074680776

4844088711 3413525033 6787731679 7709372778 6821661178 6534423173 2264637847 6978751443 3209534000 1650692130
5464768909 8505020301 5044880834 2618452087 3053097318 9492916425 3229336124 3151430657 8264070283 8984098416
0295030924 1897120971 6016492656 1341343342 2298827909 9217860426 7981245728 5345801338 2609958771 7811310216
7340256562 7440072968 3406619848 0676615805 0216918337 2368039902 7931606420 4368120799 0031626444 9146190219
4582296909 9212278855 3948783538 3056468648 8165556229 4315673128 2743908264 5061162894 2803501661 3366978240
5177015521 9626522725 4558507386 4058529983 0379180350 4328767038 0925216790 7571204061 2375963276 8567484507
9151147313 4400018325 7034492090 9712435809 4479004624 9431345502 8900680648 7042935340 3743603262 5820535790
1183956490 8935434510 1342969617 5452495739 6062149028 8728932792 5206965353 8639644322 5388327522 4996059869
7475988232 9916263545 9733244451 6375533437 7492928990 5811757863 5555562693 7426910947 1170021654 1171821975
0519831787 1371060510 6379555858 8905568852 8879890847 5091576463 9074693619 8815078146 8526213325 2473837651

FROM 40001D TO 45000D

1929901561	0918977792	2008705793	3964638274	9068069876	9168197492	3656242260	8715417610	0430608904	3779766785
1966189140	4144925270	4808819714	9880154205	7787006521	5940092897	7760133075	6847966992	9554336561	3984773806
0394368895	8876460549	8387147896	8482805384	7017308711	1776115966	3505039979	3438693391	1978988710	9156541709
1330826076	4740630571	1411098839	3880954814	3782847452	8838368079	4188843426	6622207043	8722887413	9478010177
2139228191	1992365405	5163958934	7426395382	4829609036	9002883593	2774585506	0801317988	4071624465	6399794827
5783650195	5142215513	3928197822	6984278638	3916797150	9126241054	8725700924	0700454884	8569295044	8110738087
9965474815	6891393538	0943474556	9721289198	2717702076	6613602489	5814681191	3361412125	8783895577	3571949863
1721084439	8901423948	4966592517	3138817160	2663261931	0653665350	4147307080	4414939169	3632623737	6777709585
0313255990	0957627319	5730864804	2467701212	3270205337	4266705314	2448208168	1303063973	7873664248	3672539837
4876909806	0218278578	6216512738	5635132901	4890350988	3270617258	9325753639	9397905572	9175160097	6154590447

7169226580	6315111028	0384360173	7474215247	6085152099	0161585823	1257159073	3421736576	2671423904	7827958728
1505095633	0928026684	5893764964	9770232973	6413190609	8274063353	1089792464	2421345837	4090116939	1964250459
1288134034	9881063540	0887596820	0544083643	8651661788	0557608956	8967275315	3808194207	7332597917	2784376256
6118431989	1025007491	8290864751	4979400316	0703845549	4653859460	2745244746	6812314687	9434416109	9333890899
2638411847	4252570445	7251745932	5738989565	1857165759	6148126602	0310797628	2541655905	0604247911	4016957900
3383565748	6925280074	3025623419	4982864679	1447632277	4005529460	9039401775	3633565547	1931000175	4300475047
1914489984	1040015867	9461792416	1001645471	6551337074	0739502604	4276953855	3834397550	5488710997	8520540117
5169747581	3449260794	3368954378	3221172450	6873442319	8987884412	8542064742	8097356258	0706698310	6979935260
6933921356	8588139121	4807354728	4632277849	0808700246	7776303605	5512323866	5629517885	3719673034	6347012229
3958160679	2509153217	4890308408	8651606111	9011498443	4123501246	4692802880	5996134283	5118847154	4977127847

3361766285	0621697787	1774382436	2565711779	4500644777	1837022199	9106695021	6567576440	4499794076	5037999954
8450027106	6598781360	3802314126	8369057831	9046079276	5297277694	0436130230	5178708054	6511542469	3952651271
0105292707	0306673024	4471259739	3995051462	8404767431	3637399782	5918454117	6413327906	4606365841	5292701903
0276017339	4748669603	4869497654	1752429306	0407270050	5903950314	8522921392	5755948450	7886797792	5253931765
1564161971	6844352436	9794447355	9642606333	9105512682	6061595726	2170366985	0647328126	6724521989	0605498802
8078288142	9796336696	7441248059	8219214633	9565745722	1022986775	9974673812	6069367069	1340815594	1201611596
0190237753	5255563006	0624798326	1249881288	1929373434	7686268921	9239777833	9107331065	8825681377	7172328315
3290825250	9273304785	0724977139	4483338925	5208117560	8452966590	5539409655	6854170600	1179857293	8139982583
1929367910	0391844099	2865756059	9359891000	2969864460	9747147184	7010153128	3762631146	7742091455	7404181590
8800064943	2378558393	0853082830	5476076799	5243573916	3122188605	7549673832	2431956506	5546085288	1201902363

6447127037	4863442172	7257879503	4284663129	4491631847	5347531435	0413920961	0879605773	0987201352	4840750576
3719925365	0470908582	5139368634	6386336804	2891767107	6021111598	2887553994	0120076013	9470336617	9371539630
6139863655	4922137415	9790511908	3588290097	6566473007	3387931467	8913181465	1093167615	7582135142	4860442292
4453041131	6065270097	4330088499	0346754055	1864067734	2603583409	6086055337	4736276093	5658853109	7609942383
4738222208	7292464497	6845605795	6251676557	4088410321	7313456277	3585605235	8236389532	0385340248	4227337163
9123973215	9954408284	2166663602	3296545694	7035771848	7344203422	7706653837	3875061692	1276801576	6181095420
0977083636	0436111059	2409117889	5403380214	2652394892	9686439808	9261146354	1457153519	4342850721	3534530183
1587562827	5733898268	8985235577	9929572764	5229391567	4775666760	5108788764	8453493636	0682780505	6462281359
8885879259	9409464460	4170520447	0046315137	9754317371	8775603981	5962647501	4109066588	6616218003	8266989961
9655805872	0863972117	6995219466	7898570117	9833244060	1811575658	0742841829	1061519391	7630059194	3144346051

5404771057	0054339000	1824531177	3371895585	7603607182	8605063564	7997900413	9761808955	3636696031	6219311325
0223851791	6720551806	5926351803	6251214575	9262383693	4822266589	5576994660	4919381124	8660909979	8128571823
4940066155	5219611220	7203092277	6462009993	1524427358	9488710576	6238946938	8944649509	3960330454	3408421024
6240104872	3328750081	7491798755	4387938738	1439894238	0117627008	3719605309	4383940063	7561164585	6094312951
7597713935	3960743227	9248922126	7045808183	3137641658	1826956210	5872892447	7400359470	0926866265	9651422050
6300785920	0248829186	0839743732	3538490839	6432614700	0532423540	6470420894	9921025040	4726781059	0836440074
6638002087	0126664209	4571817029	4675227854	0074508552	3777208905	8168391844	6592829417	0182882330	1497155423
5235911774	8186285929	6760504820	3864343108	7795628929	2540563894	6621948268	7110428281	6389397571	1757786915
4301650586	0296521745	9581988878	6804081103	2843273986	7198621306	2055598552	6603640504	6282152306	1545944744
8990883908	1999738747	4529698107	7620148713	4000122535	5222466954	0931521311	5337915798	0269795557	1050850747

FROM 45001D TO 50000D

3874750758	0687653764	4578252443	2638046143	0428892359	3485296105	8269382103	4980004052	4840708440	3561167817
1705128133	7880570564	3450616119	3304244407	9826037795	1198548694	5591520519	6009304127	1007277849	3015550388
9536033826	1929343797	0818743209	4991415959	3396368110	6275572952	7800425486	3060054523	8391510689	9891357882
0019411786	5356821491	1852820785	2130125518	5184937115	0342215954	2244511900	2073935396	2740020811	0465530207
9328672547	4054365271	7595893500	7163360763	2161472581	5407642053	0200453401	8357233829	2661915308	3540951202
2632916505	4426123619	1970516138	3935732669	3760156914	4299449437	4485680977	5696303129	5887191611	2929468138
4936338647	3927476012	2696415884	8900965717	0861605981	4720446742	8664208765	3347998582	2209061980	2173211614
2304194777	5499073873	8567941189	8246609130	9169177227	4207233367	6350326783	4058630193	0193242996	3972044451
7928812285	4478211953	5308989101	2534297552	4727635730	2262813820	9180743974	8671453590	7786335301	6082155991
1314144205	0914472935	3502223081	7193663509	3468658586	5631485557	5862447818	6201087118	8976065296	9899269328

1787055764	3514338206	0141077329	2610634315	2533718224	3385263520	2177354407	1528189813	7698755157	5745469397
2715048846	9793619500	4777209705	6179391382	8989845327	4262272886	4710888327	0173723258	8182446584	3624958059
2560338105	2156062061	5571329915	6084892064	3403033952	6226345145	4283678698	2880742514	2256745180	6184149564
6861116354	0497189768	2154227722	4794740335	7152743681	9409892050	1136534001	2384671429	6551867344	1537416150
4256325671	3430247655	1252192180	3578016924	0326699541	7460875924	0920700466	9340396510	1781348578	3569444076
0470232540	7555577647	2845075182	6890418293	9661133101	6013111907	7398632462	7782190236	5066037404	1606724962
4901374332	1724645409	7412995570	5291424382	0807609836	4823465973	8866913499	1978401310	8015581343	9791948528
3043673901	2482082444	8141280954	4377389832	0059864909	1595053228	5791457688	4962578665	8859991798	6752055455
8099004556	4611787552	4937012455	3217170194	2828846174	0273664997	8475508294	2280202329	0122163010	2309772151
5694464279	0980219082	6689868834	2630716092	0791408519	7695235553	4886577434	2527753119	7247430873	0436195113

9611908003	0255878387	6442060850	4473063129	9277888942	7291897271	6989057592	5244679660	1897074829	6094919064
8764693702	7507738664	3239191904	2254290235	3189233772	9316673608	6996228032	5571853089	1928440380	5071030064
7768478632	4319100022	3929785255	3723755662	1364474009	6760539439	8382357646	0699246526	0089090624	1059042154
5392790441	1529580345	3345002562	4410100635	9530039598	8644661695	9562635187	8060688513	7234627079	9732723313
4693971456	2855426154	6765063246	5676620279	2452085813	4771760852	1691340946	5203076733	9184114750	4140168924
1213198268	8156866456	1485380287	5393311602	3229255561	8941042995	3356400957	8649534093	5115266454	0244187759
4931693056	0448686420	8627572011	7231952640	5023099774	5676478384	8897346431	7215980626	7876718380	0524769688
4084989185	0861490034	3240347674	2686245952	3958903585	8213500645	0998178244	6360873177	5437885967	7672919526
1112138591	9472545140	0301180503	4378752776	6440276261	8941017576	8726804281	7662386068	0477885242	8874302591
4524707395	0546525135	3394595987	8961977891	1041890292	9438185672	0507096460	6263541732	9446495766	1265195349

5701860015	4126239622	8641389779	6733329070	5673769621	5649818450	6842263690	3678495559	7002607986	7996261019
0393312637	6855696876	7029295371	1625280055	4310078640	8728939225	7145124811	3577862766	4902425161	9902774710
9033593330	9304948380	5978566288	4478744146	9841499067	1237647895	8226329490	4679812089	9848571635	7108783119
1848630254	5016209298	0582920833	4813638405	4217200561	2198935366	9371336733	3924644161	2522319694	3471206417
3754912163	5700857369	4397305979	7097197266	6664226743	1117762176	4030686813	1035189911	2271339724	0368870009
9686292254	6465006385	2886203938	0050477827	6912835603	3725482557	9391298525	1506829969	1077542576	4748832534
1412132800	6267170940	0909822352	9657957997	8030182824	2849022147	0748111124	0186076134	1515038756	9830918652
7806588966	8236252393	7845272634	5304204188	0250844236	3190383318	3845505223	6799235775	2929106925	0432614469
5010986108	8899914658	5518818735	8252816430	2520939285	2580779697	3762084563	7482114433	9881627100	3170315133
4402309526	3519295886	8069082135	5853680161	0002137408	5115448491	2685841268	6958991741	4913382057	8492800698

2551957402	0181810564	1297250836	0703568510	5533178784	0829000041	5525118657	7945396331	7538532092	1497205266
0783126028	1961164858	0986845875	2512999740	4092797683	1766399146	5538610893	7587952214	9717317281	3151793290
4431121815	8710235187	4075722210	0123768721	9447472093	4931232410	7065080618	5623725267	3254073332	4875754482
9675734500	1932190219	9119960797	9893733836	7324257610	3938985349	2787774739	8050808001	5544764061	0535222023
2540944356	7718794565	4304067358	9649101761	0775948364	5408234861	3025471847	6485189575	8366743997	9150851285
8020607820	5544629917	2320202822	2914886959	3997299742	9747115537	1858924238	4938558585	9540743810	4882624648
7880533042	7146301194	1589896328	7926783273	2245610385	2197011130	4665871005	0008328517	7311776489	7352309266
6123458887	3102883515	6264460236	7199664455	4727608310	1187883891	5114934093	9344750073	0258558147	5619088139
8752357812	3313422798	6650352272	5367171230	7568610450	0454897036	0079569827	6263923441	0714658489	5780241408
1584052295	3693749971	0665594894	4592462866	1996355635	0652623405	3394391421	1127181069	1052290024	6574236041

FROM 50001D TO 55000D

3009369188	9255865784	6684612156	7955425660	5416005071	2766417660	5687427420	0329577160	6434486062	0123982169
8271723197	8268166282	4993871499	5449137302	0518436690	7672357740	0053932662	6227603236	5975171892	5901801104
2903842741	8550789488	7438832703	0632832799	6300720069	8012244365	1163940869	2222074532	0244624121	1558043545
4206421512	1585056896.	1573564143	1306888344	3185280853	9759277344	3365538418	8340303517	8229462537	0201578215
7373265523	1857635540	9895403323	6382319219	8921711774	4946940367	8296185920	8034038675	7583411151	8824177439
1450773663	8407188048	9358256868	5420116450	3135763335	5509440319	2367203486	5101056104	9872726472	1319865434
3545040913	1859513145	1812764373	1043897250	7004981987	0521762724	9406521461	9959232142	3144397765	4670835171
4749367986	1865527917	1582408065	1063799500	1842959387	9915835017	1580759883	7849622573	9851212981	0326379376
2183224565	9423668537	6799113140	1080431397	3233544909	0824910499	1433258432	9882103398	4698141715	7560108297
0658306521	1347076803	6806953229	7199059990	4451209087	2757762253	5104090239	2888779424	6304832803	1913271049

5478599180	1969678353	2146444118	9260631526	6181674431	9355081708	1875477050	8026540252	9410921826	4858213857
5266881555	8411319856	0022135158	8872103656	9608751506	3187533002	9421186822	2189377554	6027227291	2905042922
5978771066	7873840000	6167721546	3844129237	1193521828	4998243509	2089180168	5572798156	4218581911	9749098573
0570332667	6464607287	5743056537	2602768982	3732597450	8447964954	5648030771	5981539558	2777913937	3601717422
9960273531	0276871944	9444917939	7851446315	9731443535	1850491413	9415573293	8204854212	3508173912	5497498193
0871439661	5132942045	9193801062	3142177419	9184060180	3479498876	9105155790	5554806953	8785400664	5337598186
2846419905	2204528033	0626369562	6490910827	6271159038	5699505124	6529996062	8554438383	3032763859	9800792922
8466595035	5121124528	4087516229	0602620118	5777531374	7949362055	4964010730	0134885315	0735487353	9056029089
3352640071	3274732621	9603117734	3394367338	5759124508	1493357369	1166454128	1788171454	0230547506	6713651825
8284898099	5121391939	9563324133	6556777098	0030819102	7204099714	8687418134	6670060940	5102146269	0280449159

6465453301	0775469541	3088714165	3125448130	6119240782	1188690056	0277818242	3502269618	9344352547	6335735364
8561936325	4417756613	9817039306	3287216690	5722259745	2091929172	6219984440	9646158269	4563802395	0283712168
6446561785	2355651641	2771282691	8688615572	7162014749	3405227694	6595712198	3149433816	2211400693	6307430444
1732847861	0177774383	7977037231	7952554341	0722344551	2555589998	6461838767	6490397246	1167959018	1000350989
2864120419	5163551108	7632042676	1297982652	9425882951	1412758412	6273279079	8807559751	8515768412	6474220947
9721843309	3529726652	1001566251	4552994745	1276315509	1763673025	9462132930	1904028379	5424632325	8550301096
7069227202	2707486341	9005438302	6506812141	4213505715	4175057508	6399076739	4633514620	9082888934	9383764393
9925690060	4067311422	0933121959	3620298297	2351163259	3867722414	7791162957	2780752395	0562515816	0313335938
2311500518	6268905306	5836812998	8108663263	2719806112	7154885879	8093487912	9137074982	3057592909	1862939195
0147211975	8606727009	2547718025	7503377307	9939713453	9532646195	2699965963	8565491759	0458333585	7991020127

1320458390	3200853878	8816336376	8518208372	7885131175	2277696097	8796214237	2162545214	5912818317	9821604411
1311671406	9148271709	8101545778	1939202311	5638719508	0502467972	5792497605	7726259133	2855972637	1211201905
7207714091	4864507409	4926718035	8151575715	1405039761	0963846755	5692989703	8354731410	0223802583	4687673501
2977541327	9532060971	1545064842	1218593649	0997917766	8747744818	8287063231	5515865032	8981642282	8823274686
6106592732	1979071623	8464215348	9852476216	7890502609	9804526648	3929542357	2873439776	8049577409	1449538391
5755654854	5905897649	5198513801	0079580107	8375994577	5299196700	5476022525	5203445398	8712538780	1719607181
6407812484	7847257912	4078245443	6168234523	9570689514	2722697504	3187363326	3011103053	4233358216	0933319121
8806608268	3414289104	1517324721	6053355849	9932245487	3077882290	5252324234	8615315209	7693846104	2582849714
9634753418	3756200301	4915703279	6853018686	3157248840	1526639835	6895636346	5743532178	3493199825	5421173084
6774529708	5839507616	4582296303	2442432823	7737450517	0285606980	6788952176	8198156710	7816334052	6675953942

4926280756	9683261074	9532339053	6223090807	0814559198	3735537774	8742029039	0181429373	1152933464	4468151212
9450975965	3430628421	5319445727	1186149000	1765055817	7095302468	8752632501	1970520947	6159416768	7277844720
0019278913	7251841622	8577837922	8443908430	1181121496	3664246590	3363419454	0657183544	7719124466	2125939265
6620306888	5200555991	2123536371	8226922531	7814587925	9375044144	8933981608	6579008761	6502463519	7045828895
4817937566	8104647461	4105142498	8702521399	3687050937	2305447734	1126413548	9280684105	9107716677	8212383328
1026218558	7751312721	1793444482	0144042574	5083063944	7383637939	0628300897	3306241380	6145894142	2769474793
1665717623	1824721683	5067807648	7573420491	5576282175	8397297513	4478990696	5895325489	4033561561	3167403276
4724692125	0575911625	1529654568	5446334981	1431767025	7295661844	7754874693	7846423373	7238981920	6620485118
9437886822	4807279352	0225017965	4534375727	4163910791	9729529508	1294292220	5347717304	1844779156	7399173841
8311710362	5243957161	5271466900	5814700002	6330104526	4354786590	3290733205	4683388720	7873544476	2647925297

FROM 55001D TO 60000D

6901709120	0787418373	6735087713	3769776834	9634425241	9949951388	3150748775	3743384945	8259765560	9965559543
1804092017	8497184685	4973706962	1208852437	7013853757	6814166327	2241263442	3982152941	6453780004	9250726276
5150789085	0712659970	3670872669	2764308377	2296859851	6912230503	7462744310	8529343052	7307886528	3977335246
0174635277	0320593817	9125396915	6210636376	2588293757	1373840754	4064689647	8310070458	0613446731	2715911946
0843593582	5987782835	2665311510	6504162329	5329047772	1740835593	4972375855	2138048305	0900096466	7608830154
0612824308	7406455944	3185341375	5220166305	8121110334	5312074508	6824339432	1590435944	3031243122	7471385842
0303901060	7094031523	5556172767	9941600203	9397509989	7629335325	8555756248	0899669182	9864222677	5023601932
5797472674	2578211119	7347094023	5745722227	1212526852	3842958742	7350156366	0093188045	4933389897	4157149054
4182559738	0808715652	8143010267	0460284316	8192303925	3529779576	5862414392	7015497408	7927313105	1636119137
5770089295	6482332364	8298263024	6079758757	6774537716	0102490804	6243018565	2416175665	5600160859	1215345562

6760219268	9982855377	8725831451	4408265458	3484409478	4631787773	7479465358	0169960779	4055687011	9232860804
1130904629	3508718271	2593466871	2766694873	8998245985	2778649956	9165464029	4589350649	6433580982	4765965165
1420909867	5520380830	9203230487	3427034682	8875160407	1546653834	6196112230	1375945157	9252696743	6425319273
9003603860	8236450762	6988274976	1872357547	6762889950	7521148048	5252795084	5033958570	8381304769	3788132112
3674281319	4879502280	6632017002	2460331989	6719706491	6374117585	4851878484	0120548446	7258885140	1562725019
8217190669	6081262778	5485964818	3696214107	2171421498	6361918774	7545096503	0895709947	0934337856	9816744658
2826791194	0611956037	8453978558	3924076127	6344105766	7510243075	5981455278	6167815949	6570625597	5507430652
1085301597	9080733437	3607943286	6757890533	4836695554	8680391343	3720156498	8342208933	9997164147	9746938696
9054800891	9306713805	7171505857	3071488156	4992071408	6758259602	8760564597	8242377024	2469805328	0566327870
4192676846	7116266879	4634869504	6450742021	9373945259	2626686135	5294062478	1361206202	6364981999	9949840514

3868285258	9563422643	2870766329	9304891723	4007254717	6418868535	1372332667	8779217383	4754148002	2803392997
3579361524	1275582956	9276837231	2347989894	4627433045	4566790062	0324205163	9628258844	3085438307	2014956721
0646053323	8537203143	2421126074	2448584509	4580494081	8209276391	4000854042	2023556260	2185643489	9414543995
0410980591	8179488826	2805206644	1086319001	6885681551	6922948620	3010738897	1810077092	9059048074	9092427141
0189335428	1842999598	8169660993	8369616443	8152887721	4085268088	7574882932	5873580990	5670755817	0179491619
0611400190	8553744882	7262009366	8560447559	6557476485	6740081773	8170330738	0305476973	6097865438	5938218722
0583902344	4435088674	9986650604	0645874346	0053318274	3629617786	2518081893	1443632512	0510709469	0813586440
5192295129	3245007883	3398788429	3393424351	2634336520	4385812912	8343452973	0865290978	3300671261	7981303167
9438553572	6296998740	3595704584	5223085639	0098913179	4759487521	2639707837	5944861139	4519602867	5121056163
8976008880	0927461158	6080020780	3341591451	7970730368	3519697776	6076373785	3330120241	2011204698	8609209339

0853657732	2239241244	9051532780	9509558664	5947763448	2269986074	8132973026	3097502881	2103517723	1244650953
4965369309	0018637764	0940943498	3731325132	1862080214	8099226855	0294845466	1814715557	4447096695	3017769043
4272031892	7706047177	8452793916	0472281534	3798035396	7986142437	0956683221	4914654380	1459382927	7393396032
7540480095	5223181666	7380357183	9327570771	4204672383	8624617803	9762923771	3120958078	9363841447	9298025880
6552212926	2093623930	6373134966	4018661951	0811583471	1733120258	0586672763	9992763579	0780638188	1306915636
6274125431	2595899361	1964762610	1405563503	3995231403	2311381965	6236327198	9618372548	4533370206	2563464223
9527669435	6837676136	8711962921	8187545760	8161705303	1590728828	7007123136	6630872275	4918661395	7737305460
6599743781	0987649802	4140112421	4277366808	2751390959	3134041558	2626678951	0846776118	6659576601	6599817808
9414985754	9762843878	5610026379	6543178313	6340251358	1416115190	2096499133	5487331311	1502270068	1930135929
5959716401	9719605362	5033558479	9809634887	1803911161	2813595968	5654788683	2585643789	6173159762	0024196215

5289629790	4819822199	4622694871	3746244472	9093456470	0285376949	5885959160	6789282491	0544125159	9630078136
8367490209	3749157328	9627002865	6829344431	3423473512	3929825916	6739503425	9958689706	9726733258	2735903121
2887466604	5146148785	0346142827	7659916080	9039865257	5717263081	8334944418	2019353338	5071292345	7743755793
4406217871	1330063106	0033240539	9169368260	3746176638	5657588775	8020122936	6353270267	1006812618	2517291460
8202541892	8859352444	9107013820	6211553827	7935652969	1457650204	8643282865	5579347072	0963480737	2692141186
8954673227	6775133569	0190153723	6690368653	8916129168	8887876407	5254934942	4973342718	1178892759	9315967193
5475898809	7924525262	3636590363	2007085444	0784544797	3482918020	8204492667	0634420437	5553250505	2752283377
8887040804	0335319234	0768563010	9347772125	6390886404	1310107381	7853338316	0381352808	2811904083	2564401842
0537467929	9262203769	8718018061	1226244909	0924264198	5820861751	1771137890	5160914038	1575003366	4241560952
1632819712	2335023167	4226005679	4128140621	7219641842	7057843289	5980288233	5059828208	1966662490	3585778994

FROM 60001D TO 65000D

0333152274	8177769528	4368163008	8531769694	7836905806	7106482808	3598046698	8410981351	5865490693	3319522394
3632879239	9053481098	7830274500	1720654336	9906611778	4554364687	7236318444	6476806914	2828004551	0746866453
9280539940	9108754939	1660957316	1971503316	6968309929	4663491427	9878084225	7220697148	8755806374	8030886299
5118473187	1247772919	1007022758	8893486939	4562895158	0296537215	0409603107	7612898312	6358996489	3410247036
0366450586	8728758905	1406841238	1242473863	8542790828	2733827973	3268855049	3587430316	0274749063	1295723497
4261122151	7417153133	6186224109	1386950068	8835898962	3492763173	1647834007	7460886655	5987333821	1382992877
6911495492	1841920877	7160606847	2874673681	8861675072	2101726110	3830671787	8566948129	4878504894	3063086169
9487987031	6051588410	8282351274	1535385133	6589533294	8629494495	0618685147	7910580469	6039069372	6626703865
1290520113	7810858616	1888869479	5760741358	5534585151	7680519733	3443349523	0120395770	7396237713	1603024288
7200537320	9982530089	7761897312	9817881944	6717311606	4723147624	8457551928	7327828251	2718244680	7824215216

4695678192	9409823892	6284943760	2488522790	0362021938	6696482215	6280936053	7317804086	3727268426	6964219299
4681921490	8701707533	3610947913	8180406328	7387593848	2695355830	7739576144	7997270003	4728801827	8528138950
3217986345	2161110666	0883931405	3226944905	4555278678	9441757920	2440021450	7801920998	0446138254	7805858048
4424164047	7503153605	4906591430	0781583724	3012313751	1562284015	8386442708	9071828481	6757527123	8467824595
3433444962	2010096071	0513706084	6180118754	3120725491	3349942476	1711563332	1408934609	1565615506	0031738421
8701570226	1031019166	0388706466	1438897736	3187809407	1152752817	4689576401	5810470169	6524755774	0891644568
6777171585	0058326994	3401677202	1567677240	6812836656	5264122982	4394651331	9735919970	9403275938	5026695574
7023181320	3243716420	5861410336	0652453693	9160050644	9530601612	6782264894	2437397166	7176612310	4897503188
5732165554	9883421218	0284691252	9086101485	5278152776	2562375045	6375769497	7343368460	1560772703	5509629049
3924870884	0628106794	3622418704	7470083688	4267102255	8302403599	8416459511	2248527263	3632645114	0173952480

8619463584	0783753556	8856223171	1552094722	3065437092	6067973510	0056554938	1224575483	7285457117	9739361575
6167641692	8958052572	9752233855	8611388322	1711073622	6581621884	2443178857	4887981090	2665379342	6664216990
9140565364	3224930133	4867988154	8866286650	5234699723	5574738424	8305904236	7714327879	2316422403	8777643301
9260019228	4778313837	6325361210	2533693581	2624086866	6997382759	7736568222	7907215832	4788886423	6934639616
4363308730	1398142114	3030600873	0666164803	6789840913	3592629340	2304324974	9268878316	4360268101	1309570716
1419128306	8657732353	2639653677	3903176613	6131596555	3584999398	6005651559	2193675997	7717933019	7446881483
7110320650	3693192894	5214026509	1546518430	9936553493	3371834252	9843367991	5939417466	2239003895	2767381333
0617747629	5749438687	1697845376	7219493506	5908757119	1772087547	7107189937	9608947745	1265475750	1871194870
7387367858	9020061737	3321075693	3022163206	2843206567	1192096950	5857611739	6163232621	7708945426	2146098584
1023781321	5817727602	2227381334	9541048100	3073275107	7999489919	7796388353	0734443457	5329759142	6376840544

2264784216	0631227696	4696715647	3999043715	9033239065	6072664411	6438605404	8388471619	1210900870	1019130726
0710441141	4324197679	6828547885	5247794764	8180295973	6049439700	4795960402	9274629920	3572099761	9501403483
1538094771	4601056333	4469988208	2212058728	1510729182	9712119178	7642488035	4672316916	5418522567	2923442918
7128163232	5969654135	4858957713	3208339911	2887759172	2611527337	9010341362	0856145779	9239877832	5083550730
1998184590	2595835598	9260553299	6737704917	2245493532	9683300002	2301815172	2657578752	4058832249	0858212800
8974790932	6100762578	7704286560	0699617621	2176845478	9964407050	6624171021	3327486796	2374302291	5535820078
0141165348	0656474882	3061500339	2068983794	7662550365	4982280532	9662862117	9306284301	7049240230	1985719978
9488368971	8304380518	2174419147	6604297524	3725168343	5411217038	6313794114	2209529588	5798060152	9387527537
9903093887	1683572095	7607152219	0027937929	2786303637	2687658226	8124199338	4808166021	6037221547	1014300737
7537792699	0695871212	8928801905	2031601285	8618254944	1335382078	4883465311	6326504076	4242839087	0121015194

2319616522	6842200371	1230464300	6734420647	4771802135	3070124098	8603533991	5266792387	1101706221	8658835737
8121093517	9775604425	6346949997	8725112544	0854522274	8109148743	0725986960	2040275941	1789425812	8188215995
2359658979	1811440776	5335432175	7595255536	1581280011	6384672031	9346507296	8079907939	6371496177	4312119402
0212975731	2516525376	8017359101	5573381537	7200195244	4543620071	8484756634	1540744232	8621060997	6132434875
4884743453	9665981338	7174660930	2053507027	1952983943	2714253711	5576660002	5784423031	0734295515	3394506048
6222764966	6876240793	2435319299	2639253731	0768921353	5257232108	0889819339	1686682789	4628117047	2624501948
4097009757	6092098372	4090074717	9733407881	4182519584	2598096241	7476101382	5264395513	5259311885	0456362641
8830033853	9652435997	4169313228	9471987830	8427600401	3680747039	0409723847	3945834896	1865397905	9411859931
0356168436	8692194853	8205578039	5773881360	6795499000	8512325944	2529724486	6667668346	4140218991	5944565309
4234406506	6785194841	7766779470	4720419588	2204329538	0326310537	4948831221	8039127967	8446100139	7267538921

FROM 65001D TO 70000D

9511911783	6587662528	0836900532	4900459741	0947068772	9123282143	0463533728	3519953648	2743258331	1914445901
7809607782	8835837301	1185754365	9958982724	5319253105	8811502630	7542571493	9430244539	3187017992	3608166611
3054262539	9583389794	2971602070	3387678150	3301028012	0095997252	2222808014	2357109476	0351925544	4349299867
6781789104	5559063015	9538097618	7592035893	7341978962	3589311259	8390259831	0267193304	1892151096	8915622506
9659119828	3234555030	5908173073	5195503721	6658702880	5399213857	6037035377	1051780212	8012956684	1984140362
8727256232	1442875430	2210909472	7210734741	3497551419	0737043318	2766261772	7599688882	6027225247	1336833534
5281669277	9591328861	3817663498	5772893690	0965749562	2871030243	6259077241	2219094300	8717556926	2575806570
9912016659	6224360802	4287002454	7362036394	8412559548	8172727247	3653467783	6472019183	0399871762	7037515724
6499222894	6793232269	3619177641	6146187956	1395669956	7783068290	3165896994	3076733350	8234990790	6241002025
0613405734	4300695745	4746821756	9044165154	0636584680	4636926212	7421107539	9042188716	1276177870	1425886482

5775223889	1845995233	7629237791	5585744549	4773612955	2595222657	8636462118	3775984737	0034797140	8206994145
5807190802	1359073226	9233100831	7595106590	1912129479	5408603640	7573587502	0589020870	4579670007	0552625058
1142066390	7459215273	3094068236	4944159089	1009220296	6805233252	6619891131	1842016291	6310768940	8472356436
6808182168	6572196882	6835840278	5500782804	0434537101	8365109695	1782335743	0305048526	5373807353	1074185917
7056103973	9506264035	5442275156	1011072617	7937063472	3804990666	9221619711	9425912044	5084641746	3835899382
3994651739	5509000859	4799901360	2667426149	4290066467	1150671754	2217703877	4507673563	7421547829	0591101261
9157555870	2389570014	0511782264	6989944917	9083017954	7587676016	8094100135	8376135785	9135692445	5647764464
1786671153	9195135769	6104864922	4900834467	1548638305	4477914330	0976804868	7834818467	2733758436	8927243104
4740680768	5278625585	1650920882	6381323362	3148733336	7147645204	5087662761	4950389949	5048095604	6098960432
9123358348	8599902945	2640028499	4280878624	0398118148	8476730121	6754161106	6299955536	6819312328	7425702063

7383520200	8686369131	1733469731	7412191536	3324674532	5630871347	3027921749	5622701468	7325867891	7345583799
6435135880	0959350877	5563562488	1049385299	9007675135	5135277924	1242927748	8565888566	5132473025	1471021057
5352516511	8148509027	5047684551	8252096331	8990685276	1443513821	3662152368	8905787866	9943228881	6028377482
0355060160	2989400911	9713850179	8716836337	4413927597	3644017007	0147637066	5570350433	8121113576	4150184518
2141361982	3495159601	0647527125	7593518530	4332875537	7830575095	6742544268	4712219618	7091785607	8393614451
1383335649	1032564057	3389866717	8123972237	5193164306	1701385953	9474367843	3926709867	1245221118	9690840236
3274114966	0124348309	8929941738	0305884171	6661307304	0067588380	4321115553	7944060549	7721705942	8215148861
6567277124	0903387727	7456290971	1013488518	4374118695	6554497457	3684521806	6982911045	0580042998	8795389902
7804383596	2824094218	6055628778	8428802127	5538848037	2864001944	1614257499	9042720095	9520465417	0598104989
9675045119	3647117277	2220436102	6140797508	0968697517	6600237187	7483480161	2031023468	0567112644	7661237476

2785219024	1202569943	5347162266	6089367521	9833111813	5111465038	5489502512	0655772636	1454736044	2685949807
4396932331	2971273771	5734709971	3952291182	6534851555	8713733662	9120242714	3025037632	6950135091	1612952993
7858646813	0722648600	8270881333	5381937036	8259886789	3321238327	0532976258	5738279009	7826460545	5985551318
3668884462	8265133798	4916678394	0976135376	6251798258	2496634587	7195012438	4040359140	8492097337	5464247448
8176184070	0235695801	7741017769	6925077814	8933866725	5789856458	9851056891	9609243988	4156928069	6983352240
2256345704	9731224526	9354193837	0048431833	5719651662	6721575524	1934019330	9901831930	9196582920	9696562476
6768365964	7019595754	7393455143	3741370876	1517323677	2042273856	7427917069	8204549953	0959188724	3493952409
4441678998	8463198455	0485239366	2972079777	4528143994	1825678945	7795712552	4268260899	4086331737	1538896262
8896294021	1210888442	7376568624	5276121303	7101730078	5135715404	5330415079	5944777614	3597437803	7424366469
7324713841	0492124314	1389035790	9241603640	6314038149	8314819052	5172093710	3964026808	9948325722	9795456404

2701757722	9041732347	9607361878	7889913318	3058430693	9482596131	8713816423	4672187308	4513387721	9086975104
9428437693	2502498165	6673816260	6159417682	5250999374	1672883951	7440669325	4965340310	1452225316	1890092353
7648637848	2881344209	8700480962	2717122640	7489571939	0029185733	0746010436	0729190945	7679946149	2929042798
1687729426	4877299528	5843464777	5386906950	1489841339	2454039414	4680263625	4021186143	1703125111	7577642829
9146445334	0892097696	1699098372	6523617687	4560589470	4968170136	9749095230	7208268288	7890730190	0182534258
0534342170	5928713931	7379931424	1085264739	0948284596	4180936141	3847583113	6130576108	4623668372	3769591349
2615824516	2215521348	7924414504	1756848064	1206365201	7038633012	9532777699	0231186480	2006755690	5682295016
3549319923	0591424639	6217025329	7475731140	9422018019	9368035026	4956369558	6642590676	2685687372	1103391567
9383989576	5565193177	8830002416	1353956243	7777840801	7488193730	9502069990	0890899328	0883974303	6773659552
4891300156	6332940779	0713961546	4534088791	5103006513	2193448667	3248275907	9468078798	1942501958	2622320395

FROM 70001D TO 75000D

1312520141	0996053126	0696555404	2486705499	8678692302	1746989009	5478507256	7297879476	9888831093	4874644264
0071818316	0331655511	5342761556	2240547447	3378049246	2149521332	5852769884	7336269182	6491743389	8782478927
8468918828	0546699823	0368993978	3413747587	0258057163	4941356843	3929396068	1920617733	3179173820	8562436433
6353598634	9449689078	1064019674	0744365836	6707158692	4521182997	8938040771	3750129085	8646578905	7714268335
8276897855	4717687184	4277261205	0926648610	2051535642	8406323684	8180728794	0717127966	8200607275	5955590404
0233178749	4473464547	6062818954	1512139162	9184442976	5106694796	9354016866	0100551960	7768733539	6511614930
9375709685	5455938151	3789569039	2510149532	6562814701	1998326992	2000663928	7537471313	5236421589	2651262040
7288771657	8358405219	6460541054	3544364216	6562244565	0429990102	5658692727	9142752931	1720827939	3775132610
6052881235	3734510683	7293989358	0871243869	3859343891	7571337630	0720319760	8166044646	8393772580	6909237297
5234867029	1691042636	9262090199	6052041210	2407764819	0316014085	8635584276	0953708655	8164273995	3493465463

1450404019	9528537252	0049578052	5465625115	4109252437	9913262627	1360909940	2902262062	8367521323	0506518393
4057450112	0993414649	1843332364	6569371725	9144893241	5900624202	0612885732	9261335968	0872650004	5628284557
5745965921	2053034131	0111827501	3069615098	3551563200	4310784601	9065654938	0654252522	9161991819	9596027523
2770224985	5738824899	8827074659	3635576858	2560518068	9642853768	5077201222	0347920993	9361792682	0659014216
5615925306	7379445689	4907085326	3568196831	8617722682	4991147261	5732035807	6462981162	4401331673	7892788689
2290325933	4986179702	1994981925	7396176730	7583441709	8559222170	1718257127	7753449150	8205278430	9046194608
3521740200	5838672849	7094110232	6695392144	5461066215	0064106747	4020700918	9911951376	4669044812	6725369153
7162290791	3854039375	6007783515	3374167747	9421003840	0230895185	0994548779	0393461222	2086506016	0500351776
2648316111	5332558770	5073541279	2499098593	7347378708	1194253055	1214369797	4991495186	0535920403	8302357163
5272763087	4693219622	1900642608	8618367610	3346002255	4774778136	4101269190	6569686495	0126883762	9690723396

1276287223	0411418136	1006026404	4030035996	9889199458	2739762411	4613744804	0596970625	7676472376	6065541618
5746905272	2923822827	5186799156	9833907476	7114610302	2776606020	0612468764	7772881909	6791613354	0198814027
5799217416	7678799231	6039635694	9285151363	3647219540	6111717673	8737255572	8522940054	3617851765	0230754469
3869307873	4991103521	8253292972	6044553210	7978877114	4989887091	1511237250	6042387537	3484125708	6064069052
0584521227	5453384800	8205302450	4565176695	1857691320	0042816758	0549248117	8051983264	6032445792	8297301291
0531838563	6821206215	5312886685	6495651261	3892261367	0640939533	3457052698	6959692350	3530942245	4386527867
7673027540	4027022463	8448355323	9914751363	4410440500	9233036127	1496081355	4905315390	2100229959	5756583705
3812619656	8314428605	7956696622	1547216956	2087001372	7768536960	8407048333	2513279311	2232507148	6302069512
4539500373	5723346807	0946564830	8920980153	4878705633	4910923660	5755405086	4111521441	4814346304	3727327104
5027768661	9531078583	2333485784	0297160925	2153260925	5893265560	0672124359	4642550659	9677177038	8445396181

6328796144	6081778927	2171836908	8801267782	0743010642	2524634807	4543004764	9288555340	9062185153	6543554741
2547615276	9772667769	7727770583	1580141218	5688011705	0283652755	4321480348	8004442979	9980621579	0456416195
7212784508	9284898064	2649742709	0579129069	2178072987	6947797511	2447305991	4060506299	4689428093	1034216416
6299356148	2813099887	0745292716	0484336308	1840412646	9637925843	0941854422	1635908457	6146078558	5624738149
3142707826	6215185541	6038702068	7698046174	7400808324	3436653823	5455510944	9498431093	4947599446	7267366535
2517662706	7721941831	9197719637	8015702169	9336750837	6005716345	4643671776	7233875886	4340564487	1566964321
0412825956	4534984138	8412890420	6820470076	1559691684	3038999348	3667935425	4921032811	3363184722	5923055543
8305820694	1675629992	0133731754	8912203723	0349072681	0685344540	3599356182	3576312837	7676406310	1312533521
2141994611	8693508331	7658785204	7112364331	2267651299	6417132521	7513553261	8676819423	3879036546	8908001827
1352835848	8844411176	1234101179	9187092365	0718485785	6221021104	0097769944	5312179502	2479578069	5065329659

4038398736	9907240797	6790408267	9400761872	9547835963	4927939045	7697366164	3405359792	2192858705	7495748169
6694062334	2726197335	1813662606	3735982575	5524965098	0726012366	8283605928	3418558480	2695841377	2558970883
7899429105	4980033111	3884603401	9391661221	8669605849	1571485733	5682861495	0001909759	1125218800	3964197621
6355937574	3718011480	5594422987	3041819680	8085647265	7135476128	3162920044	9880315402	1055305970	7666636274
9328308916	8809323592	9008178741	1985738317	1926167288	3491840242	9721290434	9655269427	2640255964	1463525914
3484006758	6769035038	2320572934	1329815935	3304444649	6829441367	3234421583	8076169483	1219333119	8190610961
4295220153	6170298575	1055943264	6146850545	2684975764	8078080092	2133581137	8197749271	7685450755	3832876887
4474591593	7311624706	0109124460	9829424841	2875202244	6259447763	8749491997	8404468292	5736096853	4549843266
5368628444	8936570411	1817793806	4416165312	2360021491	8768769467	3984075171	7630751684	9856359201	4868929431
0594020245	7969622924	5666448819	6757629434	9535326382	1716133957	5779076637	0764569570	2597388004	3841580589

FROM 75001D TO 80000D

4336137106	5518599876	0075492418	7211714889	2952217377	2114608115	4344982665	4798725800	5667472405	1122007383
4592715757	2771521858	9946948117	9406444663	9943237004	4291140747	2181802248	2583773601	7346685300	7449855647
1542003612	3593397312	9144585915	2288740871	9508708632	2188372882	6282288463	1843717261	9033057771	4765156414
3822306791	8473860391	4768310814	1358275755	8536435977	2165002827	7803713422	8696887873	4979509603	1108899196
1433866640	6845069742	0787700280	5093672033	8723262963	7856038653	2164323488	1555755701	8469089074	6478791224
3637555666	8678067610	5449550172	6079114293	0831285761	2544819444	4947324481	9093795369	0082063846	3167822506
4809531810	4065702543	2760438570	3505922818	9198780658	6541218429	9217273720	9551032422	5107971807	7833042609
0867942734	2895573555	9252723805	5114404380	0123904168	7716445180	2264916816	4192740110	6451622431	1017000566
9112173318	9423400547	9596846698	0429801736	2570406733	2821299621	5368488140	4102194463	4246462207	4557564396
0452985313	0714090846	0849965376	7803793201	8991408658	1466217531	9337665970	1143306086	2500982956	6917638846

0567629729	3146491149	3704624469	3519840395	3444913514	1193667933	3019366176	6365255514	9174982307	9870722808
6085962611	2660504289	2969665356	5251668888	5572112276	8027727437	0891738963	9772257564	8905334010	3885593112
5679991516	5890250164	8696142720	7005916056	1661597024	5198905183	2969278935	5503039346	8121976158	2183980483
9605625230	9146263844	7386296039	8489243861	8729850777	5928792722	0685548072	1049781765	3286210187	4767668972
4884113956	0349480376	7270363169	2100735083	4073865261	6845074824	9644859742	8134936480	3724261167	0426687083
1925040997	6153190768	5577032742	1785010006	4419841242	0739640013	9603601583	8105659284	1368457411	9102736420
2741637234	8821452410	1347716529	6031284086	5841978795	1116511529	8278146203	7913985500	6399960326	5912485253
0849369031	3130100799	9771913622	3086601109	9929142871	2493885416	1203802041	1340188887	2196934779	0449752745
4288072803	5093058287	5442075513	4816660927	8793535665	2125562013	9988249628	4787262144	3236285367	6502591450
4683776352	8258765213	9156480972	1419296755	4938437558	2600253168	5363567313	7926247587	8049445944	1834291727

5698837622	6261846365	4527434976	6241113845	1305481449	8363117897	8448973207	6719508784	1586188796	9295581973
3250699951	4026015116	7552975057	5437810242	2389579257	8656212843	2731202200	7167305740	6928686936	3930186765
9582513264	9914595026	0917069347	5194089753	5746401683	0811798846	4524736189	5605647942	6358070562	5632811892
6966302647	9535951097	1276591362	3318086692	1535788607	8127599105	3717140220	4506186075	3748663063	5059148391
6467656723	2057145168	8617079098	4695932236	7249467375	8309960704	2589220481	5507991327	5208858378	1117685214
2693347869	2189524062	2657921043	6203488529	2626798401	3953216458	7911515790	5046057971	0838983371	8640380244
1751134722	6472547010	7947939969	5355466961	9726763255	2299146549	3349966323	4185951450	3609803440	9221220671
2567698723	4279407088	5707047429	3173329188	5238967219	7135392449	2426178641	1886377909	6281448691	7869468177
5917171506	6911148002	0759432012	0619696377	9510322708	9029566085	5622254526	0261046073	6131368869	0092817210
6819861855	3780982018	4711541636	3032626569	9283424155	0236009780	4641710852	5537612728	9053350455	0613568414

3775854429	6779770146	6029438768	7225115363	8011917581	5402812081	8255606485	4107879335	9892106442	7244898618
9616294134	1800129513	0683638609	2941000831	3667337215	3008352696	2357371753	3073865333	8204842190	3081864491
8409372394	4033405244	9095545580	1640646076	1581010301	7674884750	1766190869	2946098769	2016912021	8168829104
0870709560	9514704169	2114702741	3390052253	3408348128	7035303102	3919699978	5974139085	9360543359	9697075604
4601342424	5368249609	8772581311	0247327985	6207212657	2499003468	2938868723	0489556225	3204463602	6398542252
5841646432	4271611419	8178024825	9556354490	7219226583	8636626637	5083594431	4877635156	1457107455	2801615967
7048442714	1944351832	7569840755	2677926411	2617652506	1596523545	7187956673	1709133193	5876162825	5920783080
1852068901	5150471334	0386100310	0559148178	5211038475	4542933389	1884441205	1794396997	0194112695	1195265649
1959418997	5418393234	6474242907	0271887522	3534393673	6336632003	0723274703	7407123982	5620246626	5197409019
9762452056	1985576257	6000870817	3083288344	3818310700	5451449354	5885422678	5785519153	7229237955	5494333410

1744201696	0009069641	5612732297	7702212179	5186837635	9082255128	8164700219	9234886404	3959153018	4640047143
2118636062	2527011541	1222838027	7853891109	8490201342	7410141215	5976996543	8877197485	3764311582	2983853312
3071751132	9619045590	0793806427	6695819014	8426279912	2179294798	7348901868	4716765038	2732855205	9082984529
8062592503	5212845192	5927986593	5061329619	4679625237	3972565584	1578537445	6755899803	2405492186	9628884903
3256085145	5344391660	2262577755	1291620077	2796852629	3879375304	5418108072	9285891989	7153817973	4349618723
2927614747	8501926114	5041327487	3242970583	4084711123	3374627461	7274626582	4153242710	5932250625	5302314738
7592517247	8732288149	1455915605	0363345754	2423377916	0374952502	4930223514	8196138116	2563911415	6103268449
5807250827	3431765944	0540982697	6526934457	9863479709	7431244982	7193311386	3873159636	3612186234	9726140955
6079920628	3169994200	7205481152	5353393946	0768500199	0988655386	1433495781	6500899616	4907967814	2901148387
6456821749	1407562376	7618453775	1440314754	1120676016	0726460556	8592577993	2207033733	3398916369	5043466906

FROM 80001D TO 85000D

9482843662	9980037414	5276277165	4762382554	6170883189	8108688068	4785370553	6480469350	9588180253	6052974079
3538676511	1950793732	8208314626	8960071075	1755206144	3378411454	9950136432	4463281933	4638905093	6545714506
9008644834	4018042836	3390513578	1572739733	3453728426	3372174065	7757710798	3051755572	1036795976	9018899584
9413019599	9573017901	2401939086	8135658553	9661941371	7944876320	7986880037	1607303220	5474235722	6689680188
2123424391	8859841689	7227765219	4032493227	3147936692	3400484897	6059037958	0946960417	5427961378	2553781223
9476461478	3292697654	5162290281	7011004378	4603875654	4151739433	9600489153	1881757665	0500951697	4024156447
7129365661	4253949368	8842305174	0012992055	6854289853	8979426699	5677702708	9146513736	8922061044	1548166215
6804219838	4767308717	8759027920	9175900695	2734566820	2651337311	1518000181	4341209626	0165862982	1076663523
3617740078	3778342370	9152644063	0540718078	4335806107	2961105550	0204151316	9637304684	9213356837	2654003075
0982908936	4612047891	1147530370	4989395283	3457824082	8173864413	2271000296	8311940203	3234564208	2647327623

3830294639	3789983758	3655455991	9340866235	0909679611	3400486702	7123176526	6637107787	2511186035	4037554487
4186935197	3365662177	2359229396	7764632515	6202348757	0113795712	0962377234	3137021203	1004965152	1119760131
7641940820	3437348512	8526029133	3491512508	3119802850	1778557107	2537314913	9215709105	1309650598	8599993156
0863655477	4035518981	6673353588	0048214665	0997414337	6118277772	3351910741	2175728415	9258087259	1315074606
0256349037	7726337391	4461377038	0213183474	4730111303	2670296917	3350477016	3210661622	7830027269	2833655840
1179141944	7808748253	3607144032	9625228577	5009808599	6090409363	1263562132	8162071453	4061042241	1208301000
8587264252	1122624801	4264751942	6184325853	3867538740	5474349107	2710049754	2811594660	1713612259	0440158991
6002298278	0179603519	4080046513	5347526987	7760952783	9984368086	9089891978	3969353217	9980139135	4425527179
1022539701	0810632143	0485113782	9149851138	1969143043	4975001899	8068164441	2123273328	3071928243	6240673319
6554692677	8511931527	7511344646	8905504248	1133614349	8460484905	1258345683	2664415284	8971397237	6040328212

6602535166	9391408204	9947320486	0216277597	9177123475	1097502403	0789357599	3771509502	1751693555	8270725339
1189233407	0223832077	5858021371	7477837877	8391015234	1320984894	2345961369	2340497998	2793041444	6316270721
4796117456	9757196812	3929191374	0982925805	5619552074	3424329598	2898980529	2333664154	1925636738	0689494201
4712413405	2507220406	1794355252	5552250087	4879008656	8314542835	1677505422	9480327478	3044056438	5815919526
6675828292	9705226127	6287110401	3480178722	4801789684	0524079243	6058274246	7443076721	6452703134	5135416764
9668901274	7868010102	9513386269	8649748212	1186290403	3769156857	6240699296	3724930972	0162870720	0189835423
6903641492	7023696193	8547372480	3298550451	1208919287	9829874467	8641291594	1753167560	2533435310	6267452545
0711418148	3239880607	2971402347	2552071349	0798398982	3552687239	5090936566	7878992383	7125789762	4875599044
3228895388	3773173489	4112275707	1410959790	0479193010	4674075041	1435381782	4646307959	8955563899	1884773781
3413470702	4674736211	2048986226	9918885174	5625173251	9341352038	1158633501	2391305444	1910073628	4475675141

6105041097	3505852762	0444891909	7890198431	5485280533	9857778443	1393388399	4310444465	6692445508	8594631408
1751220331	3906815965	9251054685	8013133838	1521764182	1043342978	8826119630	4431113887	9625874609	0226130900
8499754303	9577124323	0616906262	9194039214	3974027089	4777663702	4881554993	2245882597	9020631257	4369109463
9325280624	1642476868	4954553249	3801763937	1615636847	8598237159	0238542126	5840615367	2286071317	0267474013
1145261063	7653833903	1592194346	9817605358	3803106128	8785205154	6933639241	0884676320	0956708971	8367490578
1630851581	3816196688	2222047570	4375906143	3804072585	3862083565	1769984267	7452319582	4182683698	2701602374
1493836349	6629351576	8540613973	4274647089	9685618170	1605511048	8097155485	9118617189	6680259735	4170542398
5135560018	7203350790	6094642127	1143993196	0465274240	5088222535	9773481519	1354385712	5325854049	3946010865
7937980586	2014336607	8825219717	8090258173	7087091646	0452727977	1535099103	4073642502	0386386718	2205228796
9445838765	2947951048	6607173902	2932745542	6785669776	8659399234	1683412227	4663015062	1553205026	5534146099

5249356050	8549217565	4913483095	8906536175	6938176374	7364418337	8974229700	7035452066	6317092960	7591989627
7324230902	5239744386	1014263098	6877339138	8251868431	6501027964	9114977375	8288891345	0341148865	9486702154
9210108432	8080783428	0894172980	0898329753	6940644969	9031253998	6391958160	1468995220	8806622854	0841486427
4786281975	5466292788	1462160717	1381880180	8405720847	1586890683	6919393381	8642784545	3795671927	2397972364
6516675920	1105799566	3962598535	5127635587	6814021340	9829016296	8734298507	9247184605	6874828331	3812591619
6247615690	2875901072	7331032991	4062386460	8333378638	2579263023	9159000355	7609032477	2813388873	3917809696
6601469615	0317542267	5112599331	5529674213	3363002229	6490648093	4582008181	0618021002	2766458040	0278213336
7585730190	1137175467	2763059044	3531313190	3609248909	7246427928	4555499134	9000518029	5707082919	0525567818
8991389962	5138662319	3800536113	4622429461	0248954072	4048571232	5662888893	1722116432	9478161905	5486805494
3441034090	6807160880	2822795968	6950133643	8142682521	7047287086	3010137301	1552368614	1690837567	5747637239

FROM 85001D TO 90000D

7631857570	3810944339	0564564468	5241830281	4810799837	6918512127	2019350440	4180460472	1626939445	7883770901
0597469321	9720558114	0787759897	7207200968	9382249303	2368305158	6265728111	4637996983	1375179376	2321511125
2349734305	2406221052	4423435373	2905655163	4066695061	6589287821	8707756794	1760807129	7378133518	7117931650
0331555238	2248773065	3444179453	4153952024	2444970341	0120874072	1881093882	6816751204	2299404948	1794494727
3289477011	1574139441	2284555218	2842492224	0658752689	1722727806	0711675404	6973008037	0396187877	9669488255
5614674384	3925701158	2954666135	8678671897	6612973112	6720007297	1553613027	5035561678	1776544228	7442114729
8816148027	0524380681	7653573275	5786025058	4708401320	8837932816	0087690813	0049249147	3682517035	3822196190
3901499952	3495387105	9973511434	7829233949	9187936608	6923013755	9636853237	3806703591	1442432685	6151210940
4259582639	3016780171	2866923928	3231057658	8517140202	1119695706	4[illegible]9814031	5056330451	4156441462	3163763809
9044028162	5691757648	9142569714	1635984393	1743327023	7812336938	0430128926	2637538266	7795034169	3343236075
0024817574	1808750388	4750949394	5489620974	0485442635	6371649959	4992098088	4294790363	6662975260	0324385635
2945844728	9445471662	0929749549	6616877414	1208821304	7702281611	6456044007	2363515811	4972973921	8966737382
6472047226	4222124201	6560150284	9713063327	9581430251	6013694825	5670147809	3579088965	7134926158	1613469018
0696508955	6310121218	4918058479	2272069187	1696316330	0448580201	0286065785	8591269974	6376617414	6393415956
9539554203	3146280265	1895116793	8074573315	7598460861	7370268786	7602943677	7805002446	7339133243	1669880354
0732323882	8184750105	1641331189	5370364884	2269027047	8052742490	6034920829	5475505400	3457160184	0725745369
3814553117	5354210726	5578356154	9987444748	0427323457	8800618731	4934156604	6352979779	4550753593	0479568720
9316724536	5472083816	8585560604	3801977030	7642460834	8987610134	5709394877	0029461757	9206195254	9255757109
0385251714	8852526567	1045349813	4198033906	4152987634	3695420256	0802776144	2191431892	1393908834	5431317696
8510184010	3844472348	9488695209	8194353190	6506555354	6173358140	4554483788	4752526253	9496658699	9205841765
2780125341	0338964698	1864243003	4146791380	6190280596	0785488801	0789705516	9462152287	7309010446	7462497979
9926271209	5168477956	8482583341	4022664772	1084336243	7593741610	5367340419	5473896419	7895425335	0363018614
0095153476	6961476255	6518738232	9246854735	6935802896	0115367917	8730355315	9378363082	2486151777	7054157757
6561759358	5120166929	4311113886	3582159667	6188303261	0416465171	4846979385	4226216871	6140012237	8213779774
1312689772	6671299202	5922017408	7700769562	8347393220	1088159356	2862819285	6357189338	4958850603	8531581797
6067947984	0878360975	9601497334	2057270460	3521790605	6476032855	6927627349	5182203236	1441125841	8242624771
2012035776	3888959743	1823282787	1314608053	5335744942	9762179678	9034568169	8895535185	0447832561	6380709476
9516990862	4710001974	8809205009	5219436323	7871976487	0339223811	5403634754	8862684595	6159755193	7654101150
1406700122	6927474393	8885899438	5973024541	4801061235	9080362745	8528849356	3251585384	3832424932	5266608758
8908318700	7091002373	7710657698	5056433928	8543376583	4259675065	3715005333	5144899082	9388773735	2051459333
0496265314	1514138612	4437935885	0709446880	4548697535	8170212908	4907873478	0681436632	3322819415	8273456713
5644317153	7967818058	1958524648	4008403290	9981943781	7181773023	1700398973	3050495387	3561162610	2399943325
9780126893	4326055847	1027876490	1070923443	8846340117	3555686590	3585244919	3701810416	2620850429	9258697435
8170981338	9404593447	1937493877	6242324098	5283276226	6604942385	1297094532	4558625210	3600829286	6497241749
1914198896	6129558076	7709795947	9530601311	9159011773	9431042090	4907942444	8868513086	8444937059	0902600612
0649425744	7103535476	5785924270	8130410618	5462198818	3009063458	8187038755	8562749115	8737542106	4667951346
4875867715	4383801852	1348281915	8124625993	3516019893	5595167968	9328522058	2479942103	4512715877	1633452229
9541883968	0448835529	7533612868	3722593539	0079201666	9413390911	6875880398	8828869216	0023732573	6158820716
3516271332	8105181876	0210485218	0675526648	6739089009	0719513805	8626735124	3122156916	3790227732	8705410842
0378415256	8328871804	6987952513	0732663402	7851905941	7338920358	5403956770	3561132935	4482585628	2876106106
9822972142	0961993509	3313121711	8789107876	6872044548	8760894101	7479864713	7882462153	9559333332	7556200943
9580434537	9197822805	9039595992	7436913793	7786649409	6404877784	1748336432	6840262829	3240626008	1908081804
3909145563	5193685606	3045089142	2896452199	8779884934	7477729132	7972660276	5840166789	0136490508	7411421268
6196986204	4126965282	9810870454	7986155954	5338021201	1556469799	7678573892	0186243599	3267776894	5406050821
8838227909	8336271671	2449002676	1178498264	3770330020	8184459000	9717235204	3319947082	4209877151	4449751017
0556430295	4282181967	0009202515	6158441742	0593365814	8134902693	1115170938	7226002645	8630561325	6057925609
2733226557	9346280805	6834439213	7368840565	0434307396	5740610177	7937014142	4615493070	7413608054	4210029560
0095663588	9778992676	3051771878	1943706761	4982175641	8659011616	0865408635	3915130392	0131680576	9034172596
4536923508	0641744656	2351523929	0504094799	5318407486	2151210561	8338545661	7665260639	3713658802	5216662235
7613220194	1701372664	9660732520	1077194793	1265282763	3024138051	6490717456	5964853748	3546691945	2358031530

FROM 90001D TO 95000D

1969160480	9946068149	0403781982	9732360930	0871357607	9862142542	2096419004	3679054790	4993007837	2421581954
5354183711	2936865843	0553842717	6280352791	2882112930	8351575656	5999447417	8843838156	5148434229	8587042455
9243469329	5232821803	5083337262	8379183021	6591836181	5542171574	4846577842	0134329982	5945668845	5826617197
9012180849	4803324487	8725818377	4805522268	1510113717	4536841787	0280274452	4429054745	1823467491	9564188551
2444213377	8352142386	5979925988	2032870851	0933838682	9906571994	6149062902	5742768603	8850511032	6385445404
1918495886	6538545040	5713236296	8106914681	4847869659	1668618427	5679846004	1868762298	0555629630	4595322792
3051616721	5919686758	4952363529	8935788507	7460815373	2145464298	4792310511	6763577494	9462295256	9497660359
4739624309	9534331040	4994209677	8838270027	1447849406	9037073249	1064441516	9605325656	0586778757	4174721108
2743577431	5194060757	9835636291	4332639781	2218946287	4477981198	0722564671	4664054850	1310096567	8631488009
0303749338	8753641831	6513498254	6694673316	1181233648	5439764932	5026179549	3572043054	0218297487	1251107404

0116114058	9991109306	2492312813	1163405492	6257135672	1818628932	7861388337	1802853505	6503591952	7414008695
1092616754	1476792668	0321092374	6708721360	6278332922	3864136195	9412133927	8036118276	3241060047	4097111104
8140003623	3427145144	8333464167	5466354699	7314947566	4342365949	3496845884	5515241507	5637660508	6632827424
7941360628	7604129064	4913828519	4564026431	5322585862	4043141838	6695906332	4506300039	2213192647	6259626915
1090445769	5301444054	6180378575	0303668621	2462278639	7527466678	7012100339	2984873375	0144756003	2210062235
8029343774	9550320370	1273846816	3061026570	3008722754	6296679688	0890587127	6763610662	2572235222	9739206443
0935243272	2810085997	3095132528	6306011054	9791564479	1845004618	0467624089	2892568091	2930592960	6423570210
6152464620	5023248966	5939873249	3396737695	2023991760	8984745718	4353193664	6529125848	0644801965	2016283879
5189499336	7592414856	2613699594	5307287254	5324632915	2911012876	3770605570	6095313775	2775186792	3292134955
2451330898	6796916512	9073841302	1675732386	3757582008	0363575728	0027544903	2795307990	0799442541	1087256931

8801466793	5595834676	4328688769	6661009739	5749967836	5933978463	4695994895	0610490383	6474095046	9522606385
8046758073	0699122904	7408987916	6872117147	5276447116	0440195271	8169508289	7335371485	3092893704	6384420893
2997711258	5684084660	8339934045	6890267875	1600877546	1267988015	4658565220	6121095349	0796707365	5397025761
9943137663	9960606061	1064069593	3082817187	6426043573	4253617569	4378484849	5250108266	4883951597	0049059838
0812105221	1110919433	2395113605	1446459834	2107990580	8209371646	4523127704	0231600721	3854372346	1267260997
8703856570	9199850759	5634613248	4601884098	5019428768	7902268734	5565005191	2154654406	3829253851	2763176639
2205093834	5204300773	0170299403	6261543400	1322763910	9129883278	6392041230	0445551684	0548898090	8077917463
6092439334	9126411642	4009388074	6356607262	3366958427	6458369826	8734815881	9610585718	3576746200	9650526065
9292635482	9149904576	8307210893	2458570737	0166071739	8194485028	8426039636	6074603118	4786225831	0565808708
7030556759	5861341700	7454029656	8763477417	6431051751	0367328692	4555858208	2372038601	7817394051	7513043799

4868822320	0443780431	0317092103	4261674998	0000730160	9481458637	4488778522	2730763304	9538394434	5382770608
7607635420	9844500830	6247630253	5727810327	8346176697	0544287155	3153400164	9707665719	5985041748	1990872014
9087568603	7783591994	7193433527	7294728553	7925787684	8323011018	5936580071	7291186967	6176550537	7503029303
3830706448	9128114120	2550615089	6411007623	8245744886	5518258105	8140345320	1247547232	6908754750	7078577659
7325428444	5935304499	2070014538	7489482265	5644222369	6365544194	2254413382	1222547749	7535494624	8276805333
3698328415	6138692363	4433585538	6847111143	0498248398	9918031654	5863828935	3799130535	2228334301	3795337295
4016257623	2280811384	9949187614	4141322933	7671065634	9252881452	8239506209	0223578766	8465011666	0097382753
6604054469	4165342223	9052108314	5858470355	2935221992	8272760574	8212660652	9138553034	5549744551	4703449394
8686342945	9658431024	1907859236	8022456076	3936784166	2705185551	7870290407	3557304620	6396924533	0779578224
5949710420	1880430001	8388142900	8173039450	5073427870	1312446686	0092778581	8110409115	1172937487	3627887874

9074652855	6543474888	6831064110	0510230208	7510776891	8781525622	7352515503	7953244485	7787277617	0019648537
0355516765	5209119339	3437628662	8461984402	6295252183	6785223674	7510880978	1507098978	4130862458	8152266096
3551401874	4958369269	1779904712	0726494905	7372642860	0521140358	1231076006	6995185361	2486274675	6375896225
2991164960	6687650826	1734178484	7893372950	5673900787	8617925351	4406210453	6625064046	3728815698	2323175005
9626108092	1955211150	8593029556	5496753886	2612972339	9146283584	7604862762	7027309739	2020014322	4870758233
7354915246	0856082103	2888297418	3906478869	9232736913	6004883743	6615223517	0584377055	4521081551	3361262142
9118156153	0175888257	3594892507	1088792621	2864139244	3309383797	3338678061	3179523731	5266773820	8580247014
3352700924	3803266951	7421195076	7088432634	6442749127	5589077468	6358216216	6042741315	1702124585	8605623363
1493164646	9139465624	9747174195	8354218607	7487110573	3845843368	9939645913	7406033821	5935224359	4751626239
1886853078	2282176398	3237306180	2042465604	7752794310	4796189724	2995330297	9249748168	4052893791	0449470045

FROM 95001D TO 100000D

9086499187	2727345413	5081019838	8186467360	9392571930	5119686456	0185578245	0218231065	8894379865	2243205067
7379966196	9554724405	8592241795	3006820451	7953700434	7245176289	3566770508	4902131077	3662575169	7335527462
3029430312	0359626095	3423574397	2496592110	1065781782	6108745318	8748031874	3082357369	9195156340	9571627009
9244492974	9105489851	5196586647	4014822510	6335367949	7371425102	2934188258	5117371994	4991150975	8374613010
5505064197	7215319293	5487537119	1630262030	3285886585	2848019350	9225875775	5974252765	8401172134	2323648084
0271433563	6754204637	5182552524	9443296570	4386138786	5901965738	8028684018	9408767281	6714137033	6617326501
2057865391	5780703088	7142615190	7500149257	6112927675	1930967284	5397116021	3606303090	5422439663	2067432358
2797889332	3244057791	9927848463	3339777737	6559018705	7480682867	8347965624	1461028995	0848739969	2970750432
7530299728	7229732793	4442988646	4127253481	6060377970	7298299173	0292963086	9580199631	2413304939	3504933254
1235507105	4461182591	1411164545	3471032988	1047844067	7801380771	3146540009	9386306481	2666143308	5820681139
5838319169	5455582594	2689576984	1428893743	4670841079	4631893253	9106963955	7807060212	4597489829	3564613560
7889834724	1997947856	4362042094	6134123876	1319886535	2358312996	8622689486	0840845665	5606876954	5012744866
3140505473	5351746873	0098063227	8046891224	6821460806	7276277084	0240226615	5485024008	9528916571	1761743902
0337584877	8429112896	2324705919	1874691042	0058483261	4067733375	1027195653	9946971625	1724831223	0633919328
7079838007	4848572651	6123434933	2733566644	7335855643	0235280883	9243482787	6088616494	3289399166	3992104883
0784777704	8045728491	4563033532	6507002958	8906265915	4985094079	7276756712	9795010098	2294762289	6189159144
1520032283	8787734851	3097908101	9129267227	1037788980	5396415636	2364169154	9857684083	9846886168	4375407065
1210390625	0612810766	3799047908	8796747780	6973847317	0475253442	1563903872	0123880632	3688037017	9493089549
0077633152	3063548374	2568166533	6160664198	0030188287	1237674818	9833024683	6371488309	2592833759	0227894258
8060087286	0388591688	4973069394	8020511221	7663591382	5152427867	0094406942	3551202015	6837777885	1824670025
6517085092	4962374772	6813694284	3500629388	1442998790	5301056217	3754591826	7997321773	5029368928	0652100253
9626880749	8092643458	0116557158	8670044350	3976505323	4782873273	6884086354	0002740676	7838219635	2222653929
0939807367	3913640828	9872201777	6747168118	1958561337	2158311905	4682936083	2369761134	5028175783	0202934845
9829250008	9568263027	1263295866	2921476531	4223335179	3093387951	3570953463	7718368409	2444422096	3193312956
2030557551	7340067973	7406141621	0792363342	3805646850	0920371671	5264255637	1853889571	4164197723	8742261059
6667396997	1731681694	1543509528	3193556417	7056686222	1521799115	1355639707	1433128936	5755384464	8326201206
4243380169	5586269856	1022460646	0693307938	4785881436	7407000599	7697036490	1927332882	6135329363	1124036506
9865216063	8987250267	2380874033	9674439783	0258296894	2568967418	6433613497	9475245526	2914265228	4241924308
3388103580	0537870239	9954217211	3686550275	3413622116	9314069466	9513186928	1025747959	8560514500	5021715913
3177516099	5786555198	1886193211	2821107094	4228724044	2481153406	0558959583	5581523201	2184605820	5635926993
0347885113	2068626627	5887714460	3599665610	8430725696	5005630644	8918759946	6596772847	1715395736	1210818084
1547273142	6617489331	3417463266	2354222072	6001460127	0120693463	9520564445	5432916629	8666078308	9068118790
0908152950	6362678207	5614388815	7813511346	9536630387	8412092346	9428687308	3932043233	3872775496	8052103028
2154432472	3388845215	3437272501	2858974769	1460808314	4041258681	8154004918	7772287869	8018534545	3700652665
5649170915	4295227567	0922221747	4112062720	6566229898	0603289167	2068743654	9482461086	9736722554	7404812889
2424718543	2360575341	1672850757	5520571311	5669795458	4887398742	2281358879	8584078313	5060548290	5514827852
9489112190	5383195624	2287194847	5940785939	8047901094	1940706717	6443903273	0712135887	3850499936	3883820550
1683402777	4960702768	4488028191	2220636888	6368110435	6952930065	2195528261	5269912716	3727738841	8993287130
5634646882	2739828876	3198645709	8363089177	8648708667	6185485680	0476725526	7541474285	1028145807	4031529921
9781455775	6843681110	1853174981	6701642664	7884090262	6828244482	5802753209	4549915104	5185177165	4631180490
4567985713	2575281179	1365627815	8111288816	5622858760	3087597496	3849435275	6766121689	5926148503	0785362045
2745077529	5063101248	0341804584	0594329260	7985443562	0093708091	8215239203	7179067812	1992280496	0697382387
4331262673	0306795943	9609549571	8957721791	5597300588	6936468455	7667609245	0906088202	2122357192	5453671519
1834872587	4239194108	9044411595	9932760044	5065562064	6116465566	5487594247	3692523369	5599303035	5095817626
1762318495	6190649483	9673002037	7638743693	4399982943	0209147073	6189479326	9276244518	6560239559	0537051289
7816345542	3320114975	9948962784	2432748378	8032701418	6769526211	8097500640	5149755889	6502930048	6760520801
0491537885	4139094245	3169171998	7628941277	2211294645	6829486028	1493181560	2496778879	4981377721	6229359437
8110044480	6079767242	9276249510	7841534464	2915084276	4520002042	7694706980	4177583220	9097020291	6573472515
8290463091	0359037842	9775726517	2087724474	0952267166	3060054697	1638794317	1196873484	6887381866	5675127929
8575016363	4113146275	3049901913	5646823804	3299706957	7015078933	7728658035	7127909137	6742080565	5493624646

On the Computation of Euler's Constant

By Dura W. Sweeney

1. Introduction. The computation of Euler's constant, γ, to 3566 decimal places by a procedure not previously used is described. As a part of this computation, the natural logarithm of 2 has been evaluated to 3683 decimal places. A different procedure was used in computations of γ performed by J. C. Adams in 1878 [1] and J. W. Wrench, Jr. in 1952 [2], and recently by D. E. Knuth [3]. This latter procedure is critically compared with that used in the present calculation. The new approximations to γ and ln 2 are reproduced *in extenso* at the end of this paper.

2. Evaluation of γ. A new procedure based upon the expansion of the exponential integral, $-E_i(-x)$, was used to evaluate γ rather than the classical approach used by Adams, Wrench, and Knuth. This new procedure was chosen so as to avoid the more complex programming required in the computation of high orders of Bernoulli numbers.

The exponential integral is given as

$$\begin{aligned}(1)\qquad -E_i(-x) &= \int_x^\infty \frac{e^{-t}\,dt}{t} = -\gamma - \ln x + x - \frac{x^2}{2\cdot 2!}\\ &\quad + \frac{x^3}{3\cdot 3!} - \cdots = -\gamma - \ln x + S(x).\end{aligned}$$

Its asymptotic expansion is

$$(2)\qquad -E_i(-x) = \int_x^\infty \frac{e^{-t}\,dt}{t} \cong \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} - \cdots\cdots\right) = R(x).$$

Equating these and moving γ to the left, we have

$$(3)\qquad \gamma \cong S(x) - \ln x - R(x).$$

Since the asymptotic form behaves as e^{-x}/x for large x, the difference between $S(x)$ and $\ln x$ will approximate γ to the accuracy of the number of leading zeros in the value of $R(x)$.

$$(4)\qquad \text{For}\quad x = 8192,\quad R(x) = 0.22190\cdots 10^{-3561}.$$

The value of x was chosen as a power of 2 to simplify the calculation of $\ln x$. Also, since a binary computer was to be used, many of the multiplications in the terms of $S(x)$ could be reduced to shifting operations.

3. Method of Computation. The computation of ln 2 is very rapid and straightforward on a binary computer using one of the forms of the expansion

$$(5)\qquad \ln 2 = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \cdots.$$

Received June 29, 1962.

The computation of $S(x)$ is also straightforward, but requires substantially more computer time since for $x = 8192$ almost 30,000 terms are required for convergence, and up to three times the number of digits in the final answer are required during intermediate computations to avoid truncation errors and to compensate for the loss of significant figures arising from subtractions.

The computer program was written to do the computation two different ways to establish the accuracy of the analysis, the programming, and the system operation. Two different binary-to-decimal conversion routines were also used, one with each of the computations.

The first part of the computer run used the following procedure. The individual terms of the expansion,

$$13 \ln 2 = 13\left[\frac{5}{1\cdot 1\cdot 2^3} + \frac{11}{2\cdot 3\cdot 2^5} + \frac{17}{3\cdot 5\cdot 2^7} \cdots\cdots\right], \tag{6}$$

were evaluated and summed to form ln 8192. $S(x)$ was evaluated by summing the odd and even terms of the expansion separately to avoid subtractions, thus:

$$\begin{aligned} S(x) = {} & \left(x + \frac{x^3}{3\cdot 3!} + \cdots\cdots\right) \\ & - \left(\frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} + \cdots\cdots\right) = x + \sum D_{2n+1} - \sum D_{2n}. \end{aligned} \tag{7}$$

The individual terms of these sums were computed from an intermediate value, C_{2n}, as follows:

$$C_{2n} = \frac{x^{2n+1}}{(2n)!} = \frac{x^2}{2n(2n-1)} D_{2n-2}, \quad \text{where} \tag{10}$$

$$D_{2n+1} = \frac{C_{2n}}{(2n+1)^2} \quad \text{and} \quad D_{2n} = \frac{C_{2n}}{2nx}. \tag{11}$$

The second part of the computer run used the following procedure. Ln 8192 was evaluated from the following recursion starting at $n = 12{,}300$:

$$\ln 8192 = 13B_1, \qquad B_n = \frac{1}{2}\left(\frac{1}{n} + B_{n+1}\right). \tag{12}$$

$S(x)$ was evaluated by the following recursion starting at $n = 30{,}000$:

$$S(x) = xA_1, \qquad A_n = 1 - \frac{nx}{(n+1)^2} A_{n+1}. \tag{13}$$

The complete computation was performed on the engineering model of the IBM 7094 in 58 minutes. The first part of the computer run took approximately 20 minutes. The second part took approximately 35 minutes. The remaining time was required for non-overlapped printing and punching of results. The same computation was performed again on an IBM 7090 in 114 minutes as part of the tests of the speed and compatibility of the two systems.

The computed values of ln 2 agreed to 3683 decimal places, and the tabulation is believed accurate to that number of decimals. The value of ln 2 confirms the value calculated by H. S. Uhler [4] to 330 decimal places.

The computed values of γ agreed to the same number of decimal places as ln 2, but the accuracy is limited by the value of x to 3561 decimal places. The value of $R(8192)$ given in (4) was subtracted to give the additional five decimal places shown in parentheses in the tabulation. This value of γ is believed accurate to 3566 decimal places and confirms the value calculated by D. E. Knuth to 1270 decimal places.

4. Comparison of Methods. The operating times reported by Knuth presented an opportunity to compare the two methods to determine which might be more useful in extending the value of γ to greater accuracy. An estimate of the time required shows that if the expansion of the exponential integral had been used it would have been substantially faster than the classical method for the evaluation of γ to 1271 decimal places on the Burroughs 220.

$$\text{(14)} \qquad \text{For} \quad x = 3000, \qquad R(x) < 10^{-1300}, \qquad \text{and}$$

$$\text{(15)} \qquad \begin{aligned} \ln 3000 &= \frac{7}{8} \ln 10000 + \frac{1}{2} \ln \left(1 - \frac{1}{10}\right) \\ &= \frac{7}{8} \ln 10000 - \frac{1}{2}\left(\frac{21}{1\cdot 2\cdot 10^2} + \frac{43}{3\cdot 4\cdot 10^4} + \cdots\cdots\right). \end{aligned}$$

Knuth reported a time for the evaluation of ln 10000 of approximately 18 minutes. The additional logarithm would take approximately 4 minutes more.

$S(x)$ would require approximately 10,800 terms for convergence and could probably be most efficiently computed as follows:

$$\text{(16)} \quad S(x) = \left(x + \frac{x^3}{3\cdot 3!} + \cdots\cdots\right) - \left(\frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} + \cdots\cdots\right) = \sum D_{2n-1} - \sum D_{2n},$$

where

$$\text{(17)} \qquad D_{2n-1} = \frac{(2n-2)x}{(2n-1)^2} D_{2n-2} \quad \text{and} \quad D_{2n} = \frac{(2n-1)x}{(2n)^2} D_{2n-1}.$$

The evaluation of each D_n would require a multiplication loop, a division loop, and a summation loop which would be used to evaluate each storage word of accuracy for each of the terms required for the convergence of $S(x)$. These loop operations would require less than 10 milliseconds for each word of storage. Since there are 10,800 terms, all that is required is an estimate of the accuracy or number of storage words for each term.

The upper curve in Figure 1 shows the value of $r = \log_{10}(3000^n/n \cdot n!)$. For $n = 3000$, r reaches its maximum value of almost 1300. At this value of n, the value of D_n must be known to 2600 decimal places (260 words of storage). To avoid truncation errors, at least 2600 decimal places must be carried for each $n < 3000$. As n becomes larger, the required accuracy decreases, reaching 1300 decimal places at $n \cong 8200$, and going to zero at $n \cong 10{,}800$. This is shown as the difference at a particular n between the upper and lower curves in Figure 1. If the accuracy is carried to 2600 decimal places throughout, as shown by the area between the two curves plus the area outlined by the dotted line, the computation of $S(x)$ would have taken 7.8 hours, i.e.,

$$\left(\frac{260 \times 10{,}800 \times .010}{3600}\right).$$

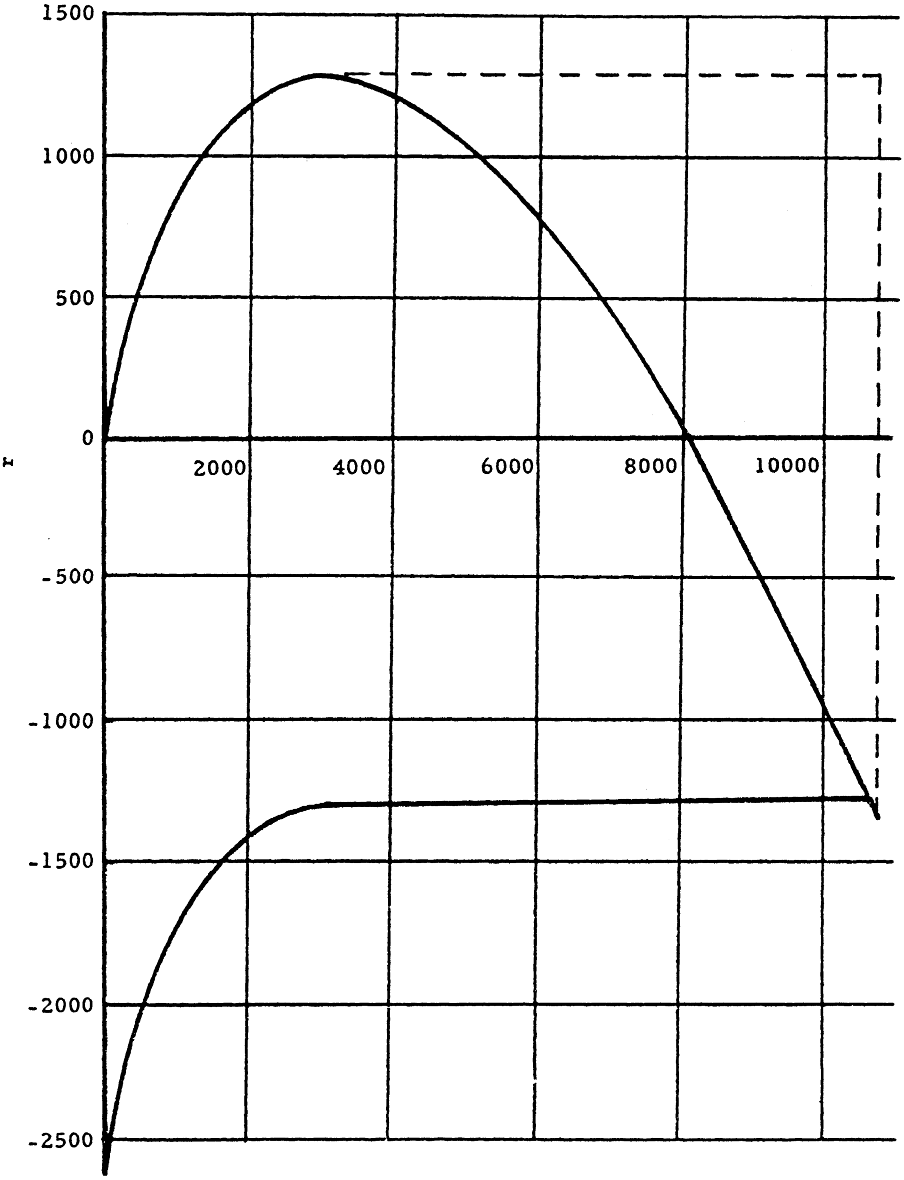


Fig. 1. $r = \log_{10}\left(\frac{3000^n}{n \cdot n!}\right)$

Since the area bounded by the two curves is less than 75% of the total area considered above, an achievable and still faster time for the evaluation of $S(x)$ would be approximately 5.8 hours. This is to be compared to the approximately 9 hours reported by Knuth for the evaluation of the sum of the first 10,000 reciprocals and the first 250 Bernoulli numbers. When the times for the evaluation of the logarithms are added, the comparison shows that the evaluation of γ by this new method would have required about two-thirds of the time reported by Knuth.

A similar comparison was attempted for the evaluation of γ to 3566 decimal places on an IBM 7094 using the classical method. This would have required the evaluation of the sum of the first 65,536 reciprocals and the first 610 Bernoulli numbers. This approach was abandoned since a "good" lower bound of the time required could not be established with reasonable effort because of the complexity in establishing the accuracy (number of words of storage) needed for each of the Bernoulli numbers used in the recursion for evaluating the next higher Bernoulli number. It also appeared that the storage capacity of the system would have been exceeded, requiring additional time and programming complexity. No auxiliary storage is required for the evaluation of γ using the expansion of the exponential integral on either computer.

It should be noted that there exists a still faster method which remains to be tried. This method will require additional programming effort, but substantially less computer time will be required. For a given x evaluate $\ln x$, $S(x)$ and e^{-x} to twice the number of decimal places which would be expected from the value of $R(x)$. Then evaluate the semi-convergent portion of $R(x)$ and multiply by the value of e^{-x}. When this value of $R(x)$ is subtracted from $S(x) - \ln x$, the accuracy of γ will be extended to that expected from the value of $R(2x)$. This method will be faster since $S(x)$ will require far fewer terms for convergence to a certain accuracy than $S(2x)$; e.g., $S(8192)$ will require approximately 36,000 terms for convergence to about 7200 decimal places, while $S(16384)$ will require almost 60,000 terms to achieve the same accuracy.

IBM Data Processing Division
Poughkeepsie, New York

1. J. C. Adams, "On the value of Euler's constant," *Proc. Roy. Soc. London*, v. 27, 1878, p. 88–94.
2. J. W. Wrench, Jr., "A new calculation of Euler's constant," *MTAC*, v. 6, 1952, p. 255.
3. D. E. Knuth, "Euler's constant to 1271 places," *Math. Comp.*, v. 16, 1962, p. 275–281.
4. H. S. Uhler, "Recalculation and extension of the modulus and of the logarithms of 2, 3, 5, 7, and 17," *Proc. Nat. Acad. Sci.*, v. 26, 1940, p. 205–212.

γ = .57721 56649 01532 86060 65120 90082 40243 10421 59335 93992
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495
14631 44724 98070 82480 96050 40144 86542 83622 41739 97644
92353 62535 00333 74293 73377 37673 94279 25952 58247 09491
60087 35203 94816 56708 53233 15177 66115 28621 19950 15079

84793 74508 57057 40029 92135 47861 46694 02960 43254 21519
05877 55352 67331 39925 40129 67420 51375 41395 49111 68510
28079 84234 87758 72050 38431 09399 73613 72553 06088 93312
67600 17247 95378 36759 27135 15772 26102 73492 91394 07984
30103 41777 17780 88154 95706 61075 01016 19166 33401 52278

93586 79654 97252 03621 28792 26555 95366 96281 76388 79272
68013 24310 10476 50596 37039 47394 95763 89065 72967 92960
10090 15125 19595 09222 43501 40934 98712 28247 94974 71956
46976 31850 66761 29063 81105 18241 97444 86783 63808 61749
45516 98927 92301 87739 10729 45781 55431 60050 02182 84409

60537 72434 20328 54783 67015 17739 43987 00302 37033 95183
28690 00155 81939 88042 70741 15422 27819 71652 30110 73565
83396 73487 17650 49194 18123 00040 65469 31429 99297 77956
93031 00503 08630 34185 69803 23108 36916 40025 89297 08909
85486 82577 73642 88253 95492 58736 29596 13329 85747 39302

37343 88470 70370 28441 29201 66417 85024 87333 79080 56275
49984 34590 76164 31671 03146 71072 23700 21810 74504 44186
64759 13480 36690 25532 45862 54422 25345 18138 79124 34573
50136 12977 82278 28814 89459 09863 84600 62931 69471 88714
95875 25492 36649 35204 73243 64109 72682 76160 87759 50880

95126 20840 45444 77992 29915 72482 92516 25127 84276 59657
08321 46102 98214 61795 19579 59095 92270 42089 89627 97125
53632 17948 87376 42106 60607 06598 25619 90102 88075 61251
99137 51167 82176 43619 05705 84407 83573 50158 00560 77457
93421 31449 88500 78641 51716 15194 56570 61704 32450 75008

16870 52307 89093 70461 43066 84817 91649 68425 49150 49672
43121 83783 87535 64894 95086 84541 02340 60162 25085 15583
86723 49441 87880 44094 07701 06883 79511 13078 72023 42639
52269 20971 60885 69083 82511 37871 28368 20491 17892 59447
84861 99118 52939 10293 09905 92552 66917 27446 89204 43869

71114 71745 71574 57320 39352 09122 31608 50868 27558 89010
94516 81181 01687 49754 70969 36667 12102 06304 82716 58950
49327 31486 08749 40207 00674 25909 18248 75962 13738 42311
44265 31350 29230 31751 72257 22162 83248 83811 24589 57438
62398 70375 76628 55130 33143 92999 54018 53134 14158 62127

88648 07611 00301 52119 65780 06811 77737 63501 68183 89733
89663 98689 57932 99145 63886 44310 37060 80781 74489 95795
83245 79418 96202 60498 41043 92250 78604 60362 52772 60229
19682 99586 09883 39013 78717 14226 91788 38195 29844 56079
16051 97279 73604 75910 25109 95779 13351 57917 72251 50254

92932 46325 02874 76779 48421 58405 07599 29040 18557 64599
01862 69267 76437 26605 71176 81336 55908 81554 81074 70000
62336 37252 88949 55463 69714 33012 00791 30855 52639 59549
78230 23144 03914 97404 94746 82594 73208 46185 24605 87766
94882 87953 01040 63491 72292 18580 08706 77069 04279 26743

γ (*continued*)

28444	69685	14971	82567	80958	41654	49185	14575	33196	40633
11993	73821	57345	08749	88325	56088	88735	28019	01915	50896
88554	68259	24544	45277	28173	05730	10806	06177	01136	37731
82462	92466	00812	77162	10186	77446	84959	51428	17901	45111
94893	42288	34482	53075	31187	01860	97612	24623	17674	97755
64124	61983	85640	14841	23587	17724	95542	24820	16151	76579
94080	62968	34242	89057	25947	39269	63863	38387	43805	47131
96764	29268	37249	07608	75073	78528	37023	04686	50349	05120
34227	21743	66897	92848	62972	90889	26789	77703	26246	23912
26188	87653	00577	86274	36060	94443	60392	80977	08133	83693
42355	08583	94112	67092	18734	41451	21878	03276	15050	94780
55466	30058	68455	63152	45460	53151	13252	81889	10792	31491
31103	23443	02450	93345	00030	76558	64874	22297	17700	33178
45391	50566	94015	99884	92916	09114	00294	86902	08848	53816
97009	55156	63470	55445	22176	40358	62939	82865	81312	38701
32535	88006	25686	62692	69977	67737	73068	32269	00916	08510
45150	02261	07180	25546	59284	93894	92775	95897	54076	15599
33782	64824	19795	06418	68143	78817	18508	85408	03679	96314
23954	00919	64388	75007	89000	00627	99794	28098	86372	99259
19777	65040	40992	20379	40427	61681	78371	56686	53066	93983
09165	24322	70595	53041	76673	66401	16792	95901	29305	37449
71830	80042	7(58486)							

ln 2 = .69314 71805 59945 30941 72321 21458 17656 80755 00134 36025
52541 20680 00949 33936 21969 69471 56058 63326 99641 86875
42001 48102 05706 85733 68552 02357 58130 55703 26707 51635
07596 19307 27570 82837 14351 90307 03862 38916 73471 12335
01153 64497 95523 91204 75172 68157 49320 65155 52473 41395

25882 95045 30070 95326 36664 26541 04239 15781 49520 43740
43038 55008 01944 17064 16715 18644 71283 99681 71784 54695
70262 71631 06454 61502 57207 40248 16377 73389 63855 06952
60668 34113 72738 73722 92895 64935 47025 76265 20988 59693
20196 50585 54764 70330 67936 54432 54763 27449 51250 40606

94381 47104 68994 65062 20167 72042 45245 29612 68794 65461
93165 17468 13926 72504 10380 25462 59656 86914 41928 71608
29380 31727 14367 78265 48775 66485 08567 40776 48451 46443
99404 61422 60319 30967 35402 57444 60703 08096 08504 74866
38523 13818 16767 51438 66747 66478 90881 43714 19854 94231

51997 35488 03751 65861 27535 29166 10007 10535 58249 87941
47295 09293 11389 71559 98205 65439 28717 00072 18085 76102
52368 89213 24497 13893 20378 43935 30887 74825 97017 15591
07088 23683 62758 98425 89185 35302 43634 21436 70611 89236
78919 23723 14672 32172 05340 16492 56872 74778 23445 35347

64811 49418 64238 67767 74406 06956 26573 79600 86707 62571
99184 73402 26514 62837 90488 30620 33061 14463 00737 19489
00274 36439 65002 58093 65194 43041 19115 06080 94879 30678
65158 87090 06052 03468 42973 61938 41289 65255 65396 86022
19412 29242 07574 32175 74890 97705 75268 71158 17051 13700

91589 42665 47859 59648 90653 05846 02586 68382 94002 28330
05382 07400 56770 53046 78700 18416 24044 18833 23279 83863
49001 56312 18895 60650 55315 12721 99398 33203 07514 08426
09147 90012 65168 24344 38935 72472 78820 54862 71552 74187
72430 02489 79454 01961 87233 98086 08316 64811 49093 06675

19339 31289 04316 41370 68139 77764 98176 97486 89038 87789
99129 65036 19270 71088 92641 05230 92478 39173 73501 22984
24204 99568 93599 22066 02204 65494 15106 13918 78857 44245
57751 02068 37030 86661 94808 96412 18680 77902 08181 58858
00016 88115 97305 61866 76199 18739 52007 66719 21459 22367

20602 53959 54365 41655 31129 51759 89940 05600 03665 13567
56905 12459 26825 74394 64831 68332 62490 18038 24240 82423
14523 06140 96380 57007 02551 38770 26817 85163 06902 55137
03234 05380 21450 19015 37402 95099 42262 99577 96474 27138
15736 38017 29873 94070 42421 79972 26696 29799 39312 70693

57472 40493 38653 08797 58721 69964 51294 46491 88377 11567
01678 59880 49818 38896 78413 49383 14014 07316 64727 65327
63591 92335 11233 38933 87095 13209 05927 21854 71328 97547
07978 91384 44546 66761 92702 88553 34234 29899 32180 37691
54973 34026 75467 58873 23677 83429 16191 81043 01160 91695

26554 78597 32891 76354 55567 42863 87746 39871 01912 43175
42558 88301 20677 92102 80341 20687 97591 43081 28330 72303
00883 49470 57924 96591 00586 00123 41561 75741 32724 65943
06843 54652 11135 02154 43415 39955 38185 65227 50221 42456
64400 06276 18330 32064 72725 72197 51529 08278 56842 13207

ln 2 (*continued*)

95988	63896	72771	19552	21881	90466	03957	00977	47065	12619
50527	89322	96088	93140	56254	33442	55239	20620	30343	94177
73579	45592	12590	19925	59114	84402	42390	12554	25900	31295
37051	92206	15064	34583	78787	30020	35414	42178	57580	13236
45166	07099	14383	14500	49858	96688	57722	21486	52882	16941
81270	48860	75897	22032	16663	12837	83291	56763	07498	72985
74638	92826	93735	09840	77804	93950	04933	99876	26475	50703
16221	61390	34845	29942	49172	48373	40613	66226	38349	36811
16841	67056	92521	47513	83930	63845	53718	62687	79732	88955
58871	63442	97562	44755	39236	63694	88877	82389	01749	81027
35655	24050	51854	77306	19440	52423	22125	59024	83308	27788
88890	59629	11972	99545	74415	62451	24859	26831	12607	46797
28163	80902	50005	65599	91461	28332	54358	11140	48482	06064
08242	24792	40385	57647	62350	31100	32425	97091	42501	11461
55848	30670	01258	31821	91534	72074	74111	94009	83557	32728
26144	27382	13970	70477	95625	96705	79023	03384	80617	13455
55368	55375	81065	74973	44479	22511	19654	61618	27896	01006
85129	65395	47965	86637	83522	47362	45460	93585	03605	06784
14391	14452	31457	78033	59179	21127	95570	50555	54514	38788
81881	53519	48593	44672	46429	49864	05062	65184	24475	39566
37833	73482	20753	32944	81306	49336	03546	10101	77464	93267
87716	71986	12073	96832	01235	96077	29024	68304	59403	13056
37763	13240	10804	20285	43590	26945	09403	07400	14933	95076
73160	28502	86973	03187	18239	98433	525			

ACTA ARITHMETICA
X (1964)

Approximations to the logarithms of certain rational numbers

by

A. BAKER (Cambridge)

1. Introduction. In a recent paper [1] methods were introduced for investigating the accuracy with which certain algebraic numbers may be approximated by rational numbers. It is the main purpose of the present paper to deduce, using similar techniques, results concerning the accuracy with which the natural logarithms of certain rational numbers may be approximated by rational numbers, or, more generally, by algebraic numbers of bounded degree.

Results of this type were first proved by Morduchai-Boltowskoj in 1923 (see [6]) but the most precise results so far established are due to Mahler [4] and Feldman [2]. Suppose that a is an algebraic number other than 0 or 1. Then the work of Mahler leads to inequalities of the form

$$|\log a - \xi| > H^{-\varkappa}$$

valid for all algebraic numbers ξ of degree n and sufficiently large height H, where $\varkappa$ is an explicit function of n, of order c^n, where c is a constant >1. For small values of n and rational a these represent the best inequalities known, but for large n very much stronger results were recently given by Feldman, indeed with $\varkappa$ of order $(n\log n)^2$.

In the present paper we shall begin by proving

THEOREM 1. *Let a, b and n be positive integers and let $a = b/a$. Suppose that $\varkappa > n$ and*

(1) $$a > \{(4\sqrt{2})^n h\}^{\varrho},$$

where $h = b-a > 0$ and

(2) $$\varrho = (n+1)(\varkappa+1)(\varkappa-n)^{-1}.$$

Then

(3) $$|x_0 + x_1 \log a + \ldots + x_n (\log a)^n| > cX^{-\varkappa}$$

for all integers $x_0, x_1, \ldots, x_n$, *where*

(4) $$X = \max(|x_0|, |x_1|, \ldots, |x_n|) > 0$$

and c *is given by*

(5) $$c = a^{-\lambda \log a} \quad \text{where} \quad \lambda = 50\varkappa(\varkappa+1).$$

It follows that for certain rational numbers a the inequality (3) holds with $\varkappa$ only slightly greater than n, and this is almost the best possible; for it is well known that (3) could not hold for all integers $x_0, x_1, \ldots, x_n$, not all zero, with any constant c, if $\varkappa$ were less than n.

From Theorem 1 we obtain as an immediate deduction the following

COROLLARY. *Suppose that the hypotheses of Theorem* 1 *hold and let* $\delta = n(\varkappa - n)$. *Then*

(i) $|\log a - \xi| > H^{-n-1-\delta}$ *for all algebraic numbers* ξ *of degree at most* n *and sufficiently large height* H.

(ii) *There are infinitely many algebraic numbers* ξ *of degree at most* n *and height* H *for which* $|\log a - \xi| < H^{-n-1+\delta}$.

The proof of Theorem 1 depends on combining a theorem of Mahler, given in [4], concerning certain polynomials in $\log x$, with a lemma of an arithmetical nature due essentially to Siegel (see [8]). The corollary is deduced by direct application of two formulae of Wirsing [9].

The condition (1) may be relaxed if we suppose that $n = 1$, $h = 1$. We prove

THEOREM 2. *For all integers* $a > 0$, p *and* $q > 0$ *we have*

(6) $$\left|\log\left(1+\frac{1}{a}\right) - \frac{p}{q}\right| > c(a)q^{-\varkappa(a)}$$

where

$$\varkappa(1) = 12{\cdot}5, \quad \varkappa(2) = 7,$$

$$\varkappa(a) = 2\frac{\log\{4\sqrt{2}a^2/(a+1)\}}{\log\{\sqrt{2}a^3/(a+1)^2\}} \quad \text{for} \quad a \geqslant 3,$$

and

(7) $$c(1) = 10^{-10^5}, \quad c(a) = (\sqrt{2}a)^{-10^4} \quad \text{for} \quad a \geqslant 2.$$

Here $\varkappa(a)$ is decreasing and tends to 2 as a tends to infinity. For $a \geqslant 15$ we see that $\varkappa(a) < 3$ and thus we obtain, for example, the following measure of irrationality for $\log\frac{15}{16}$;

$$\left|\log\frac{15}{16} - \frac{p}{q}\right| > q^{-3}$$

for all rationals p/q with q sufficiently large.

Finally, by way of application, we give a positive lower bound for the fractional part of the sum of the series

$$e^{-\theta}+e^{-2\theta}+e^{-3\theta}+\dots, \tag{8}$$

where θ is any positive rational number. Clearly we need consider only the case in which the sum ζ of (8) is greater than 1. Then the result is as follows.

THEOREM 3. *Let $\theta>0$ be a rational number with denominator $q>0$ and suppose that*

$$\zeta=(e^{\theta}-1)^{-1}>1.$$

Then the fractional part of ζ is greater than $c(1)q^{-12\cdot 5}$ where $c(1)$ is given by (7).

I am indebted to Prof. Davenport for valuable suggestions in connection with the present work.

2. Lemmas. The following lemma is due essentially to Siegel.

LEMMA 1. *Let n be a positive integer and let q_{ij} $(i,j=0,1,\dots,n)$ be $(n+1)^2$ integers with absolute values at most Q such that the matrix (q_{ij}) is non-singular. Suppose that $\xi_1,\dots,\xi_n$ are real or complex numbers and let*

$$\Phi_i=q_{i0}+q_{i1}\xi_1+\dots+q_{in}\xi_n \tag{9}$$

for $i=0,1,\dots,n$. Suppose that the Φ_i have absolute values at most Φ. If $x_0,x_1,\dots,x_n$ are integers, not all zero with absolute values at most X and

$$\Psi=x_0+x_1\xi_1+\dots+x_n\xi_n \tag{10}$$

then

$$|\Psi|\geqslant(n!Q^n)^{-1}-n\Phi XQ^{-1}. \tag{11}$$

Proof. Since (q_{ij}) is non-singular, there are n of the linear forms (9) in $1,\xi_1,\dots,\xi_n$ which together with the linear form (10) make up a linearly independent set. Without loss of generality we can take them to be the last n forms. We have

$$\begin{vmatrix}\Phi_1 & q_{11} & \cdots & q_{1n}\\ \cdot & \cdot & \cdot & \cdot\\ \Phi_n & q_{n1} & \cdots & q_{nn}\\ \Psi & x_1 & \dots & x_n\end{vmatrix}=\begin{vmatrix}q_{10} & q_{11} & \cdots & q_{1n}\\ \cdot & \cdot & \cdot & \cdot\\ q_{n0} & q_{n1} & \cdots & q_{nn}\\ x_0 & x_1 & \dots & x_n\end{vmatrix}$$

and the determinant on the right is a non-zero integer. Expanding the determinant on the left, and estimating each term, we obtain

$$n!Q^n|\Psi|+n(n!Q^{n-1}X\Phi)\geqslant 1,$$

which gives (11).

LEMMA 2. *Let a be a real number such that $1 < a \leqslant 2$ and let m, n be positive integers. Then there exist $(n+1)^2$ polynomials $A_{ij}(x)$ $(i, j = 0, 1, \ldots, n)$ in x of degree at most m with the following properties.*

(i) *The determinant of order $(n+1)$ with $A_{ij}(a)$ in the i-th row and j-th column $(i, j = 0, 1, \ldots, n)$ is not zero.*

(ii) *Each polynomial $A_{ij}(x)$ has integer coefficients with absolute values at most*

$$n!\,2^{n-\frac{3}{2}m}(m+1)^{2n+1}(4\sqrt{2})^{(n+1)m}.$$

(iii) *The $(n+1)$ functions*

$$R_i(x) = \sum_{j=0}^{n} A_{ij}(x)(\log x)^j \qquad (i = 0, 1, \ldots, n) \tag{12}$$

satisfy the inequalities

$$|R_i(a)| \leqslant n!\,2^{-\frac{3}{2}m}(e\sqrt{m})^{n+1}a^{2m+1}\{\sqrt{8}(n+1)^{-1}\log a\}^{(n+1)m}. \tag{13}$$

Proof. This is a special case of Theorem 1 of Mahler [4] (see p. 378). We have interchanged m and n and restricted the number a so that $1 < a \leqslant 2$. Then

$$n+1 > 2|\log a|$$

and thus a condition required by Mahler's Theorem is satisfied.

On the basis of Lemma 2 we introduce the following notation. Corresponding to each pair of positive integers m, n and each pair of integers a, b such that $a > 0$ and $1 < \alpha \leqslant 2$, where $\alpha = b/a$, we define numbers $q_{ij} = q_{ij}(m, n, a, b)$ by the equations

$$q_{ij} = a^m A_{ij}(\alpha) \qquad (i, j = 0, 1, \ldots, n), \tag{14}$$

where the $A_{ij}(x)$ are the polynomials given by Lemma 2 corresponding to m, n, α. Since the $A_{ij}(x)$ are of degree at most m it follows that the q_{ij} are integers. Also from (i) of Lemma 2 it is clear that the matrix (q_{ij}) is non-singular. We now put

$$\xi_i = (\log \alpha)^i \quad \text{for} \quad i = 1, 2, \ldots, n \tag{15}$$

and define the numbers $\Phi_i = \Phi_i(m, n, a, b)$ $(i = 0, 1, \ldots, n)$ by (9). Then clearly, for each i,

$$\Phi_i = a^m R_i(\alpha), \tag{16}$$

where the $R_i(x)$ are the functions given by (12) of Lemma 2. Finally we note that if the Φ_i have absolute values at most Φ, $x_0, x_1, \ldots, x_n$ are integers, not all zero, with absolute values at most X, and Ψ is given by (10) then all the hypotheses of Lemma 1 are satisfied and hence (11) holds.

3. Proof of Theorem 1 and Corollary. We note first that, from (1) and (2), $a > h$, so that α is a rational between 1 and 2 exclusive.

Let $x_0, x_1, \ldots, x_n$ be integers, not all zero, and let X be given by (4). Suppose that ξ_i is given by (15) and that Ψ is defined by (10). We proceed to prove that (3) holds, that is

$$|\Psi| > cX^{-\varkappa}. \tag{17}$$

We put

$$w = \{h(4\sqrt{2})^n(n+1)^{-1}\}^{n+1} \tag{18}$$

and suppose first that

$$X \geqslant (a/w)^{50\varkappa\log a}. \tag{19}$$

From (2) it is clear that $\varrho > n+1$ and it follows from (1) and (18) that $a > w$. Thus there is a positive integer m such that

$$(a/w)^{m-1} \leqslant n^{n+1}X < (a/w)^m. \tag{20}$$

The supposition (19) then implies that

$$m > 50\varkappa\log a. \tag{21}$$

Our next object is to calculate upper bounds for the numbers $q_{ij} = q_{ij}(m, n, a, b)$ and $\Phi_i = \Phi_i(m, n, a, b)$ defined as above. Several preliminary inequalities will be required. First we note that

$$\sum_{r=0}^{m} \alpha^r < \sum_{r=0}^{m} 2^r < 2^{m+1}. \tag{22}$$

Secondly it is clear that $\varrho > 2$ and from (1) we obtain $a > 32h$. Hence

$$\alpha^{2m+1} = \left(1+\frac{h}{a}\right)^{2m+1} < 2^{(2m+1)/8} < 2^{m/2}. \tag{23}$$

Thirdly, since $e^x > 1+x$ for each $x > 0$, it follows that

$$\log\alpha < h/a. \tag{24}$$

Next we prove that

$$2^{m/2} > n^n 2^{n+1}(m+1)^{2n+1}. \tag{25}$$

From (1) and (2) we obtain

$$\log a > n(n+1)\log(4\sqrt{2}) > 3n$$

and it follows from (21) that $m > (12n)^2$. It is then easily verified that $\log(m+1) < \sqrt{m}$ and hence

$$(2n+1)\log(m+1) < 3n\sqrt{m} < \tfrac{1}{4}m < \tfrac{3}{8}m\log 2. \tag{26}$$

Also we obtain

$$(27)\qquad n\log n+(n+1)\log 2 < 2n^2 < \tfrac{1}{8}m\log 2,$$

and then (25) is deduced by adding (26) and (27). Finally we shall require the inequality

$$(28)\qquad 2^m > n^n e^{n+1} m^{(n+1)/2}$$

which is clear from (25).

Now from (ii) of Lemma 2, (14) and (22) we see that the integers q_{ij} have absolute values at most

$$a^m (n!) 2^{n-3m/2}(m+1)^{2n+1}(4\sqrt{2})^{(n+1)m}\sum_{r=0}^{m} a^r$$
$$\leqslant a^m n^n 2^{n-m/2+1}(m+1)^{2n+1}(4\sqrt{2})^{(n+1)m},$$

and, from (25), it follows that this is less than Q where

$$(29)\qquad Q = a^m (4\sqrt{2})^{(n+1)m}.$$

From (13), (16), (23) and (24) we deduce that the Φ_i have absolute values at most

$$a^m n^n 2^{-3m/2}(e\sqrt{m})^{n+1}2^{m/2}\{\sqrt{8}(n+1)^{-1}h/a\}^{(n+1)m}$$
$$= a^{-mn}2^{-m}n^n e^{n+1}m^{(n+1)/2}\{\sqrt{8}h(n+1)^{-1}\}^{(n+1)m},$$

and, from (28), this is less than Φ where

$$(30)\qquad \Phi = a^{-mn}\{\sqrt{8}h(n+1)^{-1}\}^{(n+1)m}.$$

We now use Lemma 1. From (11) we obtain

$$(31)\qquad |\Psi| \geqslant (nQ)^{-n}\{1-n^{n+1}\Phi X Q^{n-1}\}.$$

Since from (18), (29), (30) and the right-hand inequality of (20),

$$n^{n+1}\Phi X Q^{n-1} \leqslant (a/w)^m \Phi Q^{n-1} = 2^{-(n+1)m},$$

it follows from (31) that

$$(32)\qquad |\Psi| > \tfrac{1}{2}(nQ)^{-n}.$$

We next prove that

$$(33)\qquad (a/w)^{(m-1)\varkappa} > Q^n.$$

From (1), (2) and (18) we deduce that

$$a^{\varkappa-n} > \{h(4\sqrt{2})^n\}^{(n+1)(\varkappa+1)}$$
$$> 2\{h(4\sqrt{2})^n(n+1)^{-1}\}^{(n+1)\varkappa}(4\sqrt{2})^{n(n+1)} = 2w^{\varkappa}(4\sqrt{2})^{n(n+1)},$$

and hence

$$(34)\qquad a^{m(\varkappa-n)} > 2^m w^{(m-1)\varkappa}(4\sqrt{2})^{mn(n+1)}.$$

Further, from (21) we obtain

$$(35)\qquad a^{\varkappa} = e^{\varkappa\log a} < 2^{2\varkappa\log a} < 2^{m}.$$

It follows from (34) and (35) that

$$a^{m(\varkappa-n)} > a^{\varkappa} w^{(m-1)\varkappa} (4\sqrt{2})^{mn(n+1)},$$

and this is equivalent to (33). Then from (20) and (33) we obtain

$$(n^{n+1}X)^{\varkappa} > Q^{n}$$

so that, from (32),

$$(36)\qquad |\Psi| > CX^{-\varkappa},$$

where

$$(37)\qquad C = \tfrac{1}{2} n^{-n-(n+1)\varkappa}.$$

It is clear from (1), (2) and (37) that

$$C > (2n)^{-(n+1)(\varkappa+1)} > a^{-(\varkappa+1)\log a} > c,$$

where c is given by (5), and hence (17) certainly holds for all integers $x_0, x_1, \ldots, x_n$ such that (19) is satisfied.

Now suppose that $x_0, x_1, \ldots, x_n$ are integers, not all zero, for which (19) does not hold. Then $X < u$ where u is the integer given by

$$(38)\qquad u = [(a/w)^{50\varkappa\log a}] + 1.$$

Since at least one of the $(n+1)$ integers $ux_0, ux_1, \ldots, ux_n$ has absolute value at least u, we may apply the result (36), just established, with this set of integers in place of $x_0, x_1, \ldots, x_n$ and uX in place of X. We obtain

$$u|\Psi| > C(uX)^{-\varkappa},$$

that is

$$(39)\qquad |\Psi| > Cu^{-\varkappa-1}X^{-\varkappa}.$$

Finally we show that

$$(40)\qquad Cu^{-\varkappa-1} > c.$$

From (38),

$$u^{\varkappa+1} < 2^{\varkappa+1}(a/w)^{\lambda\log a},$$

where λ is given by (5), and it follows from (1), (2) and (37) that

$$Cu^{-\varkappa-1} > (4n/w)^{-(n+1)(\varkappa+1)} a^{-\lambda\log a}.$$

It is clear from (18) that

$$w \geqslant (2\sqrt{2})^{n(n+1)} > 4n$$

and thus (40) holds as required. Then (17) follows from (39) and (40) and this completes the proof of Theorem 1.

As for the proof of the corollary, the results follow by an immediate application of two inequalities given in Wirsing [9]. With the notation of that paper, Theorem 1 implies that

$$w_n(\log a) \leqslant \varkappa.$$

Then from (3) of [9] (see p. 68) it follows that

$$w_n^*(\log a) \leqslant w_n(\log a) \leqslant \varkappa.$$

and from (7) of [9] we obtain

$$w_n^*(\log a) \geqslant w_n(\log a)/\big(w_n(\log a) - n + 1\big) \geqslant \varkappa/(\varkappa - n + 1).$$

However $\varkappa < n + \delta$ (for $n > 1$) and

$$\varkappa/(\varkappa - n + 1) = (n^2 + \delta)/(n + \delta) > n - \delta,$$

where $\delta = n(\varkappa - n)$, and, by the definition of $w_n^*(\log a)$, this proves the corollary.

4. Proof of Theorem 2. We distinguish two cases according as $a > 1$ or $a = 1$. In the first case we repeat the arguments of Theorem 1 with $n = 1$, $h = 1$ but base these arguments on stronger estimates. In the case $a = 1$ it is necessary to modify the methods of Theorem 1 and we proceed in a similar manner to Mahler [4]. Theorem 5 of [4] contains the result that (6) holds with $\varkappa(1) = 48$ and by means of suitable estimates for the numbers $q_{ij}(m, 2, 1, 2)$ and $\Phi_i(m, 2, 1, 2)$ this may be improved to $\varkappa(1) = 12{\cdot}5$. The proofs follow directly on the lines indicated and we omit the details.

5. Proof of Theorem 3. Let $a = [\zeta]$. From Theorem 2 we obtain

$$\left|\log\left(1 + \frac{1}{a}\right) - \theta\right| > c(a)q^{-12\cdot 5}, \tag{41}$$

where $c(a)$ is given by (7). We proceed to prove that $c(a)$ in (41) can be replaced by $c(1)a^{-1}$ and the required result then follows by application of the mean-value theorem. It is clear from the form of ζ that there is only one significant value of a, if $a \geqslant 4$, namely the integer nearest to $(\theta^{-1} - \frac{1}{2})$. Since also $c(a) > c(1)$ for $a < 32$, it suffices to consider the case $q > a \geqslant 32$. The proof now follows in a similar manner to that of Theorem 1 with $n = 1$, $h = 1$ and we again omit the details.

References

[1] A. Baker, *Rational approximations to certain algebraic numbers*, Proc. Lond. Math. Soc. (3)14(1964), pp. 385-398.

[2] N. I. Feldman, *Approximation by algebraic numbers to the logarithms of algebraic numbers*, Izw. Akad. Nauk. SSSR, ser. mat., 24 (1960), pp. 475-492 (in Russian).

[3] K. Mahler, *Zur Approximation der Exponentialfunktion und des Logarithmus,* Jour. reine. angew. Math. 166 (1932), (I) pp. 118-136, (II) pp. 137-150.

[4] — *On the approximation of logarithms of algebraic numbers,* Philos. Trans. Roy. Soc. Lond., ser. A, 245 (1953), pp. 371-398.

[5] — *On the approximation of π,* Proc. Akad. Wetensch. Amst., ser. A, 56 (1953), pp. 30-42.

[6] D. Morduchai-Boltowskoj, *Sur le logarithme d'un nombre algebrique,* Comptes Rendus Acad. Sci., Paris, 176 (1923), pp. 724-727.

[7] Th. Schneider, *Einführung in die transzendenten Zahlen,* Berlin, Göttingen, Heidelberg, 1957, Kap. 4.

[8] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen,* Abh. Preuss. Akad. Wiss. (1929), No 1.

[9] E. Wirsing, *Approximation mit algebraischen Zahlen beschrankten Grades.* Jour. reine. angew. Math. 206 (1961), pp. 67-77.

Reçu par la Rédaction le 14. 12. 1963

Reprinted from the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES
Vol. 55, No. 1, pp. 28–31. January, 1966.

ASYMPTOTIC DIOPHANTINE APPROXIMATIONS TO E*

BY WILLIAM W. ADAMS

UNIVERSITY OF CALIFORNIA, BERKELEY

Communicated by S. Eilenberg, November 1, 1965

1. *Statement of Results.*—Schmidt[5] proved that for almost all real numbers α, the number of solutions in integers p, q of the inequalities

$$|q\alpha - p| < 1/q \quad \text{and} \quad 1 \leqq q \leqq B$$

is asymptotic to a constant times $\log B$. One might conjecture that the classical numbers (e.g., algebraic numbers, e, π) behave like almost all numbers. Machine computations[1] were carried out for some of these numbers, and they seemed to bear out such a conjecture. Also, Lang[3] has proved that the estimate is valid when α is a real quadratic irrationality.

In this paper, we shall obtain an asymptotic estimate for e, which shows that the conjecture for e is false. The machine computations of reference 1 are misleading because the range in which they are carried out, even though going to $q \leqq 10^6$, is still too small to exhibit the proper asymptotic behavior.

The function $4^x\Gamma(x + 3/2)$ is strictly monotone-increasing. Let G be its inverse function. An application of L'Hôpital's rule shows that

$$\lim_{x \to \infty} G(x) \Big/ \frac{\log x}{\log \log x} = 1.$$

THEOREM 1. *Let $\lambda_1(B)$ be the number of solutions in integers p, q of the inequalities*

$$|qe - p| < 1/q \quad \text{and} \quad 1 \leqq q \leqq B. \tag{1}$$

Then

$$\lambda_1(B) = \frac{(2G(B))^{3/2}}{3} + O(G(B)).$$

THEOREM 2.—*Let $\lambda_2(B)$ be the number of solutions in relatively prime integers p, q of the inequalities* (1) *above. Then*

$$\lambda_2(B) = 3G(B) + O(1).$$

We note that if we let $\lambda_1{}^+(B)$ be the number of solutions in integers p, q of the inequalities $0 < qe - p < 1/q$ and $1 \leqq q \leqq B$, then a trivial modification of the proof given below shows that $\lambda_1{}^+ = {}^1/_2\lambda_1$, and similarly for $\lambda_2{}^+$.

Theorems similar to these are true for any irrational number whose continued fraction expansion is similar to that of e. However, the notation is considerably more involved and so these extensions will be reserved for a later paper.

2. *Some Facts about Continued Fractions.*—The proofs of the theorems are based on the simple continued fraction for e. We recall some easily proved facts. See Khinchin[2] for details.

Let $\alpha > 0$ be any irrational number, and denote by $[a_0, a_1, a_2, \ldots]$ the simple continued fraction expansion of α, where the a_n are integers, positive for $n \geqq 1$. Denote the nth principal convergent to α by $p_n/q_n = [a_0, a_1, \ldots, a_n]$ and the nth intermediate convergents by $(p_n + rp_{n+1})/(q_n + rq_{n+1})$, where $1 \leqq r < a_{n+2}$. We shall refer to both types of convergents simply as convergents. We know that the even convergents form a strictly increasing sequence of rational numbers tending to α, and the odd convergents form a strictly decreasing sequence tending to α. Moreover, if P/Q and P'/Q' are two successive terms in either sequence, then $PQ' - P'Q = \pm 1$. We recall that $q_n = a_n q_{n-1} + q_{n-2}$, and hence, that $q_n/q_{n-1} = [a_n, a_{n-1}, \ldots, a_1]$. Further, we have always $|q_n\alpha - p_n| < 1/q_n$. Finally, let $\alpha_n = [a_n, a_{n+1}, \ldots]$. Then we have the easily derived formula

$$|(q_n + rq_{n+1})\alpha - (p_n + rp_{n+1})| = \frac{\alpha_{n+2} - r}{q_n + \alpha_{n+2}q_{n+1}}, \tag{2}$$

for integers $0 \leqq r < a_{n+2}$.

LEMMA. *If $|\alpha - p/q| < 1/q^2$ where p, q are positive integers, then p/q is a convergent of α.*

Proof: We assume that $\alpha < p/q$, the other case being proved in a similar manner. If p/q is not a convergent, then there exist two successive convergents P/Q and P'/Q' such that $\alpha < P/Q < p/q < P'/Q'$ and $P'Q - PQ' = 1$. Thus,

$$\frac{1}{q^2} > \frac{p}{q} - \alpha > \frac{p}{q} - \frac{P}{Q} \geqq \frac{1}{qQ},$$

and

$$\frac{1}{Q'q} \leqq \frac{P'}{Q'} - \frac{p}{q} < \frac{P'}{Q'} - \frac{P}{Q} = \frac{1}{Q'Q}.$$

These estimates are contradictory.

3. *Proof of the Theorem.*—We know[4] that $e = [2,1,2,1,1,4,1,1,6,1,1,8,\ldots]$ so that for e, we have $a_0 = 2$, $a_{3m} = a_{3m-2} = 1$, and $a_{3m-1} = 2m$ for all $m \geqq 1$. We set as above, $e_n = [a_n, a_{n+1}, \ldots]$.

PROPOSITION 1. *We have for all n, $m \geqq 1$,*

$$\lambda_1(q_{3m+1}) = {}^1/_3(2m)^{3/2} + O(m)$$

$$\lambda_2(q_n) = n + O(1).$$

Proof: We first show that, with a finite number of exceptions, the only reduced fractions p/q satisfying $|e - p/q| < 1/q^2$ are the principal convergents. From this the formula for λ_2 is clear since the nth solution is simply p_n, q_n.

By the lemma we know that any solution must be a convergent, so suppose that $(p_n + rp_{n+1})/(q_n + rq_{n+1})$ with $0 \leqq r < a_{n+2}$ is a solution. We wish to show that $r = 0$. If $n = 3m - 2$ or $n = 3m - 1$, then $a_{n+2} = 1$ and so $r = 0$. Thus, we may restrict our attention to the case where $n = 3m$. To say that this convergent is a solution simply means by (2) that

$$\frac{e_{3m+2} - r}{q_{3m} + e_{3m+2}q_{3m+1}} < \frac{1}{q_{3m} + rq_{3m+1}}.$$

This condition is equivalent to $f(r) > 0$, where

$$f(r) = r^2 + \left(\frac{q_{3m}}{q_{3m+1}} - e_{3m+2}\right)r + \left(\frac{q_{3m}}{q_{3m+1}} + e_{3m+2} - e_{3m+2}\frac{q_{3m}}{q_{3m+1}}\right).$$

Thus, we must show that $f(1), f(2), \ldots, f(a_{3m+2} - 1) < 0$. Since f is a quadratic polynomial in r with leading coefficient 1, it suffices to show that $f(1) < 0$ and $f(a_{3m+2} - 1) < 0$. Well, $f(1) = 1 + (2 - e_{3m+2})q_{3m}/q_{3m+1}$, and $f(1) < 0$ follows from

$$e_{3m+2} > a_{3m+2} = 2(m+1), \qquad q_{3m+1}/q_{3m} = [1,1,2m,\ldots] < 2.$$

Furthermore,

$$f(a_{3m+2} - 1) = (e_{3m+2} - a_{3m+2})\,(2 - a_{3m+2} - q_{3m}/q_{3m+1}) + 1,$$

and $f(a_{3m+2} - 1) < 0$ follows similarly.

We must now determine which multiples of the convergents p_n, q_n are also solutions of (1). Again by (2), with $r = 0$, this condition is equivalent to the condition

$$\frac{e_{n+2}}{q_n + e_{n+2}q_{n+1}} < \frac{1}{k^2 q_n} \quad \text{or} \quad k^2 < \frac{1}{e_{n+2}} + \frac{q_{n+1}}{q_n}.$$

If $n = 3m - 1$ or $n = 3m$, then the condition implies $k^2 < 4$, so $k = 1$ is the only possibility. If $n = 3m - 2$, the condition amounts to $k^2 < 2m + O(1)$, i.e., $1 \leqq k < (2m)^{1/2} + O(1)$. For such k, we note that $kq_{3m-2} < q_{3m+1}$ (for m sufficiently large). Hence, modulo $O(m)$, we find

$$\lambda_1(q_{3m+1}) \equiv \sum_{\nu=0}^{m-1} (2\nu)^{1/2} \equiv \int_0^m (2x)^{1/2}dx \equiv {}^1/_3(2m)^{3/2}.$$

Proving the theorems now essentially amounts to obtaining m as a function of q_{3m+1}.

PROPOSITION 2. *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 4^m \Gamma(m + {}^3/_2) \leqq q_{3m+1} \leqq c_2\, 4^m \Gamma(m + {}^3/_2).$$

Proof: We note that the equations

$$q_{3m+2} = 2(m+1)q_{3m+1} + q_{3m}$$

$$q_{3m+1} = q_{3m} + q_{3m-1}$$

$$q_{3m} = q_{3m-1} + q_{3m-2}$$

may be solved to yield

$$\frac{q_{3m+1}}{q_{3m-2}} = 2(2m+1) + \frac{q_{3m-5}}{q_{3m-2}},$$

so that

$$\frac{q_{3m+1}}{q_{3m-2}} = [2(2m+1), 2(2m-1), \ldots].$$

Thus,

$$q_{3m+1} \geqq 2(2m+1)q_{3m-2} \geqq 2^2(2m+1)(2m-1)q_{3m-5} \geqq \ldots$$

and hence $q_{3m+1} \geqq c_1 4^m \Gamma(m + {}^3/_2)$ is clear. Conversely,

$$\frac{q_{3m+1}}{q_{3m-2}} \leqq 2(2m+1) + \frac{1}{2(2m-1)} = 2(2m+1)\left(1 + \frac{1}{4(2m+1)(2m-1)}\right),$$

and proceeding inductively, we see that

$$q_{3m+1} \leqq 2^m(2m+1)(2m-1)\ldots \prod_{\nu=1}^{m}\left(1 + \frac{1}{4(2\nu+1)(2\nu-1)}\right),$$

so $q_{3m+1} \leqq c_2 4^m \Gamma(m + {}^3/_2)$, where c_2 is determined by the infinite product.

To prove the theorems, we find to any given B the integer m such that $q_{3m-2} \leqq B < q_{3m+1}$. Thus,

$$c_1 4^{m-1}\Gamma(m - 1 + {}^3/_2) \leqq B < c_2 4^m \Gamma(m + {}^3/_2),$$

and

$$G(B/c_2) < m \leqq G(B/c_1) + 1.$$

Since $G(x)$ grows like $(\log x)/\log \log x$, we conclude that $m = G(B) + O(1)$, whence Theorem 1 follows at once from Proposition 1.

* This research was partially supported by National Science Foundation grant GP3990.

[1] Adams, W., and S. Lang, "Some computations in diophantine approximations," to appear in *J. Reine Angew. Math.*

[2] Khinchin, A., *Continued Fractions* (Chicago: Chicago Press, 1964).

[3] Lang, S., "Asymptotic approximations to quadratic irrationalities," *Amer. J. Math.*, **87**, 481–495 (1965).

[4] Perron, O., *Die Lehre von den Kettenbrüchen I* (Stuttgart: Teubner Verlagsgesellschaft, 1954).

[5] Schmidt, W., "A metrical theorem in diophantine approximations," *Can. J. Math.*, **11**, 619–631 (1959).

MAHLER, K.
Math. Annalen 168, 200—227 (1967)

Applications of Some Formulae by Hermite to the Approximation of Exponentials and Logarithms

To C. L. SIEGEL on his 70th birthday

K. MAHLER

While LIOUVILLE gave the first examples of transcendental numbers, the modern theory of proofs of transcendency started with Hermite's beautiful paper "Sur la fonction exponentielle" (HERMITE, 1873). In this paper, for a given system of distinct complex numbers $\omega_0, \omega_1, \ldots, \omega_m$ and of positive integers $\varrho_0, \varrho_1, \ldots, \varrho_m$ with the sum σ, HERMITE constructed a set of $m+1$ polynomials

$$\mathfrak{A}_0(z), \mathfrak{A}_1(z), \ldots, \mathfrak{A}_m(z)$$

of degrees not exceeding $\sigma-\varrho_0$, $\sigma-\varrho_1$, ..., $\sigma-\varrho_m$, respectively, such that all the functions

$$\mathfrak{A}_k(z)\, e^{\omega_l z} - \mathfrak{A}_l(z)\, e^{\omega_k z} \qquad (0 \leqq k < l \leqq m)$$

vanish at $z=0$ at least to the order $\sigma+1$. On putting $z=1$, these formulae produce simultaneous rational approximations of the numbers 1, e, $e^2, \ldots, e^m$ that are so good that they imply the linear independence of these numbers and hence the transcendency of e.

In a later paper (HERMITE, 1893), Hermite introduced a second system of polynomials

$$A_0(z), A_1(z), \ldots, A_m(z)$$

of degrees at most ϱ_0-1, $\varrho_1-1, \ldots, \varrho_m-1$, respectively, for which the sum

$$\sum_{k=0}^{m} A_k(z)\, e^{\omega_k z}$$

vanishes at $z=0$ at least to the order $\sigma-1$. On putting again $z=1$, one obtains now a linear form

$$a_0 + a_1 e + \cdots + a_m e^m$$

of small absolute value and with small integral coefficients, from which again the transcendency of e may be deduced. Surprisingly, HERMITE himself never took this step, and I was seemingly the first to use the polynomials $A_k(z)$ for this purpose (MAHLER, 1931).

In the present paper I once more wish to exhibit the usefulness of Hermite's polynomials $A_k(z)$ for the study of transcendental numbers. I shall prove a number of explicit estimates, free from any unknown constants, for the simultaneous rational approximations of powers of e or of the natural logarithms of sets of rational numbers.

1. Let $\omega_0, \omega_1, \ldots, \omega_m, \Omega$ be $m+2$ integers satisfying

$$0=\omega_0<\omega_1<\omega_2<\cdots<\omega_m=\Omega,$$

and let

$$M_k=\left|\prod_{\substack{l=0\\ l\neq k}}^{m}(\omega_k-\omega_l)\right|, \quad M=\operatorname*{lcm}_{k=0,1,\ldots,m} M_k, \quad N=\operatorname*{lcm}_{\substack{l\neq k\\ k,l=0,1,\ldots,m}}(\omega_k-\omega_l),$$

where lcm denotes the least common multiple. Let z be any complex number, ϱ a positive integer, and

$$\delta_{hk}=\begin{cases}1 & \text{if} \quad h=k,\\ 0 & \text{if} \quad h\neq k,\end{cases}$$

the Kronecker sign. Denote by C_0 and C_∞ two circles in the complex $\mathfrak{z}$-plane, both with centres at $\mathfrak{z}=0$, and of radii less than 1, and greater than Ω, respectively. Then put

$$A_{hk}(z)=\frac{1}{2\pi i}\int_{C_0}\frac{e^{z\mathfrak{z}}\,d\mathfrak{z}}{\prod_{l=0}^{m}(\mathfrak{z}+\omega_k-\omega_l)^{\varrho+\delta_{hl}}}, \quad R_h(z)=\frac{1}{2\pi i}\int_{C_\infty}\frac{e^{z\mathfrak{z}}\,d\mathfrak{z}}{\prod_{l=0}^{m}(\mathfrak{z}-\omega_l)^{\varrho+\delta_{hl}}}.$$

These definitions imply (see, e.g. MAHLER, 1931) that $A_{hk}(z)$ is a polynomial in z at most of degree ϱ; that

$$R_h(z)=\sum_{k=0}^{m}A_{hk}(z)\,r^{\omega_k z} \qquad (h=0,1,\ldots,m),$$

and that the determinant

$$D(z)=\begin{vmatrix}A_{00}(z), & \ldots, & A_{0m}(z)\\ \vdots & & \vdots\\ A_{m0}(z), & \ldots, & A_{mm}(z)\end{vmatrix}=Cz^{(m+1)\varrho},$$

where $C\neq 0$ does not depend on z.

2. By the paper quoted, $R_h(z)$ may also be written as

$$R_h(z)=z^{(m+1)\varrho}\int_0^1 dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{m-1}}dt_m\,\Phi(t)\,e^{z\Psi(t)},$$

where the expressions Φ and Ψ are defined by

$$\Phi(t)=\frac{(1-t_1)^{\varrho+\delta_{h0}-1}(t_1-t_2)^{\varrho+\delta_{h1}-1}\ldots(t_{m-1}-t_m)^{\varrho+\delta_{h,m-1}-1}t_m^{\varrho+\delta_{hm}-1}}{\prod_{l=0}^{m}(\varrho+\delta_{hl}-1)!}$$

and

$$\Psi(t)=\omega_0(1-t_1)+\omega_1(t_1-t_2)+\cdots+\omega_{m-1}(t_{m-1}-t_m)+\omega_m t_m,$$

respectively. Here the quantities

$$1-t_1, t_1-t_2, \ldots, t_{m-1}-t_m, t_m$$

are non-negative and have the sum 1. Therefore, by the theorem on the arithmetic and geometric means,

$$0 \leqq (1-t_1)(t_1-t_2)\dots(t_{m-1}-t_m)t_m \leqq (m+1)^{-(m+1)},$$

so that

$$0 \leqq \Phi(t) \leqq (m+1)^{-(m+1)}(\varrho!(\varrho-1)!^m)^{-1}.$$

Further

$$0 \leqq \Psi(t) \leqq \Omega$$

and

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} dt_m = \frac{1}{m!}.$$

It follows then from the first mean value theorem that

$$|R_h(z)| \leqq \frac{|z|^{(m+1)\varrho}\, e^{\Omega|z|}}{m!\,(m+1)^{(m+1)(\varrho-1)}\varrho!\,(\varrho-1)!^m}.$$

3. From the integral, $A_{hk}(z)$ is the polynomial

$$A_{hk}(z) = \sum_{j=0}^{\varrho} A_{hk}^{(j)} \frac{z^j}{j!}$$

where the general coefficient $A_{hk}^{(j)}$ is given by

$$A_{hk}^{(j)} = \frac{1}{2\pi i} \int_{C_0} \frac{\mathfrak{z}^j\, d\mathfrak{z}}{\prod_{l=0}^{m} (\mathfrak{z}+\omega_k-\omega_l)^{\varrho+\delta_{hl}}} \qquad (j=0, 1, \dots, \varrho).$$

If we choose for C_0 the circle

$$|\mathfrak{z}| = \frac{1}{m+1},$$

then on this circle,

$$\left|1 + \frac{\mathfrak{z}}{\omega_k-\omega_l}\right| \geqq 1 - |\mathfrak{z}| = 1 - \frac{1}{m+1} = \frac{m}{m+1} \quad \text{for} \quad k \neq l.$$

The formula for $A_{hk}^{(j)}$ may also be written as

$$A_{hk}^{(j)} = \prod_{\substack{l=0\\ l\neq k}}^{m} (\omega_k-\omega_l)^{-\varrho-\delta_{hl}} \cdot \frac{1}{2\pi i} \int_{C_0} \frac{\mathfrak{z}^{j-\varrho-\delta_{hk}}\, d\mathfrak{z}}{\prod_{\substack{l=0\\ l\neq k}}^{m} \left(1+\frac{\mathfrak{z}}{\omega_k-\omega_l}\right)^{\varrho+\delta_{hl}}}.$$

It follows therefore that

$$|A_{hk}^{(j)}| \leqq M_k^{-\varrho} \cdot \frac{1}{2\pi}\, \frac{2\pi}{m+1} \left(\frac{1}{m+1}\right)^{-(\varrho+\delta_{hk})} \left(\frac{m}{m+1}\right)^{-m\varrho-(1-\delta_{hk})},$$

and so, by $0 \leqq \delta_{hk} \leqq 1$, that

$$|A_{hk}^{(j)}| \leqq M_k^{-\varrho} m^{-m\varrho}(m+1)^{(m+1)\varrho}.$$

4. From the original integral,

$$A_{hk}(z) = \prod_{\substack{l=0\\ l \neq k}}^{m} \left(\omega_k - \omega_l + \frac{d}{dz}\right)^{-\varrho-\delta_{hl}} \frac{z^{\varrho+\delta_{hk}-1}}{(\varrho+\delta_{hk}-1)!}.$$

This formula may also be written as

$$A_{hk}(z) = \prod_{\substack{l=0\\ l \neq k}}^{m} (\omega_k - \omega_l)^{-\varrho-\delta_{hl}} \cdot \prod_{\substack{l=0\\ l \neq k}}^{m} \left(1 + \frac{1}{\omega_k - \omega_l}\,\frac{d}{dz}\right)^{-\varrho-\delta_{hl}} \frac{z^{\varrho+\delta_{hk}-1}}{(\varrho+\delta_{hk}-1)!},$$

or, what is the same,

$$A_{hk}(z) = \prod_{\substack{l=0\\ l \neq k}}^{m} (\omega_k - \omega_l)^{-\varrho-\delta_{hl}} \cdot \prod_{\substack{l=0\\ l \neq k}}^{m} \left\{ \sum_{\lambda=0}^{\infty} \binom{-\varrho-\delta_{hl}}{\lambda} (\omega_k - \omega_l)^{-\lambda} \frac{d^\lambda}{dz^\lambda} \right\} \frac{z^{\varrho+\delta_{hk}-1}}{(\varrho+\delta_{hk}-1)!}.$$

Here the binomial coefficients are integers; the differences $\omega_k - \omega_l$ are divisors of N; and hence the operator has the form

$$\prod_{\substack{l=0\\ l \neq k}}^{m} \left\{ \sum_{\lambda=0}^{\infty} \binom{-\varrho-\delta_{hl}}{\lambda} (\omega_k - \omega_l)^{-\lambda} \frac{d^\lambda}{dz^\lambda} \right\} = \sum_{\lambda=0}^{\infty} g_\lambda N^{-\lambda} \frac{d^\lambda}{dz^\lambda}$$

where

$$g_0, g_1, g_2, \dots \qquad (g_0 = 1)$$

are certain integers that also depend on h and k. It follows that

$$A_{hk}(z) = \prod_{\substack{l=0\\ l \neq k}}^{m} (\omega_k - \omega_l)^{-\varrho+\delta_{hl}} \cdot \sum_{\lambda=0}^{\varrho+\delta_{hk}-1} g_\lambda N^{-\lambda} \frac{z^{\varrho+\delta_{hk}-\lambda-1}}{(\varrho+\delta_{hk}-\lambda-1)!}.$$

Here, from the definitions of M and N, the factor

$$M^\varrho N \cdot \prod_{\substack{l=0\\ l \neq k}}^{m} (\omega_k - \omega_l)^{-\varrho-\delta_{hl}}$$

is an integer. Therefore the product

$$a_{hk}(z) = M^\varrho N^{\varrho+1} \varrho!\, A_{hk}(z), = \sum_{j=0}^{\varrho} a_{hk}^{(j)} z^j$$

say, is a polynomial in z with integral coefficients $a_{hk}^{(j)}$.

Since

$$a_{hk}(z) = M^\varrho N^{\varrho+1} \varrho! \sum_{j=0}^{\varrho} A_{hk}^{(j)} \frac{z^j}{j!},$$

these integral coefficients can be written in the form

$$a_{hk}^{(j)} = M^\varrho N^{\varrho+1} \varrho! \frac{A_{hk}^{(j)}}{j!}$$

and so satisfy the inequality

$$|a_{hk}^{(j)}| \leqq \frac{M^{\varrho} N^{\varrho+1} \varrho!\,(m+1)^{(m+1)\varrho}}{M_k^{\varrho}\, m^{m\varrho}}.$$

It is further obvious that

$$|a_{hk}(z)| \leqq \frac{M^{\varrho} N^{\varrho+1} \varrho!\,(m+1)^{(m+1)\varrho}\, e^{|z|}}{M_k^{\varrho}\, m^{m\varrho}}$$

because

$$\sum_{j=0}^{\varrho} \frac{|z|^j}{j!} < e^{|z|}.$$

In analogy to $a_{hk}(z)$ put also

$$r_h(z) = M^{\varrho} N^{\varrho+1} \varrho!\, R_h(z) \qquad (h=0, 1, \ldots, m).$$

Then

$$r_h(z) = \sum_{k=0}^{m} a_{hk}(z)\, e^{\omega_k z} \qquad (h=0, 1, \ldots, m).$$

From the identity for $D(z)$, the new determinant

$$d(z) = \begin{vmatrix} a_{00}(z), & \ldots, & a_{0m}(z) \\ \vdots & & \vdots \\ a_{m0}(z), & \ldots, & a_{mm}(z) \end{vmatrix} = c z^{(m+1)\varrho}$$

where again $c \neq 0$ is independent of z.

We note that, by the estimate for $R_h(z)$,

$$|r_h(z)| \leqq \frac{M^{\varrho} N^{\varrho+1} |z|^{(m+1)\varrho}\, e^{\Omega|z|}}{m!\,(m+1)^{(m+1)(\varrho-1)}\, (\varrho-1)!^m}.$$

5. The inequalities just proved can be simplified by means of some simple lower and upper bounds for M_k, M, and N.

First, the factors of M_k are integers distinct from one another and from zero, and of these factors k are positive and $m-k$ are negative. It follows therefore at once that

$$M_k \geqq k!\,(m-k)! = m! \binom{m}{k}^{-1} \geqq 2^{-m} m!$$

Secondly, N is the least common multiple of certain positive integers not greater than Ω, and hence

$$N \leqq \operatorname{lcm}(1, 2, \ldots, \Omega) \leqq e^{1.04\Omega}$$

where the numerical inequality is taken from the paper (ROSSER and SCHOENFELD, 1962).

Thirdly, an upper bound for M may be obtained by the following method due to B. H. NEUMANN.

For each suffix k and for each prime p let $\mu_k(p)$ denote the largest integer for which

$$p^{\mu_k(p)} | M_k.$$

Hence

$$M_k = \prod_p p^{\mu_k(p)}.$$

Since $|\omega_k - \omega_l| \leqq \Omega$, a power p^t of p cannot be a divisor of some factor $\omega_k - \omega_l$ of M_k unless

$$p^t \leqq \Omega \quad \text{and therefore} \quad p \leqq \Omega.$$

The largest possible value of t is then

$$\tau = \left[\frac{\log \Omega}{\log 2}\right],$$

because $2^{\tau+1} > \Omega$.

One counts as usual how many of the factors

$$\omega_k - \omega_l, \quad \text{where} \quad 0 \leqq l \leqq m, \ l \neq k,$$

are successively divisible by p^1, by p^2, by p^3, etc., and finally by p^τ; the sum of all these numbers is equal to $\mu_k(p)$. Now M_k has just m factors $\omega_k - \omega_l$, and so none of these numbers can exceed m. Also these factors of M_k lie in the interval from $\omega_k - \Omega$ to ω_k of length Ω, and this interval contains the multiple 0 of p^t which is not a factor of M_k. Therefore at most

$$\min\left(m, \left[\frac{\Omega}{p^t}\right]\right)$$

factors of M_k are divisible by p^t, whence

$$\mu_k(p) \leqq \sum_{t=1}^{\tau} \min\left(m, \left[\frac{\Omega}{p^t}\right]\right).$$

We replace this inequality by the weaker but more convenient one,

$$\mu_k(p) \leqq \min\left(m, \left[\frac{\Omega}{p}\right]\right) + \sum_{t=2}^{\tau} \left[\frac{\Omega}{p^t}\right], = \mu(p) \text{ say}.$$

Let

$$M^* = \prod_{p \leqq \Omega} p^{\mu(p)}.$$

Then all products M_k and so also their least common multiple M are divisors of M^*, and hence it follows that

$$M \leqq M^*.$$

6. Put now

$$\nu(p) = \sum_{t=1}^{\tau} \left[\frac{\Omega}{p^t}\right],$$

so that, by a well known formula,

$$\Omega! = \prod_{p \leqq \Omega} p^{\nu(p)}.$$

It follows that

$$M^* = \frac{\Omega!}{\Lambda}$$

where Λ denotes the product

$$\Lambda = \prod_{p \leqq \Omega} p^{\nu(p)-\mu(p)}.$$

From the definitions of $\mu(p)$ and $\nu(p)$,

$$\nu(p) - \mu(p) = \begin{cases} \left[\dfrac{\Omega}{p}\right] - m & \text{if} \quad p \leqq \dfrac{\Omega}{m}, \\ 0 & \text{if} \quad p > \dfrac{\Omega}{m}, \end{cases}$$

so that

$$\Lambda = \prod_{p \leqq \frac{\Omega}{m}} p^{\left[\frac{\Omega}{p}\right]-m}$$

and therefore also

$$\log \Lambda \geqq \sum_{p \leqq \frac{\Omega}{m}} \left(\Omega \frac{\log p}{p} - (m+1)\log p\right).$$

In the paper (ROSSER and SCHOENFELD, 1962), it is proved that

$$\sum_{p \leqq x} \frac{\log p}{p} > \log x + E - \frac{1}{2\log x} \quad \text{for} \quad x > 1$$

and

$$\sum_{p \leqq x} \log p < 1.02x \quad \text{for} \quad x \geqq 1,$$

where E is a certain constant satisfying

$$E > -1.34.$$

Assume for the moment that

$$\Omega \geqq e^2 m > m,$$

and therefore

$$2\log\frac{\Omega}{m} \geqq 4, \quad \frac{1}{2\log\dfrac{\Omega}{m}} \leqq 0.25,$$

while trivially

$$\frac{m+1}{m} \leqq 2.$$

It follows then that

$$\log \Lambda > \Omega\left(\log\frac{\Omega}{m} - 1.34 - \frac{1}{2\log\dfrac{\Omega}{m}}\right) - (m+1)\,1.02\,\frac{\Omega}{m}$$

or

$$\log \Lambda > \Omega\left\{\log\frac{\Omega}{m} - \left(1.34 + \frac{1}{2\log\frac{\Omega}{m}} + \frac{m+1}{m}1.02\right)\right\},$$

and here

$$1.34 + \frac{1}{2\log\frac{\Omega}{m}} + \frac{m+1}{m}1.02 \leqq 1.34 + 0.25 + 2.04 = 3.63 < \frac{11}{3}.$$

Hence, finally,

$$\log \Lambda > \Omega\left(\log\frac{\Omega}{m} - \frac{11}{3}\right),$$

that is,

$$\Lambda > \left(\frac{\Omega}{m}\right)^{\Omega} e^{-\frac{11}{3}\Omega}.$$

This inequality trivially is valid also for

$$\Omega < e^2 m,$$

because then

$$\Lambda \geqq 1 > e^{\left(2-\frac{11}{3}\right)\Omega} > \left(\frac{\Omega}{m}\right)^{\Omega} e^{-\frac{11}{3}\Omega}.$$

7. Thus it has been proved that always

$$M \leqq M^* \leqq \frac{\Omega!}{\Lambda} < \Omega!\left(\frac{\Omega}{m}\right)^{-\Omega} e^{\frac{11}{3}\Omega}.$$

Here

$$\Omega! < e\sqrt{\Omega}\,\Omega^{\Omega} e^{-\Omega}$$

and therefore

$$M < e\sqrt{\Omega}\,m^{\Omega} e^{\frac{8}{3}\Omega}.$$

But $\Omega \geqq 1$, hence

$$e\sqrt{\Omega} = e^{1+\frac{1}{2}\log\{1+(\Omega-1)\}} \leqq e^{\frac{1}{2}(\Omega+1)+\frac{1}{2}(\Omega-1)} = e^{\Omega},$$

and so finally

$$M < m^{\Omega} e^{\frac{11}{3}\Omega}.$$

On combining this inequality with the earlier one for N,

$$M^{\varrho}N^{\varrho+1} \leqq M^{\varrho}N^{2\varrho} < \left(m^{\Omega} e^{\frac{11}{3}\Omega}\right)^{\varrho}\left(e^{1.04\Omega}\right)^{2\varrho}$$

and hence

$$M^{\varrho}N^{\varrho+1} < m^{\Omega\varrho} e^{6\Omega\varrho}.$$

8. For the moment put

$$a=\frac{M^{\varrho}N^{\varrho+1}\varrho!\,(m+1)^{(m+1)\varrho}}{M_k^{\varrho}m^{m\varrho}},\quad r=\frac{M^{\varrho}N^{\varrho+1}}{m!\,(m+1)^{(m+1)(\varrho-1)}(\varrho-1)!^m};$$

by what has been proved in § 4,

$$\max_{h,k,j}|a_{hk}^{(j)}|\leqq a,\quad \max_{h,k}|a_{hk}(z)|\leqq a\,e^{|z|},\quad \max_{h}|r_h(z)|\leqq r|z|^{(m+1)\varrho}e^{\Omega|z|}.$$

Thus upper bounds for a and r imply upper bounds for $|a_{hk}^{(j)}|$, $|a_{hk}(z)|$, and $|r_h(z)|$. Such upper bounds are obtained as follows.

To begin with a, we apply in addition to

$$M_k\geqq 2^{-m}m! \quad\text{and}\quad M^{\varrho}N^{\varrho+1}<m^{\Omega\varrho}e^{6\Omega\varrho}$$

the formulae

$$\sqrt{2\pi\varrho}\,\varrho^{\varrho}e^{-\varrho}<\varrho!<e\sqrt{\varrho}\,\varrho^{\varrho}e^{-\varrho},\quad m!>\sqrt{2\pi m}\,m^m e^{-m}.$$

We find then that

$$a<\frac{m^{\Omega\varrho}e^{6\Omega\varrho}\cdot e\sqrt{\varrho}\,\varrho^{\varrho}e^{-\varrho}\cdot(m+1)^{(m+1)\varrho}}{(2^{-m}\cdot\sqrt{2\pi m}\,m^m e^{-m})^{\varrho}\,m^{m\varrho}}$$
$$=\left(\frac{e^2\varrho}{(2\pi)^{\varrho}}\right)^{1/2}\left(\frac{2^m e^{m-1}(m+1)^{m+1}}{m^{2m+\frac{1}{2}}}\cdot\varrho\, m^{\Omega}e^{6\Omega}\right)^{\varrho}.$$

Here

$$e^2<7.5,\quad 2\pi>6$$

and hence

$$\frac{e^2\varrho}{(2\pi)^{\varrho}}<\frac{7.5\varrho}{(1+5)^{\varrho}}\leqq\frac{7.5\varrho}{1+5\varrho}<\frac{3}{2}<4.$$

Further the function

$$\frac{2^m e^{m-1}(m+1)^{m+1}}{m^{2m+\frac{1}{2}}}$$

of m assumes it maximum when $m=2$, and this maximum has the value

$$\frac{27e}{\sqrt{32}}<13.$$

The final result is therefore

$$a<2(13\varrho\, m^{\Omega}e^{6\Omega})^{\varrho}$$

and it follows that

$$\boxed{\max_{h,k,j}|a_{hk}^{(j)}|<2(13\varrho\, m^{\Omega}e^{6\Omega})^{\varrho},\quad \max_{h,k}|a_{hk}(z)|<2(13\varrho\, m^{\Omega}e^{6\Omega})^{\varrho}e^{|z|}.}$$

8. Since

$$(\varrho-1)!>\sqrt{\frac{2\pi}{\varrho}}\,\varrho^{\varrho}e^{-\varrho},$$

we similarly find that

$$r < \frac{m^{\Omega\varrho}\, e^{6\Omega\varrho}}{\sqrt{2\pi m}\, m^m e^{-m} \cdot (m+1)^{(m+1)(\varrho-1)} \cdot \left(\sqrt{\frac{2\pi}{\varrho}}\, \varrho^\varrho e^{-\varrho}\right)^m}$$

$$= \frac{e^m \varrho^{\frac{m}{2}} m^{m\varrho}}{(2\pi)^{\frac{m+1}{2}} \cdot m^m (m+1)^{(m+1)(\varrho-1)} \cdot e^{m\varrho}} \left(\frac{m^\Omega e^{6\Omega} e^{2m}}{m^m \varrho^m}\right)^\varrho .$$

Here $m \geqq 1$ and $\varrho \geqq 1$. Further

$$m^m (m+1)^{(m+1)(\varrho-1)} = m^{m\varrho+\varrho-1}\left(1+\frac{1}{m}\right)^{(m+1)(\varrho-1)} \geqq m^{m\varrho+\varrho-1}\, e^{\varrho-1} \geqq m^{m\varrho}$$

because

$$\left(1+\frac{1}{m}\right)^{m+1} > e\,;$$

and also

$$(2\pi)^{\frac{m+1}{2}} > 1\,.$$

It follows that

$$\frac{e^m \varrho^{\frac{m}{2}} m^{m\varrho}}{(2\pi)^{\frac{m+1}{2}}\, m^m (m+1)^{(m+1)(\varrho-1)}\, e^{m\varrho}} < \left(\frac{\varrho^{\frac{1}{2}}}{e^{\varrho-1}}\right)^m \leqq 1\,,$$

since

$$e^{\varrho-1} \geqq 1 + (\varrho-1) = \varrho \geqq \varrho^{\frac{1}{2}}\,.$$

The final result is then that

$$r < \left(\frac{m^\Omega e^{6\Omega} e^{2m}}{m^m \varrho^m}\right)^\varrho \leqq \left(\frac{m^\Omega e^{8\Omega}}{m^m \varrho^m}\right)^\varrho ;$$

here we have used that

$$m \leqq \Omega \quad \text{and hence} \quad e^m \leqq e^\Omega\,.$$

Thus it has been established that

$$\boxed{\max_h |r_h(z)| < \left(\frac{m^\Omega e^{8\Omega}}{m^m \varrho^m}\right)^\varrho |z|^{(m+1)\varrho}\, e^{\Omega|z|}}$$

9. As a first application, denote by ω a positive integer and put

$$z = \frac{1}{\omega}\,.$$

Let further $q \geqq 1$, $q_1, q_2, \ldots, q_m$ be $m+1$ arbitrary integers, and let

$$\varepsilon = 2mq \max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right|$$

and

$$\varepsilon_k = 2mq\left(e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q}\right) \qquad (k = 0, 1, \ldots, m),$$

where we have put

$$q_0 = q \geqq 1.$$

Since $\omega_0 = 0$, trivially

$$\varepsilon_0 = 0,$$

and hence

$$\varepsilon = \max_{k=0,1,\ldots,m} |\varepsilon_k| = \max_{k=1,2,\ldots,m} |\varepsilon_k|.$$

The powers

$$e^{\frac{\omega_1}{\omega}}, e^{\frac{\omega_2}{\omega}}, \ldots, e^{\frac{\omega_m}{\omega}}$$

are irrational numbers, and hence

$$\varepsilon > 0.$$

We shall now establish a positive lower estimate for ε.

For this purpose we note that the $(m+1)^2$ numbers

$$\omega^\varrho a_{hk}\left(\frac{1}{\omega}\right), \quad = A_{hk} \text{ say} \qquad (h, k = 0, 1, \ldots, m)$$

are integers, with the determinant

$$\begin{vmatrix} A_{00}, & \ldots, & A_{0m} \\ \vdots & & \vdots \\ A_{m0}, & \ldots, & A_{mm} \end{vmatrix} \neq 0.$$

On putting

$$\omega^\varrho r_h\left(\frac{1}{\omega}\right) = R_h \qquad (h = 0, 1, \ldots, m),$$

we have

$$R_h = \sum_{k=0}^{m} A_{hk}\, e^{\frac{\omega_k}{\omega}} \qquad (h = 0, 1, \ldots, m).$$

The estimates in §§ 7—8 now take the form

$$\max_{h,k} |A_{hk}| < 2(13\varrho\omega m^\Omega e^{6\Omega})^\varrho e^{\frac{1}{\omega}}$$

and

$$\max_h |R_h| < \left(\frac{m^\Omega e^{8\Omega}}{m^m \varrho^m \omega^m}\right)^\varrho e^{\frac{\Omega}{\omega}}.$$

10. Since the determinant of the integers A_{hk} is distinct from zero, and since the integers

$$q_0 \geqq 1\,, \quad q_1, \ldots, q_m$$

do not all vanish, there exists a suffix h such that

$$\sum_{k=0}^{m} A_{hk} q_k \neq 0$$

and that therefore

$$\left|\sum_{k=0}^{m} A_{hk} q_k\right| \geqq 1\,.$$

With this value of h, put

$$Q = \frac{1}{q} \sum_{k=0}^{m} A_{hk} q_k\,, \quad E = \frac{1}{2mq} \sum_{k=1}^{m} A_{hk} \varepsilon_k\,.$$

From the definition of ε_k,

$$R_h = \sum_{k=0}^{m} A_{hk}\, e^{\frac{\omega_k}{\omega}} = \sum_{k=0}^{m} A_{hk}\left(\frac{q_k}{q} + \frac{\varepsilon_k}{2mq}\right) = Q + E\,.$$

Here

$$|Q| \geqq \frac{1}{q}$$

and

$$|E| \leqq \frac{\varepsilon}{2q} \max_{h,k} |A_{hk}|\,.$$

Assume now that

$$\max_{h} |R_h| \leqq \frac{1}{2q}\,.$$

It follows then that

$$|E| \geqq \frac{1}{2q}$$

and hence that

$$\varepsilon \max_{h,k} |A_{hk}| \geqq 1\,.$$

Thus the following result is obtained.

If

$$\left(\frac{m^{\Omega}\, e^{8\Omega}}{m^m\, \varrho^m\, \omega^m}\right)^{\varrho} e^{\frac{\Omega}{\omega}} \leqq \frac{1}{2q}\,,$$

then

$$\varepsilon > \{2 e^{\frac{1}{\omega}} (13 \varrho \omega m^{\Omega} e^{6\Omega})^{\varrho}\}^{-1}\,.$$

14*

This result can be slightly simplified. Since all three integers ω, Ω, ϱ are at least 1,

$$2e^{\frac{\Omega}{\omega}} \leqq 2e^{\Omega} < e^{2\Omega} \leqq e^{2\Omega\varrho},$$

so that

$$2\left(\frac{m^{\Omega} e^{8\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho} e^{\frac{\Omega}{\omega}} < \left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho}.$$

Further

$$\frac{\varepsilon}{2mq} > \{4e^{\frac{1}{\omega}} m(13\varrho\omega m^{\Omega} e^{6\Omega})^{\varrho}\}^{-1} q^{-1} > (52e\varrho\omega m^{\Omega+1} e^{6\Omega})^{-\varrho} q^{-1}.$$

Here

$$52 < e^4,$$

and so

$$\frac{\varepsilon}{2mq} > (\varrho\omega m^{\Omega+1} e^{6\Omega+5})^{-\varrho} q^{-1}.$$

Thus the following result holds.

Lemma 1. *If ϱ is chosen such that*

$$\left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho} \leqq \frac{1}{q},$$

then

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > (\varrho\omega m^{\Omega+1} e^{6\Omega+5})^{-\varrho} q^{-1}.$$

11. When applying this lemma, one naturally will choose the integer ϱ as small as possible because this improves the estimate. It is now convenient to distinguish between the two cases $\varrho = 1$ and $\varrho > 1$.

The case $\varrho = 1$ holds exactly when

$$\omega \geqq (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

and then, by the lemma,

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > (\omega m^{\Omega+1} e^{6\Omega+5})^{-1} q^{-1}.$$

Next, excluding this case, let

$$\omega < (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

so that the smallest possible value for ϱ is at least 2. This value ϱ satisfies the inequality

$$\left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho} \leqq \frac{1}{q} < \left(\frac{m^{\Omega} e^{10\Omega}}{m^m(\varrho-1)^m \omega^m}\right)^{\varrho-1}.$$

It follows that

$$\varrho\omega \leqq 2(\varrho-1)\omega < \frac{2}{m}(m^{\Omega} e^{10\Omega})^{\frac{1}{m}} q^{\frac{1}{m(\varrho-1)}},$$

and that therefore

$$\varrho\omega m^{\Omega+1} e^{6\Omega+5} < \frac{2}{m}(m^{\Omega} e^{10\Omega})^{\frac{1}{m}} m^{\Omega+1} e^{6\Omega+5} q^{\frac{1}{m(\varrho-1)}} <$$

$$< (m^{\Omega} e^{10\Omega})^{\frac{1}{m}} m^{\Omega} e^{6\Omega+6} q^{\frac{1}{m(\varrho-1)}}.$$

The lemma implies then in this case that

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > e^{(4\Omega-6)\varrho} (m^{\Omega} e^{10\Omega})^{-\frac{m+1}{m}\varrho} q^{-1-\frac{\varrho}{m(\varrho-1)}}.$$

Here we once more use that $\varrho \geqq 2$, hence that

$$q^{-1-\frac{\varrho}{m(\varrho-1)}} = q^{-1-\frac{1}{m}-\frac{1}{m(\varrho-1)}} \geqq q^{-1-\frac{1}{m}} \cdot q^{-\frac{2}{m\varrho}},$$

where, by the choice of ϱ,

$$q^{-\frac{2}{m\varrho}} \geqq \left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m} \right)^{\frac{2}{m}}.$$

Evidently $\Omega \geqq m$, and so, by this inequality,

$$q^{-\frac{2}{m\varrho}} \geqq \frac{e^{20}}{\varrho^2 \omega^2}.$$

Assume, in particular, that also $\Omega \geqq 2$. Then

$$4\Omega - 6 \geqq 2, \quad e^{(4\Omega-6)\varrho} q^{-\frac{2}{m\varrho}} \geqq \frac{e^{2\varrho+20}}{\varrho^2 \omega^2} > \frac{e^{20}}{\omega^2} \quad \text{because} \quad e^{\varrho} > \varrho.$$

Thus, in this second case, we arrive at the estimate

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > \frac{e^{20}}{\omega^2} (m^{\Omega} e^{10\Omega})^{-\frac{m+1}{m}\varrho} q^{-1-\frac{1}{m}}.$$

Our result may be expressed as follows.

Theorem 1. *Let $\omega, \omega_1, \ldots, \omega_m, q, q_1, \ldots, q_m$, and Ω be $2m+3$ integers satisfying the conditions*

$$\omega \geqq 1, \quad q \geqq 1, \quad 0 < \omega_1 < \omega_2 < \cdots < \omega_m = \Omega, \quad \Omega \geqq 2.$$

If

$$\omega \geqq (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

then

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > (\omega m^{\Omega+1} e^{6\Omega+5})^{-1} \frac{1}{q}.$$

If, however,

$$\omega < (m^{\Omega-m}\, e^{10\Omega}\, q)^{\frac{1}{m}},$$

and if ϱ denotes the smallest integer satisfying

$$\left(\frac{m^{\Omega}\, e^{10\Omega}}{m^m\, \varrho^m\, \omega^m}\right)^{\varrho} \leqq \frac{1}{q},$$

then

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > \frac{e^{20}}{\omega^2} (m^{\Omega}\, e^{10\Omega})^{-\frac{m+1}{m}\varrho}\, q^{-1-\frac{1}{m}}.$$

The interest of this theorem lies in the fact that $\omega, \omega_1, \ldots, \omega_m, q, q_1, \ldots, q_m$ may all be variable and are subject only to trivial restrictions. The assertion is particularly strong when $\omega, \omega_1, \ldots, \omega_m$ are fixed, while $q, q_1, \ldots, q_m$ are allowed to tend to infinity. For then the parameter ϱ likewise tends to infinity and is given asymptotically by

$$\varrho \sim \frac{\log q}{\log\log q}.$$

Hence a positive constant c depending only on $\omega, \omega_1, \ldots, \omega_m$ exists so that

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > q^{-1-\frac{1}{m}-\frac{c}{\log\log q}}$$

for large q.

If also $\omega, \omega_1, \ldots, \omega_m$ are variable, the theorem is much less strong. However, some consequences seem still worth of being mentioned.

12. Theorem 1 implies an analogous theorem on the simultaneous approximations of logarithms. Its proof is based on the following elementary lemma.

Lemma 2. *If x and $y > 0$ are real numbers such that*

$$|x - \log y| \leqq 1,$$

then

$$|x - \log y| \geqq e^{-x-2} |e^x - y|.$$

Proof. By the mean value theorem,

$$\frac{e^t - 1}{t} = e^{\vartheta t} \quad \text{where} \quad 0 < \vartheta < 1.$$

Hence, on putting $t = x - \log y$,

$$0 < \frac{e^x - y}{x - \log y} = y e^{\vartheta(x - \log y)} \leqq e y.$$

Here

$$\log y \leqq x + 1, \quad y \leqq e^{x+1},$$

whence the assertion.

This lemma we apply to each of the m pairs of numbers

$$x = \frac{\omega_k}{\omega}, \quad y = \frac{q_k}{q} \qquad (k = 1, 2, \ldots, m),$$

for which, evidently,

$$x \leqq \frac{\Omega}{\omega} \leqq \Omega, \quad e^{-x-2} \geqq e^{-\Omega-2}.$$

We next note that Theorem 1 remains valid if the conditions

$$0 < \omega_1 < \omega_2 < \cdots < \omega_m = \Omega$$

are replaced by the weaker hypothesis that the integers $\omega_1, \ldots, \omega_m$ are all distinct and have the maximum Ω. By combining the theorem with the lemma we obtain therefore the following result.

Theorem 2. *Let $\omega, \omega_1, \ldots, \omega_m, q, q_1, \ldots, q_m, \Omega$ be $2m+3$ positive integers satisfying the conditions*

$$\omega_k \neq \omega_l \quad \text{for} \quad k \neq l; \quad \Omega = \max_{k=1,2,\ldots,m} \omega_k \geqq 2.$$

If ω satisfies the inequality

$$\omega \geqq (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

then

$$\max_{k=1,2,\ldots,m} \left| \log \frac{q_k}{q} - \frac{\omega_k}{\omega} \right| > (\omega m^{\Omega+1} e^{7\Omega+7} q)^{-1}.$$

If, however,

$$\omega < (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

and if ϱ denotes the smallest integer satisfying

$$\left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m} \right)^{\varrho} \leqq \frac{1}{q},$$

then

$$\max_{k=1,2,\ldots,m} \left| \log \frac{q_k}{q} - \frac{\omega_k}{\omega} \right| > \frac{e^{18-\Omega}}{\omega^2} (m^{\Omega} e^{10\Omega})^{-\frac{m+1}{m}\varrho} q^{-1-\frac{1}{m}}.$$

13. We deal in detail with one special application of Theorem 2. For this purpose denote by

$$p_1 = 2, p_2 = 3, \ldots, p_m$$

the first m primes in their natural order. We apply the theorem with

$$q = 1, q_1 = p_1, \ldots, q_m = p_m$$

and choose for $\omega, \omega_1, \ldots, \omega_m$ any $m+1$ positive integers for which the fractions

$\frac{\omega_1}{\omega}, \ldots, \frac{\omega_m}{\omega}$ are approximations of $\log p_1, \ldots, \log p_m$, respectively, that are already so close that

$$\text{(A)} \qquad \max_{k=1,2,\ldots,m} \left| \log p_k - \frac{\omega_k}{\omega} \right| < \frac{1}{2} \log \frac{p_m}{p_{m-1}}.$$

Further put again

$$\Omega = \max(\omega_1, \ldots, \omega_m)$$

and assume that

$$m \geqq 10.$$

From the hypothesis (A),

$$|\log p_k - \log p_l| \geqq \log \frac{p_m}{p_{m-1}} \quad \text{for} \quad k \neq l,$$

and

$$\log p_k \geqq \log 2 > \log \frac{p_m}{p_{m-1}} \quad \text{for all } k.$$

Hence

$$\frac{\omega_{k+1}}{\omega} - \frac{\omega_k}{\omega} = \left(\frac{\omega_{k+1}}{\omega} - \log p_{k+1} \right) + (\log p_{k+1} - \log p_k) + \left(\log p_k - \frac{\omega_k}{\omega} \right)$$
$$> -\frac{1}{2} \log \frac{p_m}{p_{m-1}} + \log \frac{p_m}{p_{m-1}} - \frac{1}{2} \log \frac{p_m}{p_{m-1}} = 0$$

and

$$\frac{\omega_1}{\omega} > \log 2 - \frac{1}{2} \log \frac{p_m}{p_{m-1}} \geqq \log 2 - \frac{1}{2} \log \frac{3}{2} > 0.$$

The hypothesis (A) implies therefore that

$$0 < \omega_1 < \omega_2 < \cdots < \omega_m = \Omega.$$

It also implies that

$$\omega \geqq 2,$$

because, if ω were equal to 1, it would follow that

$$\left| \log p_1 - \frac{\omega_1}{\omega} \right| \geqq |\log 2 - 1| > \frac{1}{2} \log \frac{3}{2} \geqq \frac{1}{2} \log \frac{p_m}{p_{m-1}},$$

for all choices of the integer ω_1, contrary to (A).

Next we have $\omega_m \geqq m \geqq 10$ and therefore

$$\Omega > 2.$$

Thus all conditions of Theorem 2 are satisfied, and this theorem may be applied.

From (A),

$$\Omega < \omega\left(\log p_m + \frac{1}{2}\log\frac{p_m}{p_{m-1}}\right) = \frac{\omega}{2}\log\left(\frac{p_m^3}{p_{m-1}}\right).$$

Here, by Bertrand's law on prime numbers,

$$p_{m-1} > \frac{1}{2}p_m,$$

and by the paper (ROSSER and SCHOENFELD, 1962),

$$p_m < \sqrt{2}\, m \log m.$$

Therefore the quantity Ω allows the upper estimate

$$\Omega < \omega \log(2m \log m).$$

It follows that

$$(m^{\Omega-m} e^{10\Omega})^{\frac{1}{m}} < \frac{1}{m}\exp\left\{\frac{1}{m}(10+\log m)\cdot \omega \log(2m\log m)\right\}.$$

Here the right-hand side does not exceed 2 if

(B) $$\omega \leqq \frac{m\log(2m)}{(10+\log m)\log(2m\log m)},$$

and so, for such values of ω, the second case $\varrho \geqq 2$ of Theorem 2 cannot hold. Therefore, by this theorem,

$$\max_{k=1,2,\ldots,m}\left|\log p_k - \frac{\omega_k}{\omega}\right| > (\omega m^{\Omega+1} e^{7\Omega+7})^{-1}.$$

In this estimate,

$$\begin{aligned}\omega m^{\Omega+1} e^{7\Omega+7} &< e^7 m\omega \exp\{(7+\log m)\cdot\omega\log(2m\log m)\} \\ &\leqq e^7 m \frac{m\log(2m)}{(10+\log m)\log(2m\log m)}\exp\left\{\frac{m(7+\log m)\log(2m)}{10+\log m}\right\} \\ &< e^7 \frac{m^2}{\log m}\exp\{m\log(2m)\},\end{aligned}$$

where, by $m \geqq 10$,

$$e^7 \frac{m^2}{\log m} < 2{,}000\frac{m^2}{2} < (2m)^5.$$

Hence it follows from (B) that

$$\max_{k=1,2,\ldots,m}\left|\log p_k - \frac{\omega_k}{\omega}\right| > (2m)^{-m-5}.$$

A stronger result is obtained if ω is restricted to the smaller range

(C) $$\omega \leqq \frac{m}{(7+\log m)\log(2m\log m)}.$$

Now

$$\omega m^{\Omega+1} e^{7\Omega+7} < e^7 m \cdot \frac{m}{(7+\log m)\log(2m\log m)} \cdot e^m < \frac{e^7 m^2 e^m}{(\log m)^2},$$

where, by $m \geqq 10$,

$$\frac{e^7 m^2 e^m}{(\log m)^2} < \frac{2000 m^2 e^m}{2^2} < m^5 e^m.$$

It follows thus from (C) that

$$\max_{k=1,2,\ldots,m} \left| \log p_m - \frac{\omega_m}{\omega} \right| > m^{-5} e^{-m}.$$

The two right-hand sides

$$(2m)^{-m-5} \quad \text{and} \quad m^{-5} e^{-m}$$

in the estimates just established are smaller than the right-hand side

$$\frac{1}{2} \log \frac{p_m}{p_{m-1}}, \quad = \lambda \text{ say},$$

of the hypothesis (A). For

$$p_m < \sqrt{2}\, m \log m < m^2,$$

because

$$\log m \leqq \log 2 + \frac{1}{2}(m-2) < \frac{m}{2}.$$

Therefore

$$\lambda \geqq \frac{1}{2} \log \frac{p_m}{p_m - 1} > \frac{1}{2} \log \frac{m^2}{m^2-1} > \frac{1}{2} \log(1+m^{-2}),$$

where, by $m \geqq 10$,

$$\frac{1}{2} \log(1+m^{-2}) > \frac{1}{2}(m^{-2} - m^{-4} - m^{-6} - \cdots) > \frac{1}{3} m^{-2}.$$

Hence

$$\lambda > \frac{1}{6} m^{-2} > m^{-3},$$

giving the assertion easily.

We may then omit again the hypothesis (A), and we are also allowed in including the trivial denominator $\omega = 1$. Then, on combining the preceeding results, we obtain the following theorem.

Theorem 3. *Let $m \geqq 10$; let $p_1 = 2$, $p_2 = 3, \ldots, p_m$ be the first m primes; and let $\omega, \omega_1, \ldots, \omega_m$ be $m+1$ positive integers. Then*

$$\max_{k=1,2,\ldots,m} \left| \log p_k - \frac{\omega_k}{\omega} \right| > (2m)^{-m-5} \quad \textit{if} \quad 1 \leqq \omega \leqq \frac{m \log(2m)}{(10+\log m)\log(2m\log m)},$$

and

$$\max_{k=1,2,\ldots,m}\left|\log p_k - \frac{\omega_k}{\omega}\right| > m^{-5}e^{-m} \quad \text{if} \quad 1 \leqq \omega \leqq \frac{m}{(7+\log m)\log(2m\log m)}.$$

These two inequalities are rather weak, but it does not seem to be easy to obtain much better ones. For larger values of ω the position is worse.

14. Next put

$$\omega_1 = 1, \omega_2 = 2, \ldots, \omega_m = m, \quad \text{hence} \quad \Omega = m.$$

The general estimates for $a_{hk}^{(j)}$, $a_{hk}(z)$, and $r_h(z)$ can in this special case be a little improved. For now evidently

$$M_k = k!\,(m-k)! = m!\binom{m}{k}^{-1} \geqq 2^{-m}m!, \quad M = m!,$$

and by the paper (ROSSER and SCHOENFELD, 1962),

$$N \leqq e^{1.04m}.$$

The formulae in § 4 become therefore

$$|a_{hk}^{(j)}| \leqq 2^{m\varrho}\, e^{1.04m(\varrho+1)}\,\varrho!\, m^{-m\varrho}(m+1)^{(m+1)\varrho},$$
$$|a_{hk}(z)| \leqq 2^{m\varrho}\, e^{1.04m(\varrho+1)}\,\varrho!\, m^{-m\varrho}(m+1)^{(m+1)\varrho}\, e^{|z|},$$
$$|r_h(z)| \leqq (m!)^{\varrho-1}\, e^{1.04m(\varrho+1)}(m+1)^{-(m+1)(\varrho-1)}\{(\varrho-1)!\}^{-m}|z|^{(m+1)\varrho}\, e^{m|z|}.$$

These estimates can be further simplified if we assume from now on that m is already sufficiently large, but that ϱ may be any positive integer, small or large. For

$$\varrho! \leqq e\sqrt{\varrho}\,\varrho^{\varrho}e^{-\varrho}, \quad \varrho^{\frac{1}{\varrho}} \leqq 3^{\frac{1}{3}}, \quad \left(1+\frac{1}{m}\right)^m \leqq e,$$

while

$$(m+1)^{\frac{1}{m}} > 1$$

becomes arbitrarily close to 1. Since $2e^{1.04} < e^{1.74}$, it follows that

$$|a_{hk}^{(j)}| \leqq 2^{m\varrho}\, e^{1.04m(\varrho+1)} \cdot e\sqrt{\varrho}\,\varrho^{\varrho}\, e^{-\varrho}(m+1)^{\varrho}\left(1+\frac{1}{m}\right)^{m\varrho} \leqq$$

$$\leqq \{2e^{1.04\frac{\varrho+1}{\varrho}}\, e^{\frac{1}{m\varrho}}\, \varrho^{\frac{1}{2m\varrho}}(m+1)^{\frac{1}{m}}\}^{m\varrho}\, \varrho^{\varrho} < e^{1.75m(\varrho+1)}\,\varrho^{\varrho}$$

and hence

$$\boxed{|a_{hk}^{(j)}| < e^{1.75m(\varrho+1)}\,\varrho^{\varrho}, \quad |a_{hk}(z)| < e^{1.75m(\varrho+1)}\,\varrho^{\varrho}\, e^{|z|}.} \tag{1}$$

15. Next, the estimate for $r_h(z)$ may be written as

$$|r_h(z)| \leqq R|z|^{(m+1)\varrho}\, e^{m|z|},$$

where R denotes the expression

$$R = (m!)^{\varrho-1} e^{1.04m(\varrho+1)} (m+1)^{-(m+1)(\varrho-1)} \{(\varrho-1)!\}^{-m}$$

which does not depend on z. Since

$$m! \leqq e\sqrt{m}\, m^m e^{-m}, \quad (\varrho-1)! \geqq \sqrt{\frac{2\pi}{\varrho}}\, \varrho^{\varrho} e^{-\varrho}, \quad \left(1+\frac{1}{m}\right)^{m+1} \geqq e,$$

we find that

$$R \leqq e^{\varrho-1} m^{\frac{\varrho-1}{2}} m^{m(\varrho-1)} e^{-m(\varrho-1)} \cdot e^{1.04m(\varrho+1)} (m+1)^{-(m+1)(\varrho-1)} \cdot \left(\frac{\varrho}{2\pi}\right)^{\frac{m}{2}} \varrho^{-m\varrho} e^{m\varrho}.$$

Here

$$m^{m(\varrho-1)}(m+1)^{-(m+1)(\varrho-1)} = m^{-(\varrho-1)} \left(1+\frac{1}{m}\right)^{-(m+1)(\varrho-1)} \leqq m^{-(\varrho-1)} e^{-(\varrho-1)},$$

so that after a trivial simplification,

$$R \leqq e^{(\varrho-1)-m(\varrho-1)+1.04m(\varrho+1)+m\varrho-(\varrho-1)} m^{-\frac{\varrho-1}{2}} \left(\frac{\varrho}{2\pi}\right)^{\frac{m}{2}} \varrho^{-m\varrho} \leqq$$

$$\leqq e^{m+1.04m(\varrho+1)} m^{-\frac{\varrho-1}{2}} \left(\frac{\varrho}{2\pi}\right)^{\frac{m}{2}} \varrho^{-m\varrho}.$$

On omitting the factors that are smaller than 1,

$$R < e^{1.04m(\varrho+2)} \varrho^{-m(\varrho-\frac{1}{2})},$$

whence

$$\boxed{|r_h(z)| < e^{1.04m(\varrho+2)} \varrho^{-m(\varrho-\frac{1}{2})} |z|^{(m+1)\varrho} e^{m|z|}.} \tag{2}$$

If also ϱ is sufficiently large, this inequality can be further simplified to

$$\boxed{|r_h(z)| < e^{1.05m\varrho} \varrho^{-m\varrho} |z|^{(m+1)\varrho} e^{m|z|}.} \tag{3}$$

16. As a first application of the last estimates, let g be a very large positive integer, and let γ be the integer defined by

$$e^g = \gamma + \delta, \quad \text{where} \quad -\frac{1}{2} \leqq \delta < +\frac{1}{2}.$$

In the identity

$$r_h(z) = \sum_{k=0}^{m} a_{hk}(z) e^{kz}$$

substitute

$$z = g, \quad e^z = \gamma + \delta.$$

Then

$$r_h(g) = \sum_{k=0}^{m} a_{hk}(g)(\gamma+\delta)^k = \sum_{k=0}^{m} \sum_{l=0}^{k} a_{hk}(g) \binom{k}{l} \gamma^{k-l} \delta^l,$$

or, say,

$$r_h(g) = \sum_{l=0}^{m} b_{hl}\,\delta^l$$

where b_{hl} denotes the expression

$$b_{hl} = \sum_{k=l}^{m} a_{hk}(g) \binom{k}{l} \gamma^{k-l}.$$

In particular,

$$b_{h0} = \sum_{k=0}^{m} a_{hk}(g)\,\gamma^k.$$

Here, by § 4, the determinant $d(g)$ with the elements $a_{hk}(g)$ does not vanish. Hence a suffix h exists for which

$$b_{h0} \neq 0.$$

Let h from now on be chosen in this manner.

17. Since

$$a_{hk}(z) = \sum_{j=0}^{\varrho} a_{hk}^{(j)} z^j,$$

we have

$$b_{hl} = \sum_{k=l}^{m} \sum_{j=0}^{\varrho} a_{hk}^{(j)} g^j \binom{k}{l} \gamma^{k-l},$$

so that b_{hl} is an integer. By the estimate (1),

$$|a_{hk}^{(j)}| < e^{1.75m(\varrho+1)}\,\varrho^{\varrho}.$$

Further

$$\sum_{j=0}^{\varrho} g^j \leqq (g+1)^{\varrho}, \quad \binom{k}{l} \leqq 2^k, \quad \sum_{k=l}^{m} \binom{k}{l} \leqq \sum_{k=0}^{m} 2^k < 2^{m+1}.$$

Hence, for all suffices l,

$$|b_{hl}| < e^{1.75m(\varrho+1)}\,\varrho^{\varrho}(g+1)^{\varrho}\,2^{m+1}\,\gamma^m.$$

On the other hand, b_{h0} is a non-vanishing integer, and hence

$$|b_{h0}| \geqq 1.$$

Let us assume for the moment that

$$|\delta| < \frac{1}{3}\{e^{1.75m(\varrho+1)}\,\varrho^{\varrho}(g+1)^{\varrho}\,2^{m+1}\,\gamma^m\}^{-1}$$

and hence that

$$|\delta| < \frac{1}{3}.$$

We find then that

$$|r_h(g)| \geqq |b_{h0}| - |\delta| \sum_{l=1}^{m} |b_{hl}|\,|\delta|^{l-1} >$$

$$> 1 - \frac{1}{3}\{e^{1.75m(\varrho+1)}\varrho^{\varrho}(g+1)^{\varrho}2^{m+1}\gamma^m\}^{-1}\cdot \sum_{l=1}^{m} e^{1.75m(\varrho+1)}\varrho^{\varrho}(g+1)^{\varrho}2^{m+1}\gamma^m\cdot\left(\frac{1}{3}\right)^{l-1}.$$

Here

$$\frac{1}{3}\sum_{l=1}^{\infty}\left(\frac{1}{3}\right)^{l-1} = \frac{1}{2},$$

and so it follows that

$$|r_h(g)| > \frac{1}{2}.$$

However, if both m and ϱ are sufficiently large, then, by (3),

$$|r_h(g)| < e^{1.05m\varrho}\,\varrho^{-m\varrho}\,g^{(m+1)\varrho}\,e^{mg}.$$

If now m and ϱ are chosen so as to satisfy the inequality

$$\text{(D)} \qquad e^{1.05m\varrho}\,\varrho^{-m\varrho}\,g^{(m+1)\varrho}\,e^{mg} \leqq \frac{1}{2},$$

a contradiction arises. The assumed upper bound for δ was therefore false, and so (D) implies instead the lower bound

$$\text{(E)} \qquad |\delta| \geqq \frac{1}{3}\{e^{1.75m(\varrho+1)}\,\varrho^{\varrho}(g+1)^{\varrho}\,2^{m+1}\,\gamma^m\}^{-1}.$$

Denote by α and β two positive absolute constants to be selected immediately, and take for m and ϱ the integers

$$m = [\alpha \log g] + 1, \quad \varrho = [\beta g] + 1,$$

where, as usual, $[x]$ is the integral part of x. Then m and ϱ will exceed any given bounds as soon as g is sufficiently large, and so, under this hypothesis, we were justified in applying the formula (3).

The inequality (D) is equivalent to

$$\varrho \geqq e^{1.05}\,g^{1+\frac{1}{m}}\,e^{\frac{g}{\varrho}}\,2^{\frac{1}{m\varrho}}.$$

Here, by our choice of m and ϱ,

$$m > \alpha \log g, \quad \varrho > \beta g,$$

and therefore

$$g^{\frac{1}{m}} < e^{\frac{1}{\alpha}}, \quad e^{\frac{g}{\varrho}} < e^{\frac{1}{\beta}}.$$

The remaining factor

$$2^{\frac{1}{m\varrho}}$$

is arbitrarily close to 1 as soon as g is sufficiently large. Thus, for such g,

$$e^{1.05}\,g^{1+\frac{1}{m}}\,e^{\frac{g}{\varrho}}\,2^{\frac{1}{m\varrho}} < e^{1.06+\frac{1}{\alpha}+\frac{1}{\beta}}\,g.$$

Assume now that

$$\text{(F)} \qquad \beta \geqq e^{1.06+\frac{1}{\alpha}+\frac{1}{\beta}}.$$

The condition (D) is then satisfied because

$$\varrho > \beta g \geqq e^{1.06+\frac{1}{\alpha}+\frac{1}{\beta}} g > e^{1.05}\, g^{1+\frac{1}{m}} e^{\frac{g}{\varrho}} 2^{\frac{1}{m\varrho}}.$$

Also, for all sufficiently large g,

$$e^{1.75m(\varrho+1)} < e^{1.76\alpha\beta g \log g}, \quad \varrho^{\varrho} < (\beta g)^{1.005\beta g} < e^{1.01\beta g \log g},$$

$$(g+1)^{\varrho} < e^{1.01\beta g \log g}, \quad \gamma^m < \left(e^g + \frac{1}{2}\right)^{\alpha \log g + 1} < e^{1.01\alpha g \log g},$$

$$3 \times 2^{m+1} < e^{\alpha \log g} < e^{0.01\alpha g \log g}.$$

The lower bound (E) for δ takes therefore the form

$$|\delta| > e^{-(1.76\alpha\beta g \log g + 1.01\beta g \log g + 1.01\beta g \log g + 0.01\alpha g \log g + 1.01\alpha g \log g)},$$

that is,

$$|\delta| > g^{-(1.76\alpha\beta + 2.02\beta + 1.02\alpha)g}.$$

We finally fix the constants α and β so that (F) is satisfied, while at the same time the sum

$$\sigma = 1.76\alpha\beta + 2.02\beta + 1.02\alpha$$

becomes small. After some numerical work one is led to the values

$$\beta = 7, \quad \alpha = 1.35,$$

when

$$e^{1.06+\frac{1}{\alpha}+\frac{1}{\beta}} < e^{1.945} < 7, \quad \sigma < 32.4.$$

We arrive thus at the following result.

Theorem 4. *Let g be any sufficiently large positive integer, and let γ be the integer closest to e^g. Then*

$$|e^g - \gamma| > g^{-33g}.$$

By means of more careful estimates, the constant 33 in this theorem can be a little decreased. However, it does not seem to be easy to obtain any essential improvement of the theorem. Previously, by means of a different method, I had proved the analogous estimate with 40 instead of 33 for the constant (MAHLER, 1953).

18. We finally apply the formulae (1) and (2) to the study of the rational approximations of π.

Denote by p and q any two positive integers such that

$$\pi = \frac{p}{q} + \delta, \quad \text{where} \quad -\frac{1}{2q} \leqq \delta < +\frac{1}{2q}.$$

It is trivial that there exists arbitrarily large integers of this kind.

In the identity

$$r_h(z)=\sum_{k=0}^{m} a_{hk}(z)e^{kz}$$

put now

$$z=\frac{\pi i}{2}, \quad e^z=i,$$

so that

$$r_h\left(\frac{\pi i}{2}\right)=\sum_{k=0}^{m} a_{hk}\left(\frac{\pi i}{2}\right)i^k.$$

Here

$$a_{hk}\left(\frac{\pi i}{2}\right)=\sum_{j=0}^{\varrho} a_{hk}^{(j)}\left(\frac{i}{2}\right)^j\left(\frac{p}{q}+\delta\right)^j=\sum_{j=0}^{\varrho}\sum_{l=0}^{j}\binom{j}{l}a_{hk}^{(j)}\left(\frac{i}{2}\right)^j\left(\frac{p}{q}\right)^{j-l}\delta^l,$$

or, say,

$$a_{hk}\left(\frac{\pi i}{2}\right)=\sum_{l=0}^{\varrho} c_{hkl}\,\delta^l$$

where

$$c_{hkl}=\sum_{j=l}^{\varrho}\binom{j}{l}a_{hk}^{(j)}\left(\frac{i}{2}\right)^j\left(\frac{p}{q}\right)^{j-l}.$$

Therefore

$$r_h\left(\frac{\pi i}{2}\right)=\sum_{k=0}^{m}\sum_{l=0}^{\varrho} c_{hkl}\,i^k\delta^l=\sum_{l=0}^{\varrho} C_{hl}\,\delta^l$$

where we have put

$$C_{hl}=\sum_{k=0}^{m} c_{hkl}\,i^k.$$

In particular,

$$C_{h0}=\sum_{k=0}^{m} c_{hk0}\,i^k=\sum_{k=0}^{m}\sum_{j=0}^{\varrho} a_{hk}^{(j)}\left(\frac{i}{2}\right)^j\left(\frac{p}{q}\right)^j i^k=\sum_{k=0}^{m} a_{hk}\left(\frac{ip}{2q}\right)i^k.$$

Here, similarly as before, the determinant $d\left(\frac{ip}{2q}\right)$ with the elements $a_{hk}\left(\frac{ip}{2q}\right)$ does not vanish. Therefore, from now on, we may again assume that h is chosen so as to satisfy the inequality

$$C_{h0}\neq 0.$$

19. The expressions

$$(2q)^{\varrho}c_{hkl}=\sum_{j=l}^{\varrho}\binom{j}{l}a_{hk}^{(j)}\,2^{\varrho-j}q^{\varrho-(j-l)}i^j p^{j-l}$$

and

$$(2q)^{\varrho} C_{hl} = \sum_{k=0}^{m} (2q)^{\varrho} c_{hkl} i^k$$

are integers in the Gaussian field $Q(i)$. In particular, $(2q)^{\varrho} C_{h0}$ is a Gaussian integer different from zero, and hence its absolute value is not less than 1. Therefore

$$|C_{h0}| \geqq (2q)^{-\varrho}.$$

Assume now that m is already very large, while no such restriction need be imposed on ϱ. We are thus allowed to make use of the estimates (1) and (2). Since

$$\binom{j}{l} \leqq 2^j, \quad \binom{\varrho - l}{j - l} \geqq 1 \quad \text{for} \quad l \leqq j \leqq \varrho,$$

it follows from (1) that

$$|c_{hkl}| < \sum_{j=l}^{\varrho} \binom{\varrho - l}{j - l} 2^j e^{1.75m(\varrho+1)} \varrho^{\varrho} \left(\frac{1}{2}\right)^j \left(\frac{p}{q}\right)^{j-l}.$$

Here

$$\frac{p}{q} \leqq \pi + \delta \leqq \pi + \frac{1}{2} < 4,$$

and therefore

$$|c_{hkl}| < e^{1.75m(\varrho+1)} \varrho^{\varrho} \cdot 5^{\varrho - l}$$

and

$$|C_{hl}| < \frac{m+1}{5^l} e^{1.75m(\varrho+1)} \varrho^{\varrho} \cdot 5^{\varrho}.$$

Let now, for the moment, δ be so small that

$$|\delta| < \{(m+1) e^{1.75m(\varrho+1)} \varrho^{\varrho} 5^{\varrho} (2q)^{\varrho}\}^{-1},$$

hence that

$$|\delta| < 1.$$

Then

$$\left| r_h\left(\frac{\pi i}{2}\right) \right| \geqq |C_{h0}| - |\delta| \sum_{l=1}^{m} |C_{hl}| \, |\delta|^{l-1} >$$
$$> (2q)^{-\varrho} - \{(m+1) e^{1.75m(\varrho+1)} \varrho^{\varrho} (10q)^{\varrho}\}^{-1} \sum_{l=1}^{m} \frac{m+1}{5^l} e^{1.75m(\varrho+1)} \varrho^{\varrho} 5^{\varrho},$$

and since

$$\sum_{l=1}^{\infty} \left(\frac{1}{5}\right)^l = \frac{1}{4},$$

it follows that

$$\left| r_h\left(\frac{\pi i}{2}\right) \right| > \frac{3}{4} (2q)^{-\varrho}.$$

On the other hand, by (2),

$$\left|r_h\left(\frac{\pi i}{2}\right)\right| < e^{1.04m(\varrho+2)}\,\varrho^{-m(\varrho-\frac{1}{2})}\left(\frac{\pi}{2}\right)^{(m+1)\varrho} e^{\frac{m\pi}{2}}.$$

Hence, if m and ϱ satisfy the inequality

$$\text{(G)} \qquad e^{1.04m(\varrho+2)}\,\varrho^{-m(\varrho-\frac{1}{2})}\left(\frac{\pi}{2}\right)^{(m+1)\varrho} e^{\frac{m\pi}{2}} \leqq \frac{3}{4}(2q)^{-\varrho},$$

a contradiction arises, showing that the assumed lower bound for δ cannot be valid. The inequality (G) implies therefore that, on the contrary,

$$\text{(H)} \qquad |\delta| \geqq \{(m+1)\,e^{1.75m(\varrho+1)}\,\varrho^{\varrho}(10q)^{\varrho}\}^{-1}.$$

20. From now on let q be very large. If m is then defined by

$$m = [\lambda \log q] + 1$$

where $\lambda > 0$ is a constant to be selected immediately, also m will be arbitrarily large, as required in the preceeding proof.

The inequality (G) is equivalent to

$$\varrho \geqq \left(\frac{4}{3}\right)^{\frac{1}{m(\varrho-\frac{1}{2})}} e^{1.04\frac{\varrho+2}{\varrho-\frac{1}{2}}}\left(\frac{\pi}{2}\right)^{\frac{(m+1)\varrho}{m(\varrho-\frac{1}{2})}} e^{\frac{\pi}{2\varrho-1}}\,(2q)^{\frac{\varrho}{m(\varrho-\frac{1}{2})}}.$$

Here the factors

$$\left(\frac{4}{3}\right)^{\frac{1}{m(\varrho-\frac{1}{2})}} \quad \text{and} \quad 2^{\frac{\varrho}{m(\varrho-\frac{1}{2})}}$$

are arbitrarily close to 1 and so may be assumed to have a product

$$\left(\frac{4}{3}\right)^{\frac{1}{m(\varrho-\frac{1}{2})}} 2^{\frac{\varrho}{m(\varrho-\frac{1}{2})}} < e^{0.01\frac{\varrho+2}{\varrho-\frac{1}{2}}}.$$

We may similarly demand that

$$\left(\frac{\pi}{2}\right)^{\frac{(m+1)\varrho}{m(\varrho-\frac{1}{2})}} < \left(\frac{\pi}{2}\right)^{1.01\frac{\varrho}{\varrho-\frac{1}{2}}}.$$

Next

$$\frac{1}{m} < \frac{1}{\lambda \log q},$$

and hence

$$q^{\frac{\varrho}{m(\varrho-\frac{1}{2})}} < e^{\frac{\varrho}{\lambda(\varrho-\frac{1}{2})}}.$$

Thus (G) is for large q certainly satisfied if

$$\varrho \geqq e^{1.05\frac{\varrho+2}{\varrho-\frac{1}{2}}}\left(\frac{\pi}{2}\right)^{1.01\frac{\varrho}{\varrho-\frac{1}{2}}} e^{\frac{\pi}{2\varrho-1}}\, e^{\frac{\varrho}{\lambda(\varrho-\frac{1}{2})}},$$

that is, if

$$\varrho \geqq e^{\left(1.05\varrho + 2.1 + 1.01\varrho \log\frac{\pi}{2} + \frac{\pi}{2} + \frac{\varrho}{\lambda}\right)\frac{1}{\varrho - \frac{1}{2}}}.$$

Here

$$\frac{\pi}{2} < 1.571, \quad \log\frac{\pi}{2} < 0.452, \quad 1.01\log\frac{\pi}{2} < 0.457.$$

The condition for ϱ is therefore satisfied if

$$\varrho \geqq e^{\left(3.671 + 1.507\varrho + \frac{\varrho}{\lambda}\right)\frac{1}{\varrho - \frac{1}{2}}},$$

or equivalent to this, if

$$\frac{\varrho}{\lambda} \leqq \left(\varrho - \frac{1}{2}\right)\log\varrho - (3.671 + 1.507\varrho).$$

This inequality again is easily seen to hold if

$$\varrho = 14, \quad \lambda = 1.35.$$

On substituting the values

$$\varrho = 14, \quad m = [1.35\log q] + 1$$

in (H), we finally obtain for large q the following result.

Theorem 5. *If p and q are positive integers and q is sufficiently large, then*

$$\left|\pi - \frac{p}{q}\right| > q^{-45}.$$

This result is not as strong as the estimate

$$\left|\pi - \frac{p}{q}\right| > q^{-42} \quad \text{for} \quad q \geqq 2$$

which I have previously obtained by means of a different method (MAHLER, 1953a).

References

CH. HERMITE 1873, Oeuvres, t. III, 151—181.
CH. HERMITE 1893, Oeuvres, t. IV, 357—377.
K. MAHLER 1931, J. reine angew. Math., **166**, 118—137.
K. MAHLER 1953, Phil. Trans. Royal Soc., A, **245**, 371—398.
K. MAHLER 1953a, Proc. Acad. Amsterdam, A, **56**, 30—42.
ROSSER and SCHOENFELD 1962, Illinois J. of Math., **6**, 64—94.

K. MAHLER
Institute of Advanced Studies
Australian National University
Canberra, A.C.T., Australia

(Received May 28, 1965)

290° *Pi by probability.* The French naturalist Comte de Buffon (1707–1788) is known to mathematicians for two contributions—a translation into French of Newton's *Method of Fluxions*, and his "Essai d'arithmétique morale," which appeared in 1777 in the fourth volume of a supplement to his celebrated multi-volume *Histoire naturelle.*

It is in his "Essai" that Buffon suggested what was then the essentially new field of geometrical probability. As an example, Buffon devised his famous *needle problem* by which π may be determined by probability methods. Suppose a number of parallel lines (see Figure 35), distance d apart, are ruled on a horizontal plane, and suppose a

Eves, H.W. In Mathematical Circles: A Selection of Mathematical Stories and Anecdotes. (Boston: PWS Publishing Company, 1969 and 1971).

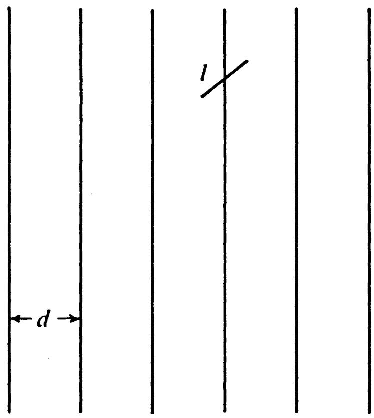


FIGURE 35

homogeneous rod of length $l < d$ is dropped at random onto the plane Buffon showed that the probability* that the rod will fall across one of the lines in the plane is given by

$$p = 2l/\pi d.$$

By actually performing this experiment a given large number of times and noting the number of successful cases, thus obtaining an empirical value for p, we may use the above formula to compute π. The best result obtained in this way was given by the Italian, Lazzerini, in 1901. From only 3408 tosses of the rod he found π correct to six decimal places! His result is so much better than those obtained by other experimenters that it is sometimes regarded with suspicion.

There are other probability methods for computing π. Thus, in 1904, R. Chartres reported an application of the known fact that if two positive integers are written down at random, the probability that they will be relatively prime is $6/\pi^2$.

* If a given event can happen in h ways and fail to happen in f ways, and if each of the $h + f$ ways is equally likely to occur, the *mathematical probability* p of the event happening is $p = h/(h + f)$.

PI

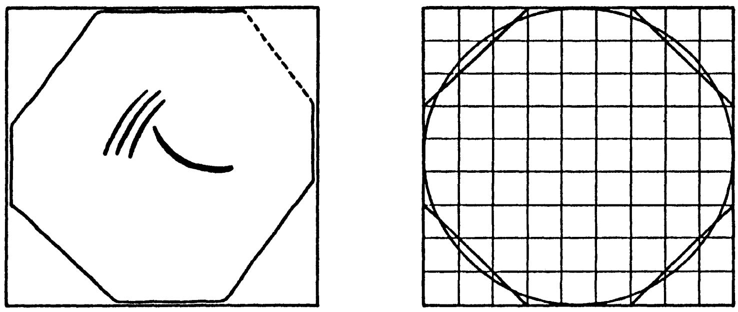
ONE of the most famous of all numbers is that universally designated today by the lower case Greek letter π; it represents, among other things, the ratio of the circumference to the diameter of a circle. It has enjoyed a long and interesting history, and over the years it has received ever better approximations.

37° *π in Biblical times.* In the ancient Orient the value of π was frequently taken very roughly as 3. Thus in the Old Testament, in II Chronicles 4:2 (and similarly in I Kings 7:28), we read: "Also he made a molten sea of ten cubits from brim to brim, round in compass and five cubits the height thereof, and a line of thirty cubits did compass it about." This tells us that the Hebrews of the time approximated the ratio of the circumference to the diameter of a circle by 3. This is equivalent to approximating the circumference of a circle by the perimeter of an inscribed regular hexagon. From this rough approximation of 3, the value of π has been determined more and more accurately, reaching, in 1967, the fantastic accuracy of a half-million decimal places.

38° *An ancient Egyptian estimation of π.* In the Rhind Papyrus, an ancient Egyptian text dating from about 1650 B.C. and composed of eighty-five mathematical problems, Problems 41, 42, 43, and 48 involve finding the area of a circle. In each of these problems the area of a circle is taken equal to the square on (8/9)ths of the circle's diameter.

Eves, H.W. In Mathematical Circles: A Selection of Mathematical Stories and Anecdotes. (Boston: PWS Publishing Company, 1969 and 1971).

It is not known how this formula for "squaring a circle" was arrived at, but a little geometrical diagram (here reproduced in Figure 8(a)) accompanying Problem 48 offers a possible clue. It may be that the diagram represents a square with its sides trisected and then the four corners cut off. If such a figure is carefully drawn (see Figure 8(b)), it



(a) (b)

FIGURE 8

looks by eye as though the area of the circle inscribed in the square is pretty well approximated by the octagon-shaped figure. If the diameter of the circle, and hence the side of the square, be taken as 9, the octagon will have an area equal to

$$81 - 4(9/2) = 63,$$

and the side of a square of the same area as the circle would be approximately $\sqrt{63}$, which, in turn, is approximately $\sqrt{64}$ or 8. That is, the side of the equivalent square is about (8/9)ths of the diameter of the given circle.

If we assume the above Egyptian formula for "squaring a circle" is correct, that is, that

$$\pi d^2/4 = 64d^2/81,$$

we arrive at

$$\pi = 256/81 = (4/3)^4 \doteq 3.1605,$$

which is not too bad an estimation of π for those remote times.

39° *An interesting rational approximation of* π. About the year

480, the early Chinese worker in mechanics, Tsu Ch'ung-chih, gave the interesting rational approximation $355/113 = 3.1415929\cdots$ of π, which is correct to six decimal places and, curiously, involves only the first three odd numbers, each twice. About 1585, Adriaen Anthoniszoon rediscovered the ancient Chinese ratio. This was apparently a lucky accident since all he showed was that

$$377/120 > \pi > 333/106.$$

He then averaged the numerators and the denominators to obtain the "exact" value of π. There is evidence that Valentin Otho, a pupil of the early table maker Rhaeticus, may have introduced this ratio for π into the Western world at the slightly earlier date of 1573.

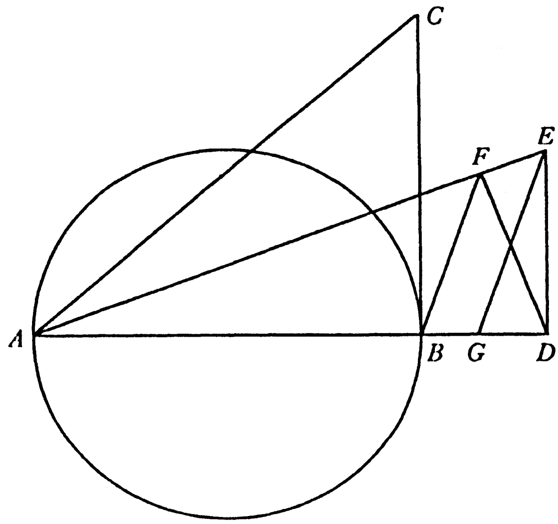


FIGURE 9

In 1849, de Gelder used the ratio 355/113 to obtain a close approximate Euclidean solution of the problem of rectifying a given circle. Let (see Figure 9) $AB = 1$ be a diameter of the given circle. Draw $BC = \frac{7}{8}$, perpendicular to AB at B. Mark off $AD = AC$ on AB produced. Draw $DE = \frac{1}{2}$, perpendicular to AD at D, and let F be the foot of the perpendicular from D on AE. Draw EG parallel to FB to cut BD in G. The reader may now care to show that

$$GB/BA = EF/FA = (DE)^2/(DA)^2 = (DE)^2/[(BA)^2 + (BC)^2],$$

whence $GB = 4^2(7^2 + 8^2) = 16/113 = 0.1415929\cdots$, the decimal part of π correct to six decimal places. The circumference of the circle would then be given very closely by three diameters plus the segment GB.

40° *The Ludolphine number.* Ludolph van Ceulen (1540–1610) of Germany computed π to thirty-five decimal places by the classical method of inscribed and circumscribed regular polygons, using polygons having 2^{62} sides. He spent a large part of his life on this task and his achievement was considered so extraordinary that the number was engraved on his tombstone, and to this day is sometimes referred to in Germany as "the Ludolphine number." Recent attempts to find the tombstone have been unsuccessful; it is probably no longer in existence.

41° *Mnemonics for* π. Among the curiosities connected with π are various mnemonics that have been devised for the purpose of recalling π to a large number of decimal places. The following, by A. C. Orr, appeared in the *Literary Digest* in 1906. One has merely to replace each word by the number of letters it contains to obtain π correct to thirty decimal places.

Now I, even I, would celebrate
In rhymes inapt, the great
Immortal Syracusan, rivaled nevermore,
Who in his wondrous lore,
Passed on before,
Left men his guidance how to circles mensurate.

A few years later, in 1914, the following similar mnemonic appeared in the *Scientific American:*

See, I have a rhyme assisting my feeble brain, its tasks ofttimes resisting.

Two other popular mnemonics are:

How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics.

May I have a large container of coffee?

42° *A brief chronology of the calculation of π by infinite series.*

1691. The Scottish mathematician James Gregory obtained the infinite series

$$\arctan x = x - x^3/3 + x^5/5 - x^7/7 + \cdots, \qquad (-1 \leqq x \leqq 1).$$

1699. Abraham Sharp found 71 correct decimal places by using Gregory's series with $x = \sqrt{(1/3)}$.

1706. John Machin obtained 100 decimal places by using Gregory's series in connection with the relation

$$\pi/4 = 4 \arctan(1/5) - \arctan(1/239).$$

1719. The French mathematician De Lagny obtained 112 correct places by using Gregory's series with $x = \sqrt{(1/3)}$.

1841. William Rutherford of England calculated π to 208 places, of which 152 were later found to be correct, by using Gregory's series in connection with the relation

$$\pi/4 = 4 \arctan(1/5) - \arctan(1/70) + \arctan(1/99).$$

1844. Zacharias Dase, the lightning calculator, found π correct to 200 places using Gregory's series in connection with the relation

$$\pi/4 = \arctan(1/2) + \arctan(1/5) + \arctan(1/8).$$

1853. Rutherford returned to the problem and obtained 400 correct decimal places.

1873. William Shanks of England, using Machin's formula, computed π to 707 places. For a long time this remained the most fabulous piece of calculation ever performed.

1948. In 1946, D. F. Ferguson of England discovered errors, starting with the 528th place, in Shanks' value of π, and in January 1947 he gave a corrected value to 710 places. In the same month J. W. Wrench, Jr., of America, published an 808-place value of π, but Ferguson soon found an error in the 723rd place. In January 1948, Ferguson and Wrench jointly published the corrected and checked value of π to 808 places. Wrench used Machin's formula, whereas Ferguson used the formula

$$\pi/4 = 3 \arctan(1/4) + \arctan(1/20) + \arctan(1/1985).$$

1949. The electronic computer, the ENIAC, at the Army Ballistic Research Laboratories in Aberdeen, Maryland, calculated π to 2037 decimal places, taking 70 hours of machine time.

1954. Nicholson and Jeenel, using NORC, calculated π to 3089 places in 13 minutes.

1958. Felton, in England, using a Ferranti PEGASUS computer, calculated π to 10,000 decimal places in 33 hours.

1958. François Genuys, in Paris, computed π to 10,000 places, using an IBM 704, in 100 minutes.

1959. Genuys computed π to 16,167 decimal places, using an IBM 704, in 4.3 hours.

1961. Wrench and Daniel Shanks, of Washington, D.C., computed π to 100,265 decimal places, using an IBM 7090, in 8.7 hours.

1966. On February 22, M. Jean Guilloud and his co-workers at the Commissariat à l'Ênergie Atomique in Paris attained an approximation to π extending to 250,000 decimal places on a STRETCH computer.

1967. Exactly one year later, the above workers found π to 500,000 places on a CDC 6600.

43° *The irrationality and the transcendentality of* π. It was in 1767 that J. H. Lambert showed that π is irrational, that is, that it is not of the form a/b, where a and b are integers, and in 1794 A. M. Legendre showed that π^2 is also irrational. In 1882, C. F. L. Lindemann proved that π is transcendental, that is, it is not a root of any polynomial equation of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are integers.

44° *The normalcy of* π. There is more to the calculation of π to a large number of decimal places than just the challenge involved. One reason for doing it is to secure statistical information concerning the "normalcy" of π. A real number is said to be *simply normal* if in its decimal expansion all digits occur with equal frequency, and it is said to be *normal* if all blocks of digits of the same length occur with equal frequency. It is not known if π (or even $\sqrt{2}$, for that matter) is normal or even simply normal. The fantastic calculations of π, starting with that on the ENIAC in 1949, were performed to secure statistical information on the matter. From counts on these extensive expansions of π, it would appear that the number is perhaps normal. The erroneous 707-place calculation of π made by William Shanks in 1873 seemed to indicate that π was not even simply normal.

The matter of the normalcy or nonnormalcy of π will never, of course, be resolved by electronic computers. We have here an example of a theoretical problem which requires profound mathematical talent and cannot be solved by computations alone. The existence of such problems ought to furnish at least a partial antidote to the disease of *computeritis*, which seems so rampant today. There is a developing feeling, not only among the general public, but also among young students of mathematics, that from now on any mathematical problem will be resolved by a sufficiently sophisticated electronic machine. These machines are merely extraordinarily fast and efficient calculators, and are invaluable only in those problems of mathematics where extensive computations can be utilized.

The elaborate calculations of π have another use in addition to furnishing statistical evidence concerning the normalcy or nonnormalcy of π. Every new automatic computing machine, before it can be adopted for day-to-day use, must be tested for proper functioning, and coders and programmers must be trained to work with the new machine. Checking into an already found extensive computation of π is frequently chosen as an excellent way of carrying out this required testing and training.

45° *Brouwer's question.* The Dutch mathematician L. E. J. Brouwer (1882–1966), seeking, for logical and philosophical purposes, a mathematical question so difficult that its answer in the following ten or twenty years would be very unlikely, finally hit upon: "In the decimal expression for π, is there a place where a thousand consecutive digits are all zero?" The answer to this question is still not known. It follows that the assertion that such a place in the decimal expression of π does exist is an example of a proposition that, to a member of the intuitionist school of philosophy of mathematics (Brouwer was a leader of this school), is neither true nor false. For, according to the tenets of that school, a proposition can be said to be true only when a proof of it has been constructed in a finite number of steps, and it can be said to be false only when a proof of this situation has been constructed in a finite number of steps. Until one or the other of these proofs is constructed, the proposition is neither true nor false, and the law of excluded middle is inapplicable. If, however, one asserts that a thousand consecutive digits are all zero somewhere in the first quintillion digits of the decimal expression for π, we have a proposition that is true or false, for the truth or falseness, though not known, can surely be established in a finite number of steps. Thus, for the intuitionists, the law of excluded middle does not hold universally; it holds for finite situations but should not be employed when dealing with infinite situations.

If π is normal, as is suspected, then a block of 1000 zeros will occur in its decimal representation not only once, but infinitely often and with an average frequency of 1 in 10^{1000}. It follows that much more is required to prove that π is normal than just to answer Brouwer's question, and this latter appears in itself to be a considerable task.

46° *π by legislation.* [The following is adapted, with permission, from the article "What's new about π?" by Phillip S. Jones, that appeared in *The Mathematics Teacher*, March, 1950, pp. 120–122.]

An often quoted but rarely documented tale about π is that of the attempt to determine its value by legislation. House Bill No. 246, Indiana State Legislature, 1897, was written by Edwin J. Goodwin, M.D., of Solitude, Posey County. It begins as follows:

> A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalities whatever on the same—
>
> Section 1. Be it enacted by the General Assembly of the State of

> Indiana: it has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side...."

The bill was referred first to the House Committee on Canals and then to the Committee on Education, which recommended its passage. It was passed and sent to the Senate, where it was referred to the Committee on Temperance, which recommended its passage. In the meantime the bill had become known and ridiculed in various newspapers. This resulted in the Senate's finally postponing indefinitely its further consideration, in spite of the backing of the State Superintendent of Public Instruction, who was anxious to assure his state textbooks of the free use of this copyrighted discovery. The detailed account of the bill together with contemporary newspaper comments makes interesting reading.

[We can incontestably establish the aberrant condition of the author of the bill by quoting the final section of the bill:

> Section 3. In further proof of the value of the author's proposed contribution to education, and offered as a gift to the State of Indiana, is the fact of his solutions of the trisection of an angle, duplication of the cube and quadrature of the circle having been already accepted as contributions to science by the *American Mathematical Monthly*, the leading exponent of mathematical thought in this country.
>
> And be it remembered that these noted problems had been long since given up by scientific bodies as unsolvable mysteries and above man's ability to comprehend.]

47° *Morbus cyclometricus*. There is a vast literature supplied by sufferers of *morbus cyclometricus*, the circle-squaring disease. The contributions, often amusing and at times almost unbelievable, would require a publication all to themselves. As a couple of samples, in addition to the one given in the preceding item, consider the following.

In 1892, a writer announced in the *New York Tribune* the rediscovery of a long lost secret that leads to 3.2 as the exact value of π. The lively discussion following this announcement won many advocates for the new value.

Again, since its publication in 1934, a great many college and public libraries throughout the United States have received, from the obliging author, complimentary copies of a thick book devoted to the demonstration that $\pi = 3\frac{13}{81}$.

There have been sufferers of *morbus cyclometricus* in rather high and responsible positions; in this country one sufferer was a college president, another was a member of the Lower House of the State of Washington, and still another served in the United States Senate.

The Lemniscate Constants

John Todd
California Institute of Technology

The lemniscate constants, and indeed some of the methods used for actually computing them, have played an enormous part in the development of mathematics. An account is given here of some of the methods used—most of the derivations can be made by elementary methods. This material can be used for teaching purposes, and there is much relevant and interesting historical material. The acceleration methods developed for the purpose of evaluating these constants are useful in other problems.

Key Words and Phrases: lemniscate, acceleration, elliptic functions, Euler transformation

CR Categories: 1.2, 5.0

This is an abbreviated version of an Invited Householder Lecture, given at the ACM Southeast Regional Meeting, Nashville, Tennessee, April 19, 1974. Another version of this paper was presented to the Bolyai János Matematikai Társulat in Keszthely, Hungary, in September 1973. Full details will be published in a monograph with the same title. The preparation of this paper has been supported in part by the National Science Foundation. Author's address: Department of Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125.

1. Introduction

The lemniscate constants are, in standard notation,

$$A = \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{\sqrt{2}} K\left(\frac{1}{2}\right) = \left(\frac{1}{4}\right)(2\pi)^{-\frac{1}{2}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2,$$

$$B = \int_0^1 \frac{t^2 dt}{\sqrt{1-t^4}} = (2\pi)^{-\frac{1}{2}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2.$$

THEOREM 1. *The length of a quadrant of the lemniscate of Bernoulli, $r^2 = \cos 2\theta$, is $2A$.*

PROOF. Trivial.

THEOREM 2. $AB = \frac{1}{4}\pi$.

An elementary proof, based essentially on Wallis' formula, is due to Euler [10]. This result was also established by Landen in 1755. It is also evident from the Γ-function representations given above.

THEOREM 3. *A is a transcendental number.*

THEOREM 4. *B is a transcendental number.*

These theorems were proved by Theodore Schneider in 1937 and 1941 respectively. (See Siegel [35, 36].)

2. Computation by Integration of Power Series

The obvious approach to the computation of A is to expand the integrand by the binomial theorem and to integrate term by term.

THEOREM 5.

$$A \equiv \sum a_n \equiv 1 + \sum_1^\infty \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{4n+1}.$$

The representation of A just given is of no practical use for computation. In fact, by the use of Wallis' formula, or Stirling's approximation for $n!$, we find that $a_n \simeq (4\pi^{\frac{1}{2}}n^{\frac{3}{2}})^{-1}$ which implies that the remainder after n terms is $O(n^{-\frac{1}{2}})$.

We can obtain an alternating series for A by writing the integrand as $(1-t^2)^{-\frac{1}{2}}(1+t^2)^{-\frac{1}{2}}$ and expanding the second factor by the binomial theorem and then integrating term by term using the fact that

$$\int_0^1 t^{2n}(1-t^2)^{-\frac{1}{2}}\, dt = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}. \tag{2.1}$$

THEOREM 6.

$$A \equiv \sum b_n \equiv \frac{\pi}{2}\left\{1 + \sum_1^\infty (-1)^n \left[\frac{(2n-1)!!}{(2n)!!}\right]^2\right\}.$$

The representation just given is of little practical use in computation since $|b_n| \simeq (2n)^{-1}$.

In view of the slow convergence of the series in Theorems 5 and 6, the question of acceleration arises. This question has been studied for a long time: among those who have contributed recently are J.R. Airey [1], W.G. Bickley and J.C.P. Miller [4], K. Nickel and D. Shanks [33], H.C. Thacher, Jr. [39], and P. Wynn [50], who has provided ALGOL algorithms in many cases. We discuss here only the Euler transform and the results of Stirling and A.A. Markoff.

Euler

It is well known that the Euler transformation is quite effective in accelerating the convergence of series such as $\sum b_n$. It is known (Knopp [24]) that if the sequence $(-1)^n b_n$ is completely monotone and

$$(b_{n+1}/b_n) \sim 1,$$

then the Euler transform will be convergent like a geometric series with common ratio 1/2. This makes computation feasible, and things can be improved by an appropriate delay in starting the transform. These matters are discussed by Dahlquist, et al. [7] and Todd [40].

The fact that the sequence $\{b_n\}$ is completely monotone can be established as follows. Since (Knopp [24]) the term-by-term product of two completely monotone sequences is completely monotone, it is enough to show that $\{b_n'\}$ is completely monotone where

$$b_n' = \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots (2n)} = \frac{(2n-1)!!}{(2n)!!}.$$

Putting $t^2 = T$ in (2.1), we find

$$b_n' = (1/\pi) \int_0^1 T^n [T(1-T)]^{-\frac{1}{2}}\, dT,$$

exhibiting b_n' as the nth moment of the distribution $[T(1-T)]^{-\frac{1}{2}}$; it follows (Widder [47, p. 108]) that $\{b_n'\}$ is completely monotone.

Stirling

The difficulty in using a power series directly was clear to Stirling, and he devised an acceleration method in 1730 which led to the value

$A = 1.31102\,87771\,46059\,87$

(which is correct to 15D) by dealing with a head of nine terms of the series of Theorem 6 and transforming the next nine terms. We shall explain this briefly.

Stirling discusses the general Beta-function

$$B(\nu, \mu) = \int_0^1 t^{\nu-1}(1-t)^{\mu-1}\, dt$$

which reduces to $4A$ when $\mu = \frac{1}{2}$, $\nu = \frac{1}{4}$. He expands the factor $(1-t)^{\mu-1}$ as a binomial series, then integrates term by term, and separates the resulting series into a head and tail

$S = [u(1) + \cdots + u(n-1)] + [u(n) + \cdots]$

which he writes as $S = S(n-1) + C(n)u(n)$ where $C(n)$ is a "converging factor" which indicates the appropriate modification of the nth term which gives a correct truncation.

Stirling proposes to find an *approximation* to $C(n)$ in the form $C(n) \doteqdot (n+p)/q$ where p,q are independent of n so that, accurately,

$S = S(n-1) + \{(n+p)u(n)/q\} + R(n+1).$

If we write $\sigma(n+1) = R(n+1) - R(n+2)$, then

$$R(n+1) = \sigma(n+1) + \sigma(n+2) + \cdots \tag{2.2}$$

and we want σ to be small. We can express σ/u as a ratio of two quadratics in n, and Stirling chooses p,q to make the n^2,n terms in the numerator vanish,

$q = \mu$, $p = (\nu - 1)/(1+\mu)$, so that the first improvement is to give

$$S \doteqdot S(n-1) + \frac{n(\mu+1) - 1(1-\nu)}{\mu(\mu+1)} u(n).$$

Stirling then repeats this process on the series (2.2)—the σ's satisfy a recurrence relation similar to that for the u's. He then finds the next improvement in the form

$$S \doteqdot S(n-1) + \frac{n(\mu+1) - 1(1-\nu)}{\mu(\mu+1)} u(n) + \frac{(n+2)(\mu+3) - 2(2-\nu)}{(\mu+2)(\mu+3)} \sigma(n+1).$$

This process can be repeated indefinitely and the convergence of the various series, e.g. (2.2), is clear.

Stirling also found B in the same way:

$B = .59907\,01173\,67796\,11.$

A.A. Markoff

Stirling's device is essentially a transformation of ordinary hypergeometric series $F(\alpha,\beta,\gamma,x)$ applied to the series Σa_n. Markoff [27] works on the alternating series Σb_n and apparently uses transformations of generalized hypergeometric series. His virtuosity in these matters was well known. Hermite wrote in 1889 (see Ogigova [30]), "Par quelle voie vous êtes parvenu à une telle transformation, je ne puis même de loin l'entrevoir, et il me faut vous laisser votre secret."

We discuss here one of the two methods given by Markoff. He begins with the remark that if

$$F(a,b) = C\left[2a+b+\alpha-1 + \frac{(2a+b+\alpha+1)\varphi(a)}{\varphi(-a-b-\alpha)} + \frac{(2a+b+\alpha+3)\varphi(a)\varphi(a+1)}{\varphi(-a-b-\alpha)\varphi(-a-b-\alpha-1)} + \cdots\right]$$

when $\varphi(t) = t^3 + \alpha t^2 + \beta t + \gamma$, with arbitrary C, a, b, α, β, γ, then $F(a,b) = F(b,a)$. In this he first puts

$$b = \tfrac{1}{2}(\delta+1),\ \alpha = \tfrac{1}{2}(\delta-1),$$

$$\varphi(t) = (t + \tfrac{1}{2}(\delta-1))t^2, \text{ and } C = D/(2a+\delta-1).$$

This gives

$$F(a,b) = D\left[1 - \left(\frac{a}{a+\delta}\right)^2 + \left(\frac{a}{a+\delta}\cdot\frac{a+1}{a+\delta+1}\right)^2 - \left(\frac{a}{a+\delta}\cdot\frac{a+1}{a+\delta+1}\cdot\frac{a+2}{a+\delta+2}\right)^2 + \cdots\right];$$

$$F(b,a) = (2a+3\delta-1)E_0 - (2a+3\delta+3)E_1 + (2a+3\delta+7)E_2 - (2a+3\delta+11)E_3 + \cdots,$$

where $E_0 = D/(4a + 2\delta - 2)$,

$$E_1 = \frac{\delta(\delta + 1)^2}{2(2a + \delta + 1)(a + \delta)^2} E_0,$$

and, generally,

$$E_{n+1} = \frac{(n + \delta)(2n + \delta + 1)^2}{2(2a + 2n + \delta + 1)(a + n + \delta)^2} E_n.$$

He then puts $a = 23/2$, $\delta = 1/2$,

$$D = \left\{\frac{1\cdot 3\cdot 5\cdot 7 \cdots 21}{2\cdot 4\cdot 6\cdot 8 \cdots 22}\right\}^2 = 0.02828\ 72353\ 30935\ 80044\ 80600 \ldots$$

and sums the first 13 terms of the E-series to 0.0147694731 6168476217 682.... Subtracting this from the head

$$1 - \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1\cdot 3\cdot 5 \cdots 19}{2\cdot 4\cdot 6 \cdots 20}\right)^2 = \sum_{n=0}^{11} (-1)^n \left(\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n)}\right)^2 = 0.8493963148\ 3575794845\ 819 \ldots$$

he finds

$$\frac{2}{\pi} A = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots (2n)}\right)^2 = 0.8346268416\ 7407318628\ 1$$

correct to 21 places.

The only proofs of the basic relation $F(a, b) = F(b, a)$ known to me are one given by H.M. Srivastava—it uses a transformation of a ${}_6F_5$ due to F.J.W. Whipple (cf *SIAM Rev.* (1974), p. 260)—and one due to R. A. Askey—this uses a transformation of a ${}_9F_8$ due to W. N. Bailey.

3. The Arithmetic-Geometric Mean Calculations of Gauss and the Carlson Sequences

As a teenager, Gauss experimented numerically with the algorithm

$a_0 = a$, $b_0 = b$, $a \geq b > 0$,

$a_{n+1} = \frac{1}{2}(a_n + b_n)$, $b_{n+1} = (a_n b_n)^{\frac{1}{2}}$, $n = 0, 1, 2, \ldots$.

It is convenient to write $c_n = (a_n^2 - b_n^2)^{\frac{1}{2}}$. The sequences $\{a_n\}$, $\{b_n\}$ converge monotonically and quadratically to a limit which we denote by $M(a, b)$. It was clear that Gauss was interested in finding the form of the function $M(a, b)$. Among his computations is the following for $M(2^{\frac{1}{2}}, 1)$:

n	a_n	b_n
0	1.41421 35623 73095 04880 2	1.00000 00000 00000 00000 0
1	1.20710 67811 86547 52440 1	1.18920 71150 02721 06671 7
2	1.19815 69480 94634 29555 9	1.19812 35214 03120 12260 7
3	1.19814 02347 93877 20908 3	1.19814 02346 77307 20579 8
4	1.19814 02347 35592 20744 1	1.19814 02347 35592 20743 9

In his Notebook [19, p. 542], on May 30, 1799, Gauss observed that $\pi/(2M(2^{\frac{1}{2}}, 1))$ coincided to 11D with A, and wrote, "Terminum medium arithmetico-geometricum inter 1 et $\sqrt{2}$ esse $= \pi/\omega$ usque ad figuram undecimam comprobavimus, qua re demonstrata prorsus novus campus in analysis certo aperietur."

Much of the work of Gauss in this area was never published by him; it was edited from his papers after his death, for his Collected Works, by various mathematicians including Schering, Klein, Fricke, Schlesinger. See Geppert [21]. It was not until several months later, on December 23, 1799, that Gauss established the relation between M and an elliptic integral.

THEOREM 7. *If* $0 \leq k \leq 1$, *then*

$$\frac{\pi}{2M(1, (1 - k^2)^{\frac{1}{2}})} = \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{\frac{1}{2}}} = \int_0^1 \frac{dt}{((1 - t^2)(1 - k^2t^2))^{\frac{1}{2}}} = K(k^2).$$

In his Notebook [19, p. 544] he wrote, "Medium Arithmetico-Geometricum ipsum est quantitas integralis. Dem[onstratum]."

There seems to be no easy proof of this theorem. All available use the following idea. In the usual notation,

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(a_n^2 \cos^2 \theta + b_n^2 \sin^2 \theta)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{(a_{n+1}^2 \cos^2 \varphi + b_{n+1}^2 \sin^2 \varphi)^{\frac{1}{2}}} \tag{3.1}$$

and so, passing to the limit,

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(a_0^2 \cos^2 \theta + b_0^2 \sin^2 \theta)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{M(a_0, b_0)},$$

which is the result required. All depends on the proof of (3.1), which can be done by the transformation

$$\sin \theta = \frac{2a_n}{(a_n + b_n) \operatorname{cosec} \varphi + (a_n - b_n) \sin \varphi}.$$

For a recent exposition of this material see Fuchs [51].

THEOREM 8. *If* $x_0 = 1$, $y_0 = 0$, *and if for* $n \geq 0$, $x_{n+1} = \frac{1}{2}(x_n + y_n)$, $y_{n+1} = (\frac{1}{2}x_n(x_n + y_n))^{\frac{1}{2}}$, *then* $\lim x_n = \lim y_n = A^{-2}$. *Further, convergence is geometric with multiplier* $\frac{1}{4}$.

PROOF. It is easy to check using the transformation $x = (1 + T)^{-\frac{1}{4}}$ that

$$A = \int_0^1 (1 - x^4)^{-\frac{1}{2}}\, dx = \frac{1}{4} \int_0^\infty (1 + T)^{-\frac{1}{2}} T^{-\frac{3}{4}}\, dT. \tag{3.2}$$

Consider the transformation of the integral

$$I(x^2, y^2) = \int_0^\infty (t + x^2)^{-\frac{1}{2}}(t + y^2)^{-\frac{1}{2}}\, dt$$

by changing the variable from t to s where $t = s(s + xy)/[s + (1/2)(x + y)^2]$. A little algebra gives

$$I(x_0^2, y_0^2) = I(x_1^2, y_1^2). \tag{3.3}$$

The existence of the common limit of $I(x_0, y_0)$ of the sequences $\{x_n\}$, $\{y_n\}$ and the rate of convergence is easy to check. Repeated use of (3.3) gives

$$I(x_0^2, y_0^2) = \int_0^\infty (t + t^2)^{-\frac{1}{2}}\, dt = 4[I(x_0, y_0)]^{-\frac{1}{2}}.$$

From (3.2), taking $x_0 = 1$, $y_0 = 0$, we find $A = [l(1, 0)]^{-\frac{1}{2}}$ as required.

Theorems 7 and 8 and many related results were obtained by a uniform method by B.C. Carlson [6].

4. Arcsl Relations

Since

$$\int_0^x \frac{dt}{(1 - t^2)^{\frac{1}{2}}} = \arcsin x, \quad \arcsin 1 = \tfrac{1}{2}\pi,$$

it is natural to introduce lemniscate functions defined in the first place by

$$\int_0^x \frac{dt}{(1 - t^4)^{\frac{1}{2}}} = \operatorname{arcsl} x, \quad \operatorname{arcsl} 1 = A,$$

$\operatorname{arccl} x + \operatorname{arcsl} x = A$.

There is no definitive account of the lemniscate functions. A Cambridge Tract announced by G.B. Matthews never appeared; there is some account of the theory in various books such as Markushevich [28], C.L. Siegel [35]. The explicit expressions in terms of the Jacobian functions with modulus $k^2 = \frac{1}{2}$ are

$\operatorname{sl} x = 2^{-\frac{1}{2}} \operatorname{sn}(2^{\frac{1}{2}} x)/\operatorname{dn}(2^{\frac{1}{2}} x)$,
$\operatorname{cl} x = \operatorname{cn}(2^{\frac{1}{2}} x)$.

It is clear that if we could obtain a relation of the form

$$A = \sum_{i=1}^{n} a_i \operatorname{arcsl} x_i, \tag{4.1}$$

where the x_i satisfy, e.g. $|x_i| \leq \frac{1}{2}$, then we would have a feasible method for evaluating A by
$\operatorname{arcsl} x = x + \frac{1}{10} x^5 + \frac{1}{24} x^9 + \cdots$
since this will mean we are working with series with effective ratio $2^{-4} = .0625$. This is similar to the classical ways of evaluating π, e.g. using

$$\tfrac{1}{4}\pi = 4 \operatorname{arccot} 5 - \operatorname{arccot} 239. \tag{4.2}$$

Gauss gave the following result:

THEOREM 9. $A = \mathit{arcsl}\, \frac{7}{23} + 2\, \mathit{arcsl}\, \frac{1}{2}$.

PROOF. This result can be written as

$$2 \int_0^{\frac{1}{2}} \frac{dt}{(1 - t^4)^{\frac{1}{2}}} = \int_{\frac{7}{23}}^{1} \frac{dT}{(1 - T^4)^{\frac{1}{2}}},$$

which can be established by using the transformation $T = (1 - 2t^2 - t^4)/(1 + 2t^2 - t^4)$.

The effective ratio of terms in the series is $2^{-4} = .0625$.

A somewhat similar result was given much earlier (1714) by Fagnano [13], who discussed the bisection of the arc of the lemniscate.

THEOREM 10. $A = 2\, \mathit{arcsl}(2^{\frac{1}{2}} - 1)^{\frac{1}{2}}$.

PROOF. This result is equivalent to

$$\int_0^1 \frac{dt}{(1 - t^4)^{\frac{1}{2}}} = 2 \int_0^{(2^{\frac{1}{2}}-1)^{\frac{1}{2}}} \frac{d\tau}{(1 - \tau^4)^{\frac{1}{2}}},$$

and can be established by the transformation $t = 2\,\tau(1 - \tau^4)^{\frac{1}{2}}/(1 + \tau^4)$.

The trigonometric analog of this theorem is: $\frac{1}{2}\pi = 2 \arcsin 2^{-\frac{1}{2}}$.

The effective ratio of the terms in the series is $(2^{\frac{1}{2}} - 1)^2 \doteqdot .1716$, so a computation of A via Theorem 10 is somewhat less convenient than that via Theorem 9. However, Fagnano showed that it is possible to divide the arc of the lemniscate into three or five parts, again at the expense of solving quadratics only: these results also lead to convenient methods of computation of A.

THEOREM 11. $A = 3\, \mathit{arcsl}\{\frac{1}{2}[1 + \sqrt{3} - \sqrt{(2\sqrt{3})}]\}$.

This can be established using the transformation employed in Theorem 9. The trigonometric analog of this theorem is: $\frac{1}{2}\pi = 3 \arcsin 3^{-\frac{1}{2}}$. The effective ratio of terms in the series is $(.43542)^4 = .03594$.

THEOREM 12.

$$A = 5\, \mathit{arcsl}\left[\frac{1 - \{(6\sqrt{5} - 13) + (340 - 152\sqrt{5})^{\frac{1}{2}}\}^{\frac{1}{2}}}{1 + \{(6\sqrt{5} - 13) + (340 - 152\sqrt{5})^{\frac{1}{2}}\}^{\frac{1}{2}}}\right]^{\frac{1}{2}}$$

PROOF. Use the fact established by Gauss:

$$\operatorname{sl} 5\varphi = \frac{s(5 - 2s^4 + s^8)(1 - 12s^4 - 26s^8 + 52s^{12} + s^{16})}{(1 - 2s^4 + 5s^8)(1 + 52s^4 - 26s^8 + 12s^{12} + s^{16})},$$

where $s = \operatorname{sl} \varphi$, and ideas of Fagnano and Watson.

The trigonometric analog of this theorem is: $\frac{1}{2}\pi = 5 \arcsin (\sqrt{5} - 1)/4$. The effective ratio of terms in the series is $\operatorname{sl}^4 \frac{1}{5} A \doteqdot .0047$.

Using the lemniscate trigonometry it is natural to investigate relations of the form (4.1) in the same way as Lehmer [26] has evaluated arctan relations typified by (4.2) as an efficient means of calculating $\frac{1}{4}\pi$. See also Størmer [38], Todd [41, 42]. For instance, the arcsl relation $A = 2 \operatorname{arcsl} \frac{1}{3} + \operatorname{arcsl} (31/49)$ is less efficient than that given in Theorem 9.

5. Quadrature Formulas

Another obvious approach to the evaluation of A is to use appropriate quadrature formulas.

We mention here briefly the Gauss-Chebyshev quadrature [40]:

$$A = \frac{1}{2} \int_{-1}^{1} \frac{1}{(1 + t^2)^{\frac{1}{2}}} \cdot \frac{dt}{(1 - t^2)^{\frac{1}{2}}}$$
$$\doteqdot Q_n = .5 \sum_{r=1}^{n} \frac{(\pi/n)}{(1 + \cos^2 ((2n - 1)\pi/2n))^{\frac{1}{2}}}.$$

It is awkward to estimate the error here, but there are some possibilities (see e.g. G. Freud [17]).

In view of the singularity at $t = 1$, Romberg methods are not directly applicable. Appropriate modifications have been developed by L. Fox [15, 16] and J.A. Shanks [34].

6. Theta Series

This seems to be the most powerful method. Formulas involving ϑ-functions have proved effective in many computational problems, e.g. that of Ewald [12] on crystal structure, and especially in calculations involving elliptic functions and integrals.

The idea is that the Jacobian elliptic functions, sn, cn, dn, and the complete elliptic integrals, K, K', for a given modulus k^2 can be expressed in terms of ϑ-functions with q determined by the equation
$q = \exp(-\pi K'/K)$.

In the lemniscate case, $k^2 = \frac{1}{2} = k'^2$ and so $K = K'$ (the period parallelogram is a square), which implies that $q = e^{-\pi} = .04321\ldots$.

We shall give an outline of a direct proof of our representations rather than relying on the general theory of elliptic functions.

Enlightened experimentation (cf. von Dávid [8]) suggests that we consider the series
$\alpha(x) = 1 + 2x + 2x^4 + 2x^9 + \cdots$,
$\beta(x) = 1 - 2x + 2x^4 - 2x^9 + \cdots$.
The following two results are needed.

THEOREM 13. *If* $|x| < 1$, *then*
$[\alpha(x)]^2 + [\beta(x)]^2 = 2[\alpha(x^2)]^2$, $\alpha(x)\beta(x) = [\beta(x^2)]^2$.

THEOREM 14. *For any a,b such that $0 \le b \le a$ there is an x, $|x| < 1$, such that $\alpha(x)/\beta(x) = (a/b)^{\frac{1}{2}}$. Indeed $x = \exp(-\pi M(a_0, b_0)/M(a_0, c_0))$. In particular, for $a = \sqrt{2}$, $b = 1$, we have $x = e^{-\pi}$.*

Theorem 13 is essentially combinatorial in nature and expresses some of the basic relations among the ϑ-functions. For a proof see e.g. van der Pol [31].

Theorem 14 depends essentially on Theorem 15.

THEOREM 15. *This $M(1,x) \log x^{-1} \to \frac{1}{2}\pi$, as $x \to 0$.*

Barna [2] has given a neat proof of this.

The point of these results is that they permit a parameterization of the Gaussian algorithm. Specifically, given $a_0 \ge b_0$, Theorem 14 implies the existence of an x, $|x| < 1$, such that
$a_0 = M(a_0, b_0)(\alpha(x))^2$, $b_0 = M(a_0, b_0)(\beta(x))^2$, and then repeated application of Theorem 13 gives

$$a_n = M(a_0, b_0)(\alpha(x^{2^n}))^2, \quad b_n = M(a_0, b_0)(\beta(x^{2^n}))^2. \quad (6.1)$$

It will now be more convenient to use the standard ϑ-function notation in which $\alpha(q) = \vartheta_3(q)$, $\beta(q) = \vartheta_4(q)$, $\vartheta_2(q) = 2q^{\frac{1}{4}}[1 + q^2 + q^6 + q^{12} + \cdots]$.

From (6.1) we have, in the lemniscate case,

$$A = (\pi/2a_n)\{\vartheta_3(\exp(-2^n\pi))\}^2. \quad (6.2)_n$$

$$A = (\pi/2b_n)\{\vartheta_4(\exp(-2^n\pi))\}^2. \quad (6.3)_n$$

$$A = (\pi/2c_n)\{\vartheta_2(\exp(-2^n\pi))\}^2. \quad (6.4)_n$$

In (6.2) and (6.3) the factor in braces is a correction factor, which brings the a_n, b_n to their limit $\pi/(2A)$. Since the argument in the ϑ series can be made arbitrarily small by choice of n, we can optimize the computation of A by balancing the work done in computing a_n, b_n with that done in estimating the sums of the series. Since ϑ_3, ϑ_4 differ only by the signs of alternate terms, use of (6.2) *and* (6.3) gives a convenient check. The formula (6.4) does not appear to be so convenient as the right-hand side approaches the form 0/0.

THEOREM 16. $A = \frac{1}{2}\pi\{\vartheta_4(e^{-\pi})\}^2 = \frac{1}{2}\pi\{\vartheta_2(e^{-\pi})\}^2$.

These were given by Gauss [18]; they come from $(6.3)_0$ and the fact that $\vartheta_2 = \vartheta_4$ in the lemniscate case.

THEOREM 17. $A = \frac{1}{4}\pi 2^{\frac{1}{2}}\{\vartheta_3(e^{-\pi})\}^2$.

This was used by Wrench [48] in his fundamental calculations to 164 + D. It comes from $(6.2)_0$ and the fact that $\vartheta_3 = 2^{\frac{1}{4}}\vartheta_4$ in the lemniscate case.

If we take $(6.4)_1$ where
$a_1 = (1 + 2^{\frac{1}{2}})/2$, $b_1 = 2^{\frac{1}{4}}$, $c_1 = (2^{\frac{1}{2}} - 1)/2$, we get
$A = (\pi/(2^{\frac{1}{2}} - 1))\{\vartheta_2(e^{-2\pi})\}^2$. If we take $(6.4)_2$ we get

$$A = (2\pi/(2^{\frac{1}{4}} - 1)^2)\{\vartheta_2(e^{-4\pi})\}^2. \quad (6.6)$$

This result appeared essentially in a Cambridge examination in 1881 (Whittaker and Watson [46]).

7. Three Additional Representations

THEOREM 18.

$$A = 2^{\frac{1}{2}}\pi \exp(-\pi/3)\left\{\sum_{-\infty}^{\infty} (-1)^n \exp(-2\pi(3n^2 + n))\right\}^2.$$

This was given by Lehmer [25] without proof and used by him for 50D calculations. It can be proved using the product representations of the ϑ-functions such as $\vartheta_4(q) = \Pi(1 - q^{2n+1})^2(1 - q^{2n})$ and the fact that $\sum(-1)^n x^{(3n^2+n)/2} = \Pi(1 - x^n)$. Proof of these results are given e.g. in Hardy and Wright [22].

THEOREM 19.

$$A^4 = \frac{11}{48} - \frac{5\pi}{8} + \frac{15}{8}\sum_{1}^{\infty} cosech^4\, n\pi.$$

This result appeared in Muckenhoupt [29].

THEOREM 20.

$$(1 - A^2\pi^{-2})^2 = 1 - \pi^{-1} + 6\sum_{0}^{\infty} cosech^4\,(2n + 1)\pi.$$

This result has been derived by Carlson (unpublished) from results of Kiyek and Schmidt [23].

8. Historical Notes

G.C. di Fagnano (1682–1766) was called for advice, in 1743, by Pope Benedict XIV, when it was discovered that St. Peter's was threatened with collapse. After restoring the foundations Fagnano was honored by being created a marquis and by having his Collected Works published. In his portrait, in his birthplace Senigallia, Fagnano has a diagram of a lemniscate in his hand; a lemniscate also appears on the title page of his book [13] *Produzioni Matematiche* and beneath it the words "Multifariam Divisa Atque Dimensa. Deo Veritatis Gloria." This book reached Euler on December 23, 1751, exactly 48 years before Gauss' proof of Theorem 7.

The part played by Fagnano in the founding of the theory of elliptic functions is now fully recognized. (See e.g. C.L. Siegel [36].) For further information see G.N. Watson [45].

J. Stirling (1692–1770) held a Snell Exhibition at Oxford. This award was restricted to students from Glasgow, but there is no record that he ever attended that university; it has been pointed out that a kinsman was rector of the university at the time.

Stirling was arrested in Oxford in 1715 for cursing the king; he was tried and found "not guilty." He lost his scholarships because he refused to sign an oath of allegiance. He was offered a chair in Italy but apparently rejected it on account of religious difficulties. He spent the last half of his life as manager of a mine in Scotland.

The English translation of his book [37], which is really about finite differences, seems very rare; a copy is in the Bodleian Library (Rignaud d. 158). For further historical information see C. Tweedie [44].

A.A. Markoff (1856–1922) with his brother W.A. Markoff (1871–1897) began work in the constructive theory of functions, established by P.L. Chebyshev. His book [27] on finite differences is one of the classics; he later made contributions to the theory of probability, introducing what are now called Markoff processes in 1907. A.A. Markoff, who was born in 1903, and is well known for his work in logic and in particular for his book on *The Theory of Algorithms*, is a son.

Acknowledgments. I am grateful for helpful discussions on parts of this paper with, among others, B.C. Carlson, Basil Gordon, J.W. Wrench Jr., and D.W. Boyd.

References

1. Airey, J.R. The "converging factor" in asymptotic series and the calculation of Bessel, Laguerre and other functions. *Phil. Mag. 7*, 24 (1937), 521–552.
2. Barna, B. Ein Limessatz aus der Theorie des arithmetisch-geometrischen Mittel. *J. Reine Angew. Math. 172* (1934), 86–88.
3. Bickley, W.G., and Miller, J.C.P. The numerical summation of slowly convergent series of positive terms. *Phil. Mag. 7*, 22 (1936), 754–767.
4. ——. Manuscript, "The numerical summation of series," 1945.
5. Brezinski, C. Accélération de la convergence en analyse numérique. Mimeographed lecture notes, Lille, France, 1973.
6. Carlson, B.C. Algorithms involving arithmetic and geometric means. *Amer. Math. Monthly 78* (1971), 496–505.
7. Dahlquist, G., Gustafson, S.A., and Siklósi, K. Convergence acceleration from the point of view of linear programming. *BIT 5* (1965), 1–16.
8. von Dávid, L. Arithmetisch-geometrisches Mittel und Modulfunktion. *J. Reine Angew. Math. 159* (1928), 154–170.
9. Euler, L. *Opera Omnia*, Vol. 1, 11, 1913, Leipzig.
10. ——. *Introductio in Analysis Infinitorum*, I, Ch. 16, 1748.
11. ——. *Institutiones Calculi Integralis*, 1748, pp. 228–236.
12. Ewald, P.P. Die Berechnung optischer und elektrostatischer Gitterpotentiale. *Ann. Phys.* (1921), 253–287.
13. Fagnano, G.C. *Opere Matematiche*, 3 vols., 1911–12.
14. ——. *Produzioni Matematiche*, 1750.
15. Fox, L. Comments on singularities in numerical integration and the solution of differential equations, In *Numerical Methods*, P. Rósza (Ed.), Coll. Math. Soc. J. Bolyai, 3, 1968, pp. 61–91.
16. ——. Romberg integration for a class of singular integrands. *Computer J. 7* (1967), 87–93.
17. Freud, G. Error estimates for Gauss-Jacobi quadrature formulae, pp. 93–112. In *Topics in Numerical Analysis*, J.J.H Miller (Ed.), London, 1973.
18. Gauss, C.F. *Werke*, 3, Leipzig, 1876.
19. ——. *Werke* 10, Leipzig, 1917.
20. Geppert, H. (Ed.) Ostwald's Klassiker der exakten Wissenschaften, # 225, Leipzig, 1927.
21. ——. Wie Gauss zur elliptischen Modulfunktion kam. *Deutsche Math. 5* (1940), 158–175.
22. Hardy, G.H., and Wright, E.M. *Theory of Numbers*. Clarendon Press, Oxford, 1938.
23. Kiyek, K., and Schmidt, H. Auswertung einiger spezieller unendlicher Reihen aus dem Bereich der elliptischen Funktionen. *Arch. Math. 18* (1967), 438–443.
24. Knopp, K. *Theory and Application of Infinite Series*, 2nd ed. Blackie, London, 1948.
25. Lehmer, D.H. The lemniscate constant. *MTAC 3* (1948/9), 550–551.
26. ——. On arccotangent relations for π. *Amer. Math. Monthly 45* (1938), 657–664.
27. Markoff, A. *Differenzenrechnung*. Tr. of Russian edition of 1889–91. Leipzig, 1896.
28. Markushevich, A.I. *The Remarkable Sine Functions*. American Elsevier, New York, 1966.
29. Muckenhoupt, B. The norm of a discrete singular transform. *Studia Math. 25* (1964/5), 97–102.
30. Ogigova, H. Les lettres de Ch. Hermite à A. Markoff. 1885–1889. *Rev. d'histoire des sciences et de leurs applications. 20* (1967), 1–32. Letter dated 11 December 1889.
31. van der Pol, B. Demonstration élémentaire de le relation $\theta_3^4 = \theta_0^4 + \theta_2^4$ entre les différentes fonctions de Jacobi. *Enseignement Math.* 1 (1955), 259–262.
32. Reichardt, H. (Ed.) *C.F. Gauss*. Gedenkband, Leipzig, 1957.
33. Shanks, D. Nonlinear transformations of divergent and slowly convergent series. *J. Math. Phys. 36* (1955), 1–62.
34. Shanks, J.A. Romberg tables for singular integrands. *Computer J. 15* (1972), 360–361.
35. Siegel, C.L. *Topics in Complex Function Theory, Vol. I.* Wiley, New York, 1969.
36. ——. *Transcendental Numbers*, Princeton U. Press, 1949.
37. Stirling, J. *Methodus Differentialis*, London, 1730. English trans. by F. Holliday, London, 1749.
38. Størmer, C. Sur un problème curieux de la théorie des nombres concernant les fonctions elliptiques. *Arch. Math. Naturvid. B47*, # 5 (1948), 83–85.
39. Thacher, H.C., Jr. Numerical application of the generalized Euler transformation pp. 627–631 in Information Processing 74, J. Rosenfeld (Ed.), North-Holland Pub. Co., Amsterdam, 1974.
40. Todd, John. Optimal parameters in two programs. *Bull. IMA 6* (1970), 31–35.
41. ——. A problem on arctangent relations. *Amer. Math. Monthly. 56* (1949), 517–528.
42. ——. Table of arctangents of rational numbers. U.S. Nat. Bur. Standards, Appl. Math. Series 11, 1951, 1965, U.S. Government Printing Office, Washington, D.C.
43. ——. *Introduction to the Constructive Theory of Functions*. Academic Press, New York, 1963.
44. Tweedie, C. *James Stirling*. Oxford, 1922.
45. Watson, G.N. The marquis and the land-agent—a tale of the eighteenth century. *Math. Gazette 17* (1933), 5–16.
46. Whittaker, E.T. and Watson, G.N. *Modern Analysis*. Cambridge U. Press, 1927.
47. Widder, D.V. *The Laplace Transform*. Princeton U. Press, 1941.
48. Wrench, J.W., Jr. Manuscript (1955).
49. ——. *MTAC 4* (1948/9), 201–203.
50. Wynn, P. Acceleration techniques in numerical analysis, with particular reference to problems in one independent variable. Proc. IFIP Congress, 62, Munich, North-Holland Pub. Co., Amsterdam, pp. 149–156.

MATHEMATICS OF COMPUTATION, VOLUME 30, NUMBER 135
JULY 1976, PAGES 565–570

Computation of π Using Arithmetic-Geometric Mean

By Eugene Salamin

Abstract. A new formula for π is derived. It is a direct consequence of Gauss' arithmetic-geometric mean, the traditional method for calculating elliptic integrals, and of Legendre's relation for elliptic integrals. The error analysis shows that its rapid convergence doubles the number of significant digits after each step. The new formula is proposed for use in a numerical computation of π, but no actual computational results are reported here.

1. Introduction. This paper announces the discovery of a new formula for π. It is based upon the arithmetic-geometric mean, a process whose rapid convergence doubles the number of significant digits at each step. The arithmetic-geometric mean, together with π as a known quantity, is the basis of Gauss' method for the calculation of elliptic integrals. But with the help of an elliptic integral relation of Legendre, Gauss' method can be turned around to express π in terms of the arithmetic-geometric mean. The resulting algorithm retains the property of doubling the number of digits at each step.

The proof of the main result (Theorem 1a) from first principles can be conducted on the elementary calculus level. The references cited here for the theorems of Landen, Gauss and Legendre have been chosen to achieve this goal, thus allowing the widest possible reader audience comprehension.

The formula presented in this paper is proposed as a method for the numerical computation of π. It has not yet been tested on a calculation of nontrivial length, although such a calculation is currently in progress [2].

2. The Arithmetic-Geometric Mean. Let a_0, b_0, c_0 be positive numbers satisfying $a_0^2 = b_0^2 + c_0^2$. Define a_n, the sequence of arithmetic means, and b_n, the sequence of geometric means, by

$$a_n = \tfrac{1}{2}(a_{n-1} + b_{n-1}), \qquad b_n = (a_{n-1} b_{n-1})^{1/2}.$$

Also, define a positive sequence c_n:

$$c_n^2 = a_n^2 - b_n^2.$$

Two relations easily follow from these definitions.

$$c_n = \tfrac{1}{2}(a_{n-1} - b_{n-1}). \tag{1}$$

$$c_n^2 = 4a_{n+1} c_{n+1}. \tag{2}$$

Received May 27, 1975; revised November 3, 1975.

AMS (MOS) subject classifications (1970). Primary 10A30, 10A40, 33A25; Secondary 41A25.

Key words and phrases. π, arithmetic-geometric mean, elliptic integral, Landen's transformation, Legendre's relation, fast Fourier transform multiplication.

The arithmetic-geometric mean is the common limit

$$\operatorname{agm}(a_0, b_0) = \lim a_n = \lim b_n.$$

Because of the rapidity of convergence of the arithmetic-geometric mean, as exhibited by Eq. (2), the formula to be derived should be regarded as a plausible candidate for the numerical computation of π.

3. Elliptic Integrals. The complete elliptic integrals are the functions

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{-1/2}\, dt, \qquad E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{1/2}\, dt.$$

Also, if $k^2 + k'^2 = 1$, then $K'(k) = K(k')$ and $E'(k) = E(k')$ are two more elliptic integrals.

There is also a symmetric form of these integrals:

$$I(a, b) = \int_0^{\pi/2} (a^2 \cos^2 t + b^2 \sin^2 t)^{-1/2}\, dt,$$

$$J(a, b) = \int_0^{\pi/2} (a^2 \cos^2 t + b^2 \sin^2 t)^{1/2}\, dt.$$

It is clear that

$$I(a, b) = a^{-1} K'(b/a), \qquad J(a, b) = aE'(b/a).$$

4. Landen's Transformation and the Computation of Elliptic Integrals. Using the notation developed in Section 2 of this paper, these transformations are [6, Section 25.15],

$$I(a_n, b_n) = I(a_{n+1}, b_{n+1}), \tag{3}$$

$$J(a_n, b_n) = 2J(a_{n+1}, b_{n+1}) - a_n b_n I(a_{n+1}, b_{n+1}). \tag{4}$$

From Eq. (3) it follows that

$$I(a_0, b_0) = \pi/2 \operatorname{agm}(a_0, b_0), \tag{5}$$

and, after a little work, Eq. (4) yields

$$J(a_0, b_0) = \left(a_0^2 - \tfrac{1}{2} \sum_{j=0}^{\infty} 2^j c_j^2\right) I(a_0, b_0). \tag{6}$$

For $a_0 = 1, b_0 = k'$, the integrals in Eqs. (5) and (6) are equal to $K(k)$ and $E(k)$, respectively, while for $a_0 = 1, b_0 = k$, they equal $K'(k)$ and $E'(k)$. This is the well-known method of Gauss for the numerical calculation of elliptic integrals [5, pp. 78–80], [1, Section 17.6].

5. Legendre's Relation. This relation is [4, Article 171], [1, Eq. 17.3.13],

$$K(k)E'(k) + K'(k)E(k) - K(k)K'(k) = \pi/2. \tag{7}$$

Equivalently,

$$a^2 I(a, b) J(a', b') + a'^2 I(a', b') J(a, b) - a^2 a'^2 I(a, b) I(a', b') = (\pi/2) a a', \tag{8}$$

where a, b, a', b' are subject to the restriction $(b/a)^2 + (b'/a')^2 = 1$.

6. New Expression for π. Take $a_0 = a'_0 = 1$, $b_0 = k$, $b'_0 = k'$. As in Section 2, define the sequences a_n, b_n, c_n, a'_n, b'_n, c'_n. In Eq. (8) eliminate the J integrals by use of (6), and then eliminate the I integrals by use of (5). Lo and behold, the resulting equation can be solved for π!

THEOREM 1a.

$$\pi = \frac{4\,\mathrm{agm}(1, k)\mathrm{agm}(1, k')}{1 - \sum_{j=1}^{\infty} 2^j(c_j^2 + c_j'^2)}. \tag{9}$$

The $j = 0$ term in the summation has been eliminated by use of $c_0^2 + c_0'^2 = k'^2 + k^2 = 1$. It is best to compute c_j from Eq. (1).

Theorem 1a is a one-dimensional continuum of formulae for π. This provides for an elegant and simple computational check. For example, π could be calculated starting with $k = k' = 2^{-1/2}$, and then checked using $k = 4/5$, $k' = 3/5$. The symmetric choice, $k = k'$, causes the two agm sequences to coincide, thus halving the computational burden.

THEOREM 1b.

$$\pi = \frac{4(\mathrm{agm}(1, 2^{-1/2}))^2}{1 - \sum_{j=1}^{\infty} 2^{j+1} c_j^2}.$$

7. Error Analysis. Although Theorem 1a is true for all complex values of k (except for a discrete set), the error analysis will assume real k and k'. Then $0 < k, k' < 1$. Let n square roots be taken in the process of computing agm $=$ agm$(1, k)$, and n' square roots in computing agm$'$ $=$ agm$(1, k')$. Then no further square roots are needed to calculate the approximation

$$\pi_{nn'} = \frac{4a_{n+1}a'_{n'+1}}{1 - \sum_{j=1}^{n} 2^j c_j^2 - \sum_{j=1}^{n'} 2^j c_j'^2}. \tag{10}$$

A rough estimate shows that a_{n+1} differs from agm by c_{n+2}, and that the finite sum differs from the infinite sum by $2^{n+3}c_{n+2}$. Thus, the numerator and denominator in (10) have been truncated for compatible error contributions, and the denominator error is dominant.

To obtain rigorous error bounds, introduce the auxiliary quantity $\bar{\pi}_{nn'}$ whose denominator is taken from (10), but whose numerator is taken from (9). The first step is to establish the existence of $e_{nn'}$, $\bar{e}_{nn'}$ such that

$$0 < \pi - \bar{\pi}_{nn'} < e_{nn'}, \tag{11}$$

$$0 < \pi_{nn'} - \bar{\pi}_{nn'} < \bar{e}_{nn'}, \qquad \bar{e}_{nn'} < e_{nn'}. \tag{12}$$

These three inequalities imply that $|\pi - \pi_{nn'}| < e_{nn'}$.

The left-hand inequalities in (11) and (12) are obvious. From the general inequality $(1/x) - (1/(x + y)) < y/x^2$, valid for positive x and y, it follows that

$$\pi - \bar{\pi}_{nn'} < \frac{\pi^2}{4\ \mathrm{agm}\ \mathrm{agm}'}\left(\sum_{n+1}^{\infty} 2^j c_j^2 + \sum_{n'+1}^{\infty} 2^j c_j'^2\right).$$

This establishes (11), with error bound

$$e_{nn'} = \frac{\pi^2}{2 \text{ agm agm}'} \left(\sum_{n+2}^{\infty} 2^j a_j c_j + \sum_{n'+2}^{\infty} 2^j a'_j c'_j \right). \tag{13}$$

Proceeding to the next inequality, we first get

$$\pi_{nn'} - \bar{\pi}_{nn'} < \frac{\pi}{\text{agm agm}'} (a_{n+1} a'_{n'+1} - \text{agm agm}').$$

Substitute $a_{n+1} = \text{agm} + s$, $a'_{n'+1} = \text{agm}' + s'$, where

$$s = \sum_{n+2}^{\infty} c_j, \qquad s' = \sum_{n'+2}^{\infty} c'_j,$$

and use agm < 1, agm$' < 1$ to get

$$\pi_{nn'} - \bar{\pi}_{nn'} < \pi(s + s' + ss')/\text{agm agm}'.$$

Also, since $s < 1$, $s' < 1$, it follows that $ss' < (s + s')/2$. Thus, inequality (12) is established with error bound

$$\bar{e}_{nn'} = \frac{3}{2} \frac{\pi}{\text{agm agm}'} \left(\sum_{n+2}^{\infty} c_j + \sum_{n'+2}^{\infty} c'_j \right). \tag{14}$$

Finally, a term-by-term comparison of (13) and (14), using $2^j a_j > 1$ and $\pi > 3$, shows that $\bar{e}_{nn'} < e_{nn'}$.

At this point, a needed inequality is derived.

$$a_j < a_j + b_j = 2a_{j+1},$$

$$2a_j c_{j+1} < 4a_{j+1} c_{j+1} = c_j^2 = (a_{j-1} - a_j) c_j,$$

$$a_j(c_j + 2c_{j+1}) < a_{j-1} c_j. \tag{15}$$

Consider the first summation in (13), but with the upper limit ∞ replaced by finite N. Perform the following sequence of operations, each of which increases the sum. First, replace a_N by a_{N-1}. Next, repeatedly apply (15) to the pair of highest-indexed terms in the sum. At the end, we are left with the single term $2^{n+2} a_{n+1} c_{n+2} < 2^{n+2} c_{n+2}$, which is thus an upper bound for the initial summation. Since N was arbitrary, the infinite summation also has this upper bound. Therefore,

$$e_{nn'} < \frac{2\pi^2}{\text{agm agm}'} (2^n c_{n+2} + 2^{n'} c'_{n'+2}). \tag{16}$$

An upper bound for c_{n+2} is needed now. It is convenient to use the abbreviations

$$x_n = \log c_n, \qquad g_n = \log(4a_n).$$

Equation (2) gives x_n as the solution to an inhomogeneous linear difference equation.

$$x_n = 2^n \left(x_0 - \sum_{j=1}^{n} 2^{-j} g_j \right).$$

By rearrangement,

$$x_n = 2^n \left(x_0 - g_1 + \sum_{j=1}^{n-1} 2^{-j} (g_j - g_{j+1}) \right) + g_n.$$

Using $g_j - g_{j+1} > 0$, $g_n < \log 4$, and $x_0 - g_1 = (1/2)\log(c_1/4a_1)$, we get

$$x_n < 2^{n-1}\left[\sum_{j=1}^{\infty} 2^{-j+1}\log(a_j/a_{j+1}) - \log(4a_1/c_1)\right] + \log 4. \tag{17}$$

For the purpose of an error analysis, the expression within brackets could be calculated numerically for any case of interest. However, it can be evaluated in closed form [7, p. 14] and is equal to $-\pi K'(k)/K(k) = -\pi\,\mathrm{agm}/\mathrm{agm}'$. Then

$$x_n < -\pi(\mathrm{agm}/\mathrm{agm}')2^{n-1} + \log 4.$$

Substituting this into (16) yields the final result.

THEOREM 2a.

$$|\pi - \pi_{nn'}| < \frac{8\pi^2}{\mathrm{agm}\ \mathrm{agm}'}\left[2^n \exp\left(-\pi\frac{\mathrm{agm}}{\mathrm{agm}'}2^{n+1}\right) + 2^{n'}\exp\left(-\pi\frac{\mathrm{agm}'}{\mathrm{agm}}2^{n'+1}\right)\right].$$

In the symmetric case, with $\pi_n = \pi_{nn}$, Theorem 2a simplifies to

THEOREM 2b.

$$|\pi - \pi_n| < (\pi^2 2^{n+4}/\mathrm{agm}^2)\exp(-\pi 2^{n+1}).$$

The number of valid decimal places is then

THEOREM 2c.

$$-\log_{10}|\pi - \pi_n| > (\pi/\log 10)2^{n+1} - n\log_{10}2 - 2\log_{10}(4\pi/\mathrm{agm}).$$

8. Numerical Computation. Raphael Finkel, Leo Guibas and Charles Simonyi are currently engaged in calculating π using the method proposed in this paper [2]. The operations of multiprecision division and square root are reduced to multiplication using a Newton's method iteration. The multiplications are then performed by the Schönhage-Strassen fast Fourier transform algorithm [10], [8, p. 274]. The computation, to be run on the Illiac IV computer, is expected to yield 33 million bits of π in an estimated run time of four hours. This run time is determined by disc input-output, and the actual computation is estimated to be only a couple of minutes. Alas, they do not plan to convert to decimal.

9. Concluding Remarks. The main result of this paper, Theorem 1a, directly follows from Gauss' method for calculating elliptic integrals, Eqs. (5) and (6), which was known in 1818 [3, pp. 352, 360], and from Legendre's elliptic integral relation, Eq. (7), which was known in 1811 [9, p. 61]. It is quite surprising that such an easily derived formula for π has apparently been overlooked for 155 years. The author made his discovery in December of 1973.

The series summation which was used to simplify Eq. (17) was also discovered by Gauss [3, p. 377]. An interesting consequence of this result of Gauss is that e^π can be expressed as a rapidly convergent infinite product. If $a_0 = 1$, $b_0 = 2^{-1/2}$, then

$$e^\pi = 32\prod_{j=0}^{\infty}\left(\frac{a_{j+1}}{a_j}\right)^{2^{-j+1}}$$

Charles Stark Draper Laboratory
Cambridge, Massachusetts 02139

1. M. ABRAMOWITZ & I. A. STEGUN (Editors), *Handbook of Mathematical Functions, With Formulas, Graphs and Mathematical Tables*, Nat. Bur. Standards, Appl. Math. Ser., no. 55, Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1966. MR 34 #8607.

2. R. FINKEL, L. GUIBAS & C. SIMONYI, private communication.

3. K. F. GAUSS, *Werke*, Bd. 3, Gottingen, 1866, pp. 331–403.

4. A. G. GREENHILL, *The Applications of Elliptic Functions*, Dover, New York, 1959. MR 22 #2724.

5. H. HANCOCK, *Elliptic Integrals*, Dover, New York, 1917.

6. H. JEFFREYS & B. S. JEFFREYS, *Methods of Mathematical Physics*, 3rd ed., Cambridge Univ. Press, London, 1962.

7. L. V. KING, *On the Direct Numerical Calculation of Elliptic Functions and Integrals*, Cambridge Univ. Press, London, 1924.

8. D. KNUTH, *The Art of Computer Programming*. Vol. 2: *Seminumerical Algorithms*, Addison-Wesley, Reading, Mass., 1969. MR 44 #3531.

9. A. M. LEGENDRE, *Exercices de calcul intégral*. Vol. 1, 1811.

10. A. SCHÖNHAGE & V. STRASSEN, "Schnelle Multiplikation grosser Zahlen," *Computing* (*Arch. Elektron. Rechnen*), v. 7, 1971, pp. 281–292. MR 45 #1431.

Fast Multiple-Precision Evaluation of Elementary Functions

RICHARD P. BRENT

Australian National University, Canberra, Australia

ABSTRACT. Let $f(x)$ be one of the usual elementary functions (exp, log, artan, sin, cosh, etc.), and let $M(n)$ be the number of single-precision operations required to multiply n-bit integers. It is shown that $f(x)$ can be evaluated, with relative error $O(2^{-n})$, in $O(M(n)\log(n))$ operations as $n \to \infty$, for any floating-point number x (with an n-bit fraction) in a suitable finite interval. From the Schönhage-Strassen bound on $M(n)$, it follows that an n-bit approximation to $f(x)$ may be evaluated in $O(n \log^2(n) \log\log(n))$ operations. Special cases include the evaluation of constants such as π, e, and e^π. The algorithms depend on the theory of elliptic integrals, using the arithmetic-geometric mean iteration and ascending Landen transformations.

KEY WORDS AND PHRASES: multiple-precision arithmetic, analytic complexity, arithmetic-geometric mean, computational complexity, elementary function, elliptic integral, evaluation of π, exponential, Landen transformation, logarithm, trigonometric function

CR CATEGORIES: 5.12, 5.15, 5.25

1. *Introduction*

We consider the number of operations required to evaluate the elementary functions $\exp(x)$, $\log(x)$,[1] $\operatorname{artan}(x)$, $\sin(x)$, etc., with relative error $O(2^{-n})$, for x in some interval $[a, b]$, and large n. Here, $[a, b]$ is a fixed, nontrivial interval on which the relevant elementary function is defined. The results hold for computations performed on a multitape Turing machine, but to simplify the exposition we assume that a standard serial computer with a random-access memory is used.

Let $M(x)$ be the number of operations required to multiply two integers in the range $[0, 2^{\lfloor x \rfloor})$. We assume the number representation is such that addition can be performed in $O(M(n))$ operations, and that $M(n)$ satisfies the weak regularity condition

$$M(\alpha n) \le \beta M(n), \tag{1.1}$$

for some α and β in $(0, 1)$, and all sufficiently large n. Similar, but stronger, conditions are usually assumed, either explicitly [11] or implicitly [15]. Our assumptions are certainly valid if the Schönhage-Strassen method [15, 19] is used to multiply n-bit integers (in the usual binary representation) in $O(n \log(n) \log\log(n))$ operations.

The elementary function evaluations may be performed entirely in fixed point, using integer arithmetic and some implicit scaling scheme. However, it is more convenient to assume that floating-point computation is used. For example, a sign and magnitude representation could be used, with a fixed length binary exponent and an n-bit binary fraction. Our results are independent of the particular floating-point number system used, so long as the following conditions are satisfied.

Author's address: Computer Centre, Australian National University, Box 4, Canberra, ACT 2600, Australia.

[1] $\operatorname{Log}(x)$ denotes the natural logarithm.

Journal of the Association for Computing Machinery, Vol. 23, No. 2, April 1976, pp. 242-251.

1. Real numbers which are not too large or small can be approximated by floating-point numbers, with a relative error $O(2^{-n})$.
2. Floating-point addition and multiplication can be performed in $O(M(n))$ operations, with a relative error $O(2^{-n})$ in the result.
3. The precision n is variable, and a floating-point number with precision n may be approximated, with relative error $O(2^{-m})$ and in $O(M(n))$ operations, by a floating-point number with precision m, for any positive $m < n$.

Throughout this paper, a *floating-point number* means a number in some representation satisfying conditions 1 to 3 above, not a single-precision number. We say that an operation is performed with *precision* n if the result is obtained with a relative error $O(2^{-n})$. It is assumed that the operands and result are approximated by floating-point numbers.

The main result of this paper, established in Sections 6 and 7, is that all the usual elementary functions may be evaluated, with precision n, in $O(M(n)\log(n))$ operations. Note that $O(M(n)n)$ operations are required if the Taylor series for $\log(1 + x)$ is summed in the obvious way. Our result improves the bound $O(M(n)\log^2(n))$ given in [4], although the algorithms described there may be faster for small n.

Preliminary results are given in Sections 2 to 5. In Section 2 we give, for completeness, the known result that division and extraction of square roots to precision n require $O(M(n))$ operations. Section 3 deals briefly with methods for approximating simple zeros of nonlinear equations to precision n, and some results from the theory of elliptic integrals are summarized in Section 4. Since our algorithms for elementary functions require a knowledge of π to precision n, we show, in Section 5, how this may be obtained in $O(M(n)\log(n))$ operations. An amusing consequence of the results of Section 6 is that e^{π} may also be evaluated, to precision n, in $O(M(n)\log(n))$ operations.

From [4, Th. 5.1], at least $O(M(n))$ operations are required to evaluate $\exp(x)$ or $\sin(x)$ to precision n. It is plausible to conjecture that $O(M(n)\log(n))$ operations are necessary.

Most of this paper is concerned with order of magnitude results, and multiplicative constants are ignored. In Section 8, though, we give upper bounds on the constants. From these bounds it is possible to estimate how large n needs to be before our algorithms are faster than the conventional ones.

After this paper was submitted for publication, Bill Gosper drew my attention to Salamin's paper [18], where an algorithm very similar to our algorithm for evaluating π is described. A fast algorithm for evaluating $\log(x)$ was also found independently by Salamin (see [2 or 5]).

Apparently similar algorithms for evaluating elementary functions are given by Borchardt [3], Carlson [8, 9], and Thacher [23]. However, these algorithms require $O(M(n)n)$ or $O(M(n)n^{\frac{1}{2}})$ operations, so our algorithms are asymptotically faster.

We know how to evaluate certain other constants and functions almost as fast as elementary functions. For example, Euler's constant $\gamma = 0.5772\ldots$ can be evaluated with $O(M(n)\log^2 n)$ operations, using Sweeney's method [22] combined with binary splitting [4]. Similarly for $\Gamma(a)$, where a is rational (or even algebraic): see Brent [7]. Related results are given by Gosper [13] and Schroeppel [20]. It is not known whether any of these upper bounds are asymptotically the best possible.

2. *Reciprocals and Square Roots*

In this section we show that reciprocals and square roots of floating-point numbers may be evaluated, to precision n, in $O(M(n))$ operations. To simplify the statement of the following lemma, we assume that $M(x) = 0$ for all $x < 1$.

LEMMA 2.1. *If* $\gamma \in (0, 1)$, *then* $\sum_{j=0}^{\infty} M(\gamma^j n) = O(M(n))$ *as* $n \to \infty$.

PROOF. If α and β are as in (1.1), there exists k such that $\gamma^k \le \alpha$. Thus, $\sum_{j=0}^{\infty} M(\gamma^j n) \le k\sum_{j=0}^{\infty} M(\alpha^j n) \le kM(n)/(1 - \beta) + O(1)$, by repeated application of (1.1). Since $M(n) \to \infty$ as $n \to \infty$, the result follows.

In the following lemma, we assume that $1/c$ is in the allowable range for floating-point numbers. Similar assumptions are implicit below.

LEMMA 2.2. *If c is a nonzero floating-point number, then $1/c$ can be evaluated, to precision n, in $O(M(n))$ operations.*

PROOF. The Newton iteration

$$x_{i+1} = x_i(2 - cx_i) \tag{2.1}$$

converges to $1/c$ with order 2. In fact, if $x_i = (1 - \epsilon_i)/c$, substitution in (2.1) gives $\epsilon_{i+1} = \epsilon_i^2$. Thus, assuming $|\epsilon_0| < \frac{1}{2}$, we have $|\epsilon_i| < 2^{-2^i}$ for all $i \geq 0$, and x_k is a sufficiently good approximation to $1/c$ if $k \geq \log_2 n$. This assumes that (2.1) is satisfied exactly, but it is easy to show that it is sufficient to use precision n at the last iteration ($i = k - 1$), precision slightly greater than $n/2$ for $i = k - 2$, etc. (Details, and more efficient methods, are given in [4, 6].) Thus the result follows from Lemma 2.1. Since $x/y = x(1/y)$, it is clear that floating-point division may also be done in $O(M(n))$ operations.

LEMMA 2.3. *If $c \geq 0$ is a floating-point number, then $c^{\frac{1}{2}}$ can be evaluated, to precision n, in $O(M(n))$ operations.*

PROOF. If $c = 0$ then $c^{\frac{1}{2}} = 0$. If $c \neq 0$, the proof is similar to that of Lemma 2.2, using the Newton iteration $x_{i+1} = (x_i + c/x_i)/2$.

LEMMA 2.4. *For any fixed $k > 0$, $M(kn) = O(M(n))$ as $n \to \infty$.*

PROOF. Since we can add integers less than 2^n in $O(M(n))$ operations, we can add integers less than 2^{kn} in $O(kM(n)) = O(M(n))$ operations. The multiplication of integers less than 2^{kn} can be split into $O(k^2)$ multiplications of integers less than 2^n, and $O(k^2)$ additions, so it can be done in $O(k^2M(n)) = O(M(n))$ operations.

3. *Solution of Nonlinear Equations*

In Section 6 we need to solve nonlinear equations to precision n. The following lemma is sufficient for this application. Stronger results are given in [4, 6].

LEMMA 3.1. *If the equation $f(x) = c$ has a simple root $\zeta \neq 0$, f' is Lipschitz continuous near ζ, and we can evaluate $f(x)$ to precision n in $O(M(n)\phi(n))$ operations, where $\phi(n)$ is a positive, monotonic increasing function, for x near ζ, then ζ can be evaluated to precision n in $O(M(n)\phi(n))$ operations.*

PROOF. Consider the discrete Newton iteration

$$x_{i+1} = x_i - h_i(f(x_i) - c)/(f(x_i + h_i) - f(x_i)). \tag{3.1}$$

If $h_i = 2^{-n/2}$, $x_i - \zeta = O(2^{-n/2})$, and the right side of (3.1) is evaluated with precision n, then a standard analysis shows that $x_{i+1} - \zeta = O(2^{-n})$. Since a sufficiently good starting approximation x_0 may be found in $O(1)$ operations, the result follows in the same way as in the proof of Lemma 2.2, using the fact that Lemma 2.1 holds with $M(n)$ replaced by $M(n)\phi(n)$. The assumption $\zeta \neq 0$ is only necessary because we want to obtain ζ with a relative (not absolute) error $O(2^{-n})$.

Other methods, e.g. the secant method, may also be used if the precision is increased appropriately at each iteration. In our applications there is no difficulty in finding a suitable initial approximation x_0 (see Section 6).

4. *Results on Elliptic Integrals*

In this section we summarize some classical results from elliptic integral theory. Most of the results may be found in [1], so proofs are omitted. Elliptic integrals of the first and second kind are defined by

$$F(\psi, \alpha) = \int_0^{\psi} (1 - \sin^2\alpha \sin^2\theta)^{-\frac{1}{2}} d\theta \tag{4.1}$$

and

$$E(\psi, \alpha) = \int_0^{\psi} (1 - sin^2\alpha \; sin^2\theta)^{\frac{1}{2}} d\theta, \tag{4.2}$$

respectively. For our purposes we may assume that α and ψ are in $[0, \pi/2]$. The complete elliptic integrals, $F(\pi/2, \alpha)$ and $E(\pi/2, \alpha)$, are simply written as $F(\alpha)$ and $E(\alpha)$, respectively.

Legendre's Relation. We need the identity of Legendre [17]:

$$E(\alpha)F(\pi/2 - \alpha) + E(\pi/2 - \alpha)F(\alpha) - F(\alpha)F(\pi/2 - \alpha) = \pi/2, \tag{4.3}$$

and, in particular, the special case

$$2E(\pi/4)F(\pi/4) - (F(\pi/4))^2 = \pi/2. \tag{4.4}$$

Small Angle Approximation. From (4.1) it is clear that

$$F(\psi, \alpha) = \psi + O(\alpha^2) \tag{4.5}$$

as $\alpha \rightarrow 0$.

Large Angle Approximation. From (4.1),

$$F(\psi, \alpha) = F(\psi, \pi/2) + O(\pi/2 - \alpha)^2, \tag{4.6}$$

uniformly for $0 \leq \psi \leq \psi_0 < \pi/2$, as $\alpha \rightarrow \pi/2$. Also, we note that

$$F(\psi, \pi/2) = \log \tan(\pi/4 + \psi/2). \tag{4.7}$$

Ascending Landen Transformation. If $0 < \alpha_i < \alpha_{i+1} < \pi/2$, $0 < \psi_{i+1} < \psi_i \leq \pi/2$,

$$\sin \alpha_i = \tan^2(\alpha_{i+1}/2), \tag{4.8}$$

and

$$\sin(2\psi_{i+1} - \psi_i) = \sin \alpha_i \sin \psi_i , \tag{4.9}$$

then

$$F(\psi_{i+1}, \alpha_{i+1}) = [(1 + \sin \alpha_i)/2]F(\psi_i , \alpha_i). \tag{4.10}$$

If $s_i = \sin \alpha_i$ and $v_i = \tan(\psi_i/2)$, then (4.8) gives

$$s_{i+1} = 2s^{\frac{1}{2}}/(1 + s_i), \tag{4.11}$$

and (4.9) gives

$$v_{i+1} = w_3/(1 + (1 + w_3^2)^{\frac{1}{2}}), \tag{4.12}$$

where

$$w_3 = \tan \psi_{i+1} = (v_i + w_2)/(1 - v_i w_2), \tag{4.13}$$

$$w_2 = \tan(\psi_{i+1} - \psi_i/2) = w_1/(1 + (1 - w_1^2)^{\frac{1}{2}}), \tag{4.14}$$

and

$$w_1 = \sin(2\psi_{i+1} - \psi_i) = 2s_i v_i/(1 + v_i^2). \tag{4.15}$$

Arithmetic-Geometric Mean Iteration. From the ascending Landen transformation it is possible to derive the arithmetic-geometric mean iteration of Gauss [12] and Lagrange [16]: if $a_0 = 1$, $b_0 = \cos \alpha > 0$,

$$a_{i+1} = (a_i + b_i)/2, \tag{4.16}$$

and

$$b_{i+1} = (a_i b_i)^{\frac{1}{2}}, \tag{4.17}$$

then

$$\lim_{i\to\infty} a_i = \pi/[2F(\alpha)]. \tag{4.18}$$

Also, if $c_0 = \sin \alpha$ and

$$c_{i+1} = a_i - a_{i+1}, \tag{4.19}$$

then

$$E(\alpha)/F(\alpha) = 1 - \sum_{i=0}^{\infty} 2^{i-1} c_i^2. \tag{4.20}$$

An Infinite Product. Let s_i, a_i, and b_i be as above, with $\alpha = \pi/2 - \alpha_0$, so $s_0 = b_0/a_0$. From (4.11), (4.16), and (4.17), it follows that $s_i = b_i/a_i$ for all $i \geq 0$. Thus, $(1 + s_i)/2 = a_{i+1}/a_i$, and

$$\prod_{i=0}^{\infty} [(1 + s_i)/2] = \lim_{i\to\infty} a_i = \pi/[2F(\pi/2 - \alpha_0)] \tag{4.21}$$

follows from (4.18). (Another connection between (4.11) and the arithmetic-geometric mean iteration is evident if $s_0 = (1 - b_0^2/a_0^2)^{\frac{1}{2}}$. Assuming (4.11) holds for $i < 0$, it follows that $s_{-i} = (1 - b_i^2/a_i^2)^{\frac{1}{2}}$ for all $i \geq 0$. This may be used to deduce (4.18) from (4.10).)

5. *Evaluation of π*

Let $a_0 = 1$, $b_0 = c_0 = 2^{-\frac{1}{2}}$, $A = \lim_{i\to\infty} a_i$, and $T = \lim_{i\to\infty} t_i$, where a_i, b_i, and c_i are defined by (4.16), (4.17), and (4.19) for $i \geq 1$, and $t_i = \frac{1}{2} - \sum_{j=0}^{i} 2^{j-1} c_j^2$. From (4.4), (4.18), and (4.20), we have

$$\pi = A^2/T. \tag{5.1}$$

Since $a_i > b_0 > 0$ for all $i \geq 0$, and $c_{i+1} = a_i - a_{i+1} = a_{i+1} - b_i$, (4.17) gives $b_{i+1} = [(a_{i+1} + c_{i+1})(a_{i+1} - c_{i+1})]^{\frac{1}{2}} = a_{i+1} - O(c_{i+1}^2)$, so $c_{i+2} = O(c_{i+1}^2)$. Thus, the process converges with order at least 2, and $\log_2 n + O(1)$ iterations suffice to give an error $O(2^{-n})$ in the estimate of (5.1). A more detailed analysis shows that $a_{i+1}^2/t_i < \pi < a_i^2/t_i$ for all $i \geq 0$, and also $a_i^2/t_i - \pi \sim 8\pi \exp(-2^i\pi)$ and $\pi - a_{i+1}^2/t_i \sim \pi^2 2^{i+4} \exp(-2^{i+1}\pi)$ as $i \to \infty$. The speed of convergence is illustrated in Table I.

From the discussion above, it is clear that the following algorithm, given in pseudo-Algol, evaluates π to precision n.

Algorithm for π

```
A ← 1; B ← 2^(-1/2); T ← 1/4; X ← 1;
while A − B > 2^(−n) do
   begin Y ← A; A ← (A + B)/2; B ← (BY)^(1/2);
         T ← T − X(A − Y)^2; X ← 2X
   end;
return A^2/T [or, better, (A + B)^2/(4T)].
```

TABLE I. CONVERGENCE OF APPROXIMATIONS TO π

i	$\pi - a_{i+1}^2/t_i$	$a_i^2/t_i - \pi$
0	2.3′−1	8.6′−1
1	1.0′−3	4.6′−2
2	7.4′−9	8.8′−5
3	1.8′−19	3.1′−10
4	5.5′−41	3.7′−21

Since $\log_2 n + O(1)$ iterations are needed, it is necessary to work with precision $n + O(\log\log(n))$, even though the algorithm is numerically stable in the conventional sense. From Lemmas 2.2–2.4, each iteration requires $O(M(n))$ operations, so π may be evaluated to precision n in $O(M(n)\log(n))$ operations. This is asymptotically faster than the usual $O(n^2)$ methods [14, 21] if a fast multiplication algorithm is used. A high-precision computation of π by a similar algorithm is described in [10]. Note that, because the arithmetic-geometric mean iteration is not self-correcting, we cannot obtain a bound $O(M(n))$ in the same way as for the evaluation of reciprocals and square roots by Newton's method.

6. *Evaluation of exp(x) and log(x)*

Suppose $\delta > 0$ fixed, and $m \in [\delta, 1 - \delta]$. If $\sin \alpha_0 = m^{1/2}$, we may evaluate $F(\alpha_0)$ to precision n in $O(M(n)\log(n))$ operations, using (4.18) and the arithmetic-geometric mean iteration, as for the special case $F(\pi/4)$ described in Section 5. (When using (4.18) we need π, which may be evaluated as described above.) Applying the ascending Landen transformation (4.8)–(4.10) with $i = 0, 1, \cdots, k - 1$ and $\psi_0 = \pi/2$ gives

$$F(\psi_k, \alpha_k) = \left\{\sum_{i=0}^{k-1} [(1 + \sin\alpha_i)/2]\right\} F(\alpha_0). \tag{6.1}$$

Since $s_0 = \sin \alpha_0 = m^{1/2} \geq \delta^{1/2} > 0$, it follows from (4.11) that $s_i \to 1$ as $i \to \infty$. In fact, if $s_i = 1 - \epsilon_i$, then $\epsilon_{i+1} = 1 - s_{i+1} = 1 - 2(1 - \epsilon_i)^{1/2}/(2 - \epsilon_i) = \epsilon_i^2/8 + O(\epsilon_i^3)$, so $s_i \to 1$ with order 2. Thus, after $k \sim \log_2 n$ iterations we have $\epsilon_k = O(2^{-n})$, so $\pi/2 - \alpha_k = O(2^{-n/2})$ and, from (4.6) and (4.7),

$$F(\psi_k, \alpha_k) = \log\tan(\pi/4 + \psi_k/2) + O(2^{-n}). \tag{6.2}$$

Assuming $k > 0$, the error is uniformly $O(2^{-n})$ for all $m \in [\delta, 1 - \delta]$, since $\psi_k \leq \psi_1 < \pi/2$.

Define the functions

$$U(m) = \left\{\sum_{i=0}^{\infty} [(1 + \sin\alpha_i)/2]\right\} F(\alpha_0) \tag{6.3}$$

and

$$T(m) = \tan(\pi/4 + \psi_\infty/2), \tag{6.4}$$

where $\psi_\infty = \lim_{i\to\infty} \psi_i$. Since $s_i \to 1$ with order 2, the infinite product in (6.3) is convergent, and $U(m)$ is analytic for all $m \in (0, 1)$. Taking the limit in (6.1) and (6.2) as n (and hence k) tends to ∞, we have the fundamental identity

$$U(m) = \log T(m). \tag{6.5}$$

Using (4.11)–(4.15), we can evaluate $U(m) = \{\prod_{i=0}^{k-1} [(1 + s_i)/2]\} F(\alpha_0) + O(2^{-n})$ and $T(m) = (1 + v_k)/(1 - v_k) + O(2^{-n})$, to precision n, in $O(M(n)\log(n))$ operations. The algorithms are given below in pseudo-Algol.

Algorithm for U(m)

```
A ← 1; B ← (1 − m)^½;
while A − B > 2^(−n/2) do
    begin C ← (A + B)/2; B ← (AB)^½; A ← C end;
A ← π/(A + B); S ← m^½;
while 1 − S > 2^(−n/2) do
    begin A ← A(1 + S)/2; S ← 2S^½/(1 + S) end;
return A(1 + S)/2.
```

Algorithm for T(m)

```
V ← 1; S ← m^½;
while 1 − S > 2^(−n) do
```

```
    begin W ← 2SV/(1 + V²);
        W ← W/(1 + (1 − W²)^½);
        W ← (V + W)/(1 − VW);
        V ← W/(1 + (1 + W²)^½);
        S ← 2S^½/(1 + S)
    end;
return (1 + V)/(1 − V).
```

Properties of $U(m)$ and $T(m)$. From (4.21) and (6.3),

$$U(m) = (\pi/2)F(\alpha_0)/F(\pi/2 - \alpha_0), \tag{6.6}$$

where $\sin \alpha_0 = m^{1/2}$ as before. Both $F(\alpha_0)$ and $F(\pi/2 - \alpha_0)$ may be evaluated by the arithmetic-geometric mean iteration, which leads to a slightly more efficient algorithm for $U(m)$ than the one above, because the division by $(1 + S)$ in the final "while" loop is avoided. From (6.5) and (6.6), we have the special cases $U(\frac{1}{2}) = \pi/2$ and $T(\frac{1}{2}) = e^{\pi/2}$. Also, (6.6) gives

$$U(m)U(1 - m) = \pi^2/4, \tag{6.7}$$

for all $m \in (0, 1)$.

Although we shall avoid using values of m near 0 or 1, it is interesting to obtain asymptotic expressions for $U(m)$ and $T(m)$ as $m \to 0$ or 1. From the algorithm for $T(m)$, $T(1 - \epsilon) = 4\epsilon^{-1/2} - \epsilon^{1/2} + O(\epsilon^{3/2})$ as $\epsilon \to 0$. Thus, from (6.5), $U(1 - \epsilon) = L(\epsilon) - \epsilon/4 + O(\epsilon^2)$, where $L(\epsilon) = \log(4/\epsilon^{1/2})$. Using (6.7), this gives $U(\epsilon) = \pi^2/[4L(\epsilon)] + O(\epsilon/L^2)$, and hence $T(\epsilon) = \exp(\pi^2/[4L(\epsilon)]) + O(\epsilon/L^2)$. Some values of $U(m)$ and $T(m)$ are given in Table II.

Evaluation of $\exp(x)$. To evaluate $\exp(x)$ to precision n, we first use identities such as $\exp(2x) = (\exp(x))^2$ and $\exp(-x) = 1/\exp(x)$ to reduce the argument to a suitable domain, say $1 \le x \le 2$ (see below). We then solve the nonlinear equation

$$U(m) = x, \tag{6.8}$$

obtaining m to precision n, by a method such as the one described in Section 3. From Lemma 3.1, with $\phi(n) = \log(n)$, this may be done in $O(M(n)\log(n))$ operations. Finally, we evaluate $T(m)$ to precision n, again using $O(M(n)\log(n))$ operations. From (6.5) and (6.8), $T(m) = \exp(x)$, so we have computed $\exp(x)$ to precision n. Any preliminary transformations may now be undone.

Evaluation of $\log(x)$. Since we can evaluate $\exp(x)$ to precision n in $O(M(n)\log(n))$ operations, Lemma 3.1 shows that we can also evaluate $\log(x)$ in $O(M(n)\log(n))$ operations, by solving the equation $\exp(y) = x$ to the desired accuracy. A more direct method is to solve $T(m) = x$ (after suitable domain reduction), and then evaluate $U(m)$.

Further details. If $x \in [1, 2]$ then the solution m of (6.8) lies in (0.10, 0.75), and it may be verified that the secant method, applied to (6.8), converges if the starting approximations are $m_0 = 0.2$ and $m_1 = 0.7$. If desired, the discrete Newton method or some other locally convergent method may be used after a few iterations of the secant method have given a good approximation to m.

TABLE II. THE FUNCTIONS $U(m)$ AND $T(m)$

m	$U(m)$	$T(m)$	m	$U(m)$	$T(m)$
0.01	0.6693	1.9529	0.60	1.7228	5.6004
0.05	0.8593	2.3615	0.70	1.9021	6.6999
0.10	0.9824	2.6710	0.80	2.1364	8.4688
0.20	1.1549	3.1738	0.90	2.5115	12.3235
0.30	1.2972	3.6591	0.95	2.8714	17.6617
0.40	1.4322	4.1878	0.99	3.6864	39.8997
0.50	1.5708	4.8105			

Similarly, if $x \in [3, 9]$, the solution of $T(m) = x$ lies in $(0.16, 0.83)$, and the secant method converges if $m_0 = 0.2$ and $m_1 = 0.8$.

If $x = 1 + \epsilon$ where ϵ is small, and for domain reduction the relation

$$\log(x) = \log(\lambda x) - \log(\lambda) \tag{6.9}$$

is used, for some $\lambda \in (3, 9)$, then $\log(\lambda x)$ and $\log(\lambda)$ may be evaluated as above, but cancellation in (6.9) will cause some loss of precision in the computed value of $\log(x)$. If $|\epsilon| > 2^{-n}$, it is sufficient to evaluate $\log(\lambda x)$ and $\log(\lambda)$ to precision $2n$, for at most n bits are lost through cancellation in (6.9). On the other hand, there is no difficulty if $|\epsilon| \le 2^{-n}$, for then $\log(1 + \epsilon) = \epsilon(1 + O(2^{-n}))$. When evaluating $\exp(x)$, a similar loss of precision never occurs, and it is sufficient to work with precision $n + O(\log \log(n))$, as in the evaluation of π (see Section 5). To summarize, we have proved:

THEOREM 6.1. *If $-\infty < a < b < \infty$, then $O(M(n) \log(n))$ operations suffice to evaluate $\exp(x)$ to precision n, uniformly for all floating-point numbers $x \in [a, b]$, as $n \to \infty$; and similarly for $\log(x)$ if $a > 0$.*

7. *Evaluation of Trigonometric Functions*

Suppose $\delta > 0$ fixed, and $x \in [\delta, 1]$. Let $s_0 = \sin \alpha_0 = 2^{-n/2}$ and $v_0 = \tan(\psi_0/2) = x/(1 + (1 + x^2)^{\frac{1}{2}})$, so $\tan \psi_0 = x$. Applying the ascending Landen transformation, as for (6.1), gives

$$F(\psi_k, \alpha_k) = \left\{\prod_{i=0}^{k-1} [(1 + s_i)/2]\right\} F(\psi_0, \alpha_0). \tag{7.1}$$

Also, from (4.5) and the choice of s_0,

$$F(\psi_0, \alpha_0) = \operatorname{artan}(x) + O(2^{-n}). \tag{7.2}$$

From (4.11), $s_{i+1} \ge s_i^{\frac{1}{2}}$, so there is some $j \le \log_2 n + O(1)$ such that $s_j \in [\frac{1}{4}, \frac{4}{5}]$. Since $s_i \to 1$ with order 2, there is some $k \le 2 \log_2 n + O(1)$ such that $1 - s_k = O(2^{-n})$. From (4.6) and (4.7), $F(\psi_k, \alpha_k) = \log \tan(\pi/4 + \psi_k/2) + O(2^{-n})$. Thus, from (7.1) and (7.2),

$$\operatorname{artan}(x) = \left\{\prod_{i=0}^{k-1} [2/(1 + s_i)]\right\} \log \tan(\pi/4 + \psi_k/2) + O(2^{-n}). \tag{7.3}$$

If we evaluate $\tan(\pi/4 + \psi_k/2)$ as above, and use the algorithm of Section 6 to evaluate the logarithm in (7.3), we have $\operatorname{artan}(x)$ to precision n in $O(M(n) \log(n))$ operations. The algorithm may be written as follows.

Algorithm for $\operatorname{artan}(x)$, $x \in [\delta, 1]$

```
S ← 2^(-n/2); V ← x/(1 + (1 + x²)^½); Q ← 1;
while 1 − S > 2^(-n) do
    begin Q ← 2Q/(1 + S);
        W ← 2SV/(1 + V²);
        W ← W/(1 + (1 − W²)^½);
        W ← (V + W)/(1 − VW);
        V ← W/(1 + (1 + W²)^½);
        S ← 2S^½/(1 +S)
    end;
return Q log((1 + V)/(1 − V)).
```

After k iterations, $Q \le 2^k$, so at most $2 \log_2 n + O(1)$ bits of precision are lost because V is small. Thus it is sufficient to work with precision $n + O(\log(n))$, and Lemma 2.4 justifies our claim that $O(M(n) \log(n))$ operations are sufficient to obtain $\operatorname{artan}(x)$ to precision n.

If x is small, we may use the same idea as that described above for evaluating $\log(1 + \epsilon)$: work with precision $3n/2 + O(\log(n))$ if $x > 2^{-n/2}$, and use $\operatorname{artan}(x)$

$= x(1 + O(2^{-n}))$ if $0 \leq x \leq 2^{-n/2}$. (Actually, it is not necessary to increase the working precision if $\log((1 + V)/(1 - V))$ is evaluated carefully.)

Using the identity $\mathrm{artan}(x) = \pi/2 - \mathrm{artan}(1/x)$ $(x > 0)$, we can extend the domain to $[0, \infty)$. Also, since $\mathrm{artan}(-x) = -\mathrm{artan}(x)$, there is no difficulty with negative x. To summarize, we have proved the following theorem.

THEOREM 7.1. *$O(M(n) \log(n))$ operations suffice to evaluate $\mathrm{artan}(x)$ to precision n, uniformly for all floating-point numbers x, as $n \to \infty$.*

Suppose $\theta \in [\delta, \pi/2 - \delta]$. From Lemma 3.1 and Theorem 7.1, we can solve the equation $\mathrm{artan}(x) = \theta/2$ to precision n in $O(M(n) \log(n))$ operations, and thus evaluate $x = \tan(\theta/2)$. Now $\sin\theta = 2x/(1 + x^2)$ and $\cos\theta = (1 - x^2)/(1 + x^2)$ may easily be evaluated. For arguments outside $[\delta, \pi/2 - \delta]$, domain reduction techniques like those above may be used. Difficulties occur near certain integer multiples of $\pi/2$, but these may be overcome (at least for the usual floating-point number representations) by increasing the working precision. We state the following theorem for $\sin(x)$, but similar results hold for the other trigonometric functions (and also, of course, for the elliptic integrals and their inverse functions).

THEOREM 7.2. *If $[a, b] \subseteq (-\pi, \pi)$, then $O(M(n) \log(n))$ operations suffice to evaluate $\sin(x)$ to precision n, uniformly for all floating-point numbers $x \in [a, b]$, as $n \to \infty$.*

8. *Asymptotic Constants*

So far we have been concerned with order of magnitude results. In this section we give upper bounds on the constants K such that $w(n) \leq (K + o(1))M(n) \log_2 n$, where $w(n)$ is the number of operations required to evaluate π, $\exp(x)$, etc., to precision n. The following two assumptions will be made.

1. For all $\gamma > 0$ and $\epsilon > 0$, the inequality $M(\gamma n) \leq (\gamma + \epsilon)M(n)$ holds for sufficiently large n.

2. The number of operations required for floating-point addition, conversion between representations of different precision (at most n), and multiplication or division of floating-point numbers by small integers is $o(M(n))$ as $n \to \infty$.

These assumptions certainly hold if a standard floating-point representation is used and $M(n) \sim n (\log(n))^\alpha (\log\log(n))^\beta$ for some $\alpha \geq 0$, provided $\beta > 0$ if $\alpha = 0$.

The following result is proved in [4]. The algorithms used are similar to those of Section 2, but slightly more efficient.

THEOREM 8.1. *Precision-n division of floating-point numbers may be performed in $(4 + o(1))M(n)$ operations as $n \to \infty$, and square roots may be evaluated in $(11/2 + o(1))M(n)$ operations.*

Using Theorem 8.1 and algorithms related to those of Sections 5–7, the following result is proved in [5].

THEOREM 8.2. *π may be evaluated to precision n in $(15/2 + o(1))M(n) \log_2 n$ operations as $n \to \infty$. If π and $\log 2$ are precomputed, the elementary function $f(x)$ can be evaluated to precision n in $(K + o(1))M(n) \log_2 n$ operations, where*

$$K = \begin{cases} 13 & \text{if } f(x) = \log(x) \text{ or } \exp(x), \\ 34 & \text{if } f(x) = \mathrm{artan}(x), \sin(x), \text{etc.,} \end{cases}$$

and x is a floating-point number in an interval on which $f(x)$ is defined and bounded away from 0 and ∞.

For purposes of comparison, note that evaluation of $\log(1 + x)$ or $\log((1 + x)/(1 - x))$ by the usual series expansion requires $(c + o(1))M(n)n$ operations, where c is a constant of order unity (depending on the range of x and the precise method used). Since $13 \log_2 n < n$ for $n \geq 83$, the $O(M(n) \log(n))$ method for $\log(x)$ should be faster than the $O(M(n)n)$ method for n greater than a few hundred.

ACKNOWLEDGMENTS. This paper was written while the author was visiting the Computer Science Department, Stanford University. The comments of R.W. Gosper, D.E. Knuth, and D. Shanks on various drafts were very useful.

REFERENCES

1. ABRAMOWITZ, M., AND STEGUN, I.A. Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards, Washington, D.C., 1964; Dover, 1965, Ch. 17.
2. BEELER, M., GOSPER, R.W., AND SCHROEPPEL, R. Hakmem. Memo No. 239, M.I.T. Artificial Intelligence Lab., M.I.T., Cambridge, Mass., 1972, pp. 70–71.
3. BORCHARDT, C.W. *Gesammelte Werke*. Berlin, 1888, pp. 455–462.
4. BRENT, R.P. The complexity of multiple-precision arithmetic. Proc. Seminar on Complexity of Computational Problem Solving (held at Australian National U., Dec. 1974), Queensland U. Press, Brisbane, Australia 1975, pp. 126–165.
5. BRENT, R.P. Multiple-precision zero-finding methods and the complexity of elementary function evaluation. Proc. Symp. on Analytic Computational Complexity, J.F. Traub, Ed., Academic Press, New York, 1976, pp. 151–176.
6. BRENT, R.P. *Computer Solution of Nonlinear Equations*. Academic Press, New York (to appear), Ch. 6.
7. BRENT, R.P. A Fortran multiple-precision arithmetic package. Submitted to a technical journal.
8. CARLSON, B.C. Algorithms involving arithmetic and geometric means. *Amer. Math. Monthly 78* (May 1971), 496–505.
9. CARLSON, B.C. An algorithm for computing logarithms and arctangents. *Math. Comput. 26* (April 1972), 543–549.
10. FINKEL, R., GUIBAS, L., AND SIMONYI, C. Manuscript in preparation.
11. FISCHER, M.J., AND STOCKMEYER, L.J. Fast on-line integer multiplication. *J. Comput. System Scis. 9* (Dec. 1974), 317–331.
12. GAUSS, C.F. *Carl Friedrich Gauss Werke, Bd. 3*. Göttingen, 1876, pp. 362–403.
13. GOSPER, R.W. Acceleration of series. Memo No. 304, M.I.T. Artificial Intelligence Lab., M.I.T., Cambridge, Mass., 1974.
14. GUILLOUD, J., AND BOUYER, M. 1,000,000 decimals de pi. Unpublished manuscript.
15. KNUTH, D.E. *The Art of Computer Programming, Vol. 2*. Addison-Wesley, Reading, Mass., 1969. Errata and addenda: Rep. CS 194, Computer Sci. Dep., Stanford U., Stanford, Calif., 1970.
16. LAGRANGE, J.L. *Oeuvres de Lagrange, Tome 2*. Gauthier-Villars, Paris, 1868, pp. 267-272.
17. LEGENDRE, A.M. *Exercices de Calcul Integral, Vol. 1*. Paris, 1811, p. 61.
18. SALAMIN, E. Computation of π using arithmetic-geometric mean. To appear in *Math. Comput.*
19 SCHÖNHAGE, A., AND STRASSEN, V. Schnelle Multiplikation grosser Zahlen. *Computing 7* (1971), 281–292.
20. SCHROEPPEL, R. Unpublished manuscript dated May 1975.
21. SHANKS, D., AND WRENCH, J.W. Calculation of π to 100,000 decimals. *Math. Comput. 16* (1962), 76–99.
22. SWEENEY, D.W. On the computation of Euler's constant. *Math. Comput. 17* (1963), 170–178.
23. THACHER, H.C. Iterated square root expansions for the inverse cosine and inverse hyperbolic cosine. *Math. Comput. 15* (1961), 399–403.

RECEIVED MARCH 1975; REVISED SEPTEMBER 1975

A NOTE ON THE IRRATIONALITY OF $\zeta(2)$ AND $\zeta(3)$

F. BEUKERS

1. *Introduction*

At the "Journées Arithmétiques" held at Marseille-Luminy in June 1978, R. Apéry confronted his audience with a miraculous proof for the irrationality of $\zeta(3) = 1^{-3}+2^{-3}+3^{-3}+\ldots$. The proof was elementary but the complexity and the unexpected nature of Apéry's formulas divided the audience into believers and disbelievers. Everything turned out to be correct however. Two months later a complete exposition of the proof was presented at the International Congress of Mathematicians in Helsinki in August 1978 by H. Cohen. This proof was based on the lecture of Apéry, but contained ideas of Cohen and Don Zagier. For a more extensive record of this little history I refer to A. J. van der Poorten [1]. Apéry's proof will be published in *Acta Arithmetica*.

In this note we give another proof for the irrationality of $\zeta(3)$ which is shorter and, I think, more elegant. This proof is achieved by means of double and triple integrals, the shape of which is motivated by Apéry's formulas. Like Apéry's proof it also works for $\zeta(2)$, which is of course already known to be transcendental since it equals $\pi^2/6$. Most of the integrals that appear in the proof are improper. The manipulations with these integrals can be justified if one replaces $\int_0^1$ by $\int_\varepsilon^{1-\varepsilon}$ and by letting ε tend to zero.

2. Throughout this paper we denote the lowest common multiple of $1, 2, \ldots, n$ by d_n. The value of d_n can be estimated by

$$d_n = \prod_{\substack{\text{Prime}\\ p\leqslant n}} p^{[\log n/\log p]} < \prod_{\substack{\text{Prime}\\ p\leqslant n}} p^{\log n/\log p} = n^{\pi(n)},$$

and the latter number is smaller than 3^n for sufficiently large n.

LEMMA 1. *Let r and s be non-negative integers. If $r > s$ then,*

(a) $$\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}\,dx\,dy$$

is a rational number whose denominator is a divisor of $d_r^{\,2}$.

(b) $$\int_0^1\int_0^1 -\frac{\log xy}{1-xy}\,x^r y^s\,dx\,dy$$

is a rational number whose denominator is a divisor of $d_r^{\,3}$.

Received 18 November, 1978.

[BULL. LONDON MATH. SOC., 11 (1979), 268–272]

If $r = s$, *then*

(c) $$\int_0^1\int_0^1 \frac{x^r y^r}{1-xy}\,dxdy = \zeta(2) - \frac{1}{1^2} - \ldots - \frac{1}{r^2},$$

(d) $$\int_0^1\int_0^1 -\frac{\log xy}{1-xy}\,x^r y^r\,dxdy = 2\left\{\zeta(3) - \frac{1}{1^3} - \ldots - \frac{1}{r^3}\right\}.$$

Remark. In case $r = 0$, we let the sums $1^{-2}+\ldots+r^{-2}$ and $1^{-3}+\ldots+r^{-3}$ vanish.

Proof. Let σ be any non-negative number. Consider the integral

$$\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy}\,dxdy. \tag{1}$$

Develop $(1-xy)^{-1}$ into a geometrical series and perform the double integration. Then we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)}. \tag{2}$$

Assume that $r > s$. Then we can write this sum as

$$\sum_{k=0}^{\infty} \frac{1}{r-s}\left\{\frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1}\right\} = \frac{1}{r-s}\left\{\frac{1}{s+1+\sigma} + \ldots + \frac{1}{r+\sigma}\right\}. \tag{3}$$

If we put $\sigma = 0$ then assertion (a) follows immediately. If we differentiate with respect to σ and put $\sigma = 0$, then integral (1) changes into

$$\int_0^1\int_0^1 \frac{\log xy}{1-xy}\,x^r y^s\,dxdy,$$

and summation (3) becomes

$$\frac{-1}{r-s}\left\{\frac{1}{(s+1)^2} + \ldots + \frac{1}{r^2}\right\}.$$

Assertion (b) now follows straight away.
Assume $r = s$, then by (1) and (2),

$$\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy}\,dxdy = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)^2}.$$

By putting $\sigma = 0$ assertion (c) becomes obvious. Differentiate with respect to σ and put $\sigma = 0$. Then we obtain

$$\int_0^1\int_0^1 \frac{\log xy}{1-xy}\,x^r y^r\,dxdy = \sum_{k=0}^{\infty} \frac{-2}{(k+r+1)^3},$$

which proves assertion (d).

THEOREM 1. *$\zeta(2)$ is irrational.*

Proof. For a positive integer n consider the integral

$$\int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\,dx\,dy, \tag{4}$$

where $P_n(x)$ is the Legendre-type polynomial given by $n!P_n(x) = \left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$.

Note that $P_n(x) \in \mathbb{Z}[x]$. In this proof we shall denote the double integration by the single sign $\int$. It is clear from Lemma 1 that integral (4) equals $(A_n+B_n\zeta(2))d_n^{-2}$ for some $A_n \in \mathbb{Z}$ and $B_n \in \mathbb{Z}$. After an n-fold partial integration with respect to x integral (4) changes into

$$(-1)^n \int \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}}\,dx\,dy. \tag{5}$$

It is a matter of straightforward computation to show that

$$\frac{y(1-y)x(1-x)}{1-xy} \leqslant \left\{\frac{\sqrt{5}-1}{2}\right\}^5 \quad \text{for all} \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1.$$

Hence integral (4) is bounded in absolute value by

$$\left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \int \frac{1}{1-xy}\,dx\,dy = \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2).$$

Since integral (5) is non-zero we have

$$0 < |A_n+B_n\zeta(2)|d_n^{-2} < \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2),$$

and hence

$$0 < |A_n+B_n\zeta(2)| < d_n^2 \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2) < 9^n \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2) < \left\{\frac{5}{6}\right\}^n$$

for sufficiently large n. This implies the irrationality of $\zeta(2)$, for if $\zeta(2)$ was rational the expression in modulus signs would be bounded below independently of n.

THEOREM 2. *$\zeta(3)$ is irrational.*

Proof. Consider the integral

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy} P_n(x)P_n(y)\,dx\,dy, \tag{6}$$

where $n!P_n(x) = \left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$. It is clear from Lemma 1 that integral (6) equals $(A_n+B_n\zeta(3))d_n^{-3}$ for some $A_n \in \mathbb{Z}$, $B_n \in \mathbb{Z}$. By noticing that

$$\frac{-\log xy}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}\,dz,$$

integral (6) can be written as

$$\int \frac{P_n(x)P_n(y)}{1-(1-xy)z}\,dxdydz,$$

where $\int$ denotes the triple integration. After an n-fold partial integration with respect to x our integral changes into

$$\int \frac{(xyz)^n(1-x)^n\,P_n(y)}{(1-(1-xy)z)^{n+1}}\,dxdydz.$$

Substitute

$$w = \frac{1-z}{1-(1-xy)z}.$$

We obtain

$$\int (1-x)^n(1-w)^n\,\frac{P_n(y)}{1-(1-xy)w}\,dxdydw.$$

After an n-fold partial integration with respect to y we obtain

$$\int \frac{x^n(1-x)^n\,y^n(1-y)^n\,w^n(1-w)^n}{(1-(1-xy)w)^{n+1}}\,dxdydw. \tag{7}$$

It is straightforward to verify that the maximum of

$$x(1-x)y(1-y)w(1-w)(1-(1-xy)w)^{-1}$$

occurs for $x = y$ and then that

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leqslant (\sqrt{2}-1)^4 \quad \text{for all} \quad 0 \leqslant x, y, w \leqslant 1.$$

Hence integral (6) is bounded above by

$$(\sqrt{2}-1)^{4n}\int \frac{1}{1-(1-xy)w}\,dxdydw = (\sqrt{2}-1)^{4n}\int_0^1\int_0^1 \frac{-\log xy}{1-xy}\,dxdy = 2(\sqrt{2}-1)^{4n}\,\zeta(3).$$

Since integral (7) is not zero we have

$$0 < |A_n + B_n \zeta(3)| d_n^{-3} < 2\zeta(3)(\sqrt{2}-1)^{4n},$$

and hence

$$0 < |A_n + B_n \zeta(3)| < 2\zeta(3) d_n^3 (\sqrt{2}-1)^{4n} < 2\zeta(3) 27^n (\sqrt{2}-1)^{4n} < \left(\frac{4}{5}\right)^n$$

for sufficiently large n, which implies the irrationality of $\zeta(3)$.

Reference

1. A. J. van der Poorten, "A proof that Euler missed . . . Apéry's proof of the irrationality of $\zeta(3)$". *To appear.*

Department of Mathematics,
University of Leiden,
Postbus 9512,
2300 RA Leiden,
Netherlands.

A Proof that Euler Missed ...

Apéry's Proof of the Irrationality of $\zeta(3)$

An Informal Report

Alfred van der Poorten

R. Apéry
Département de Mathematiques
et de Mécanique
Université de Caen

1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 "Sur l'irrationalité de $\zeta(3)$." Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

Exercise
Prove the following amazing claims:

① For all $a_1, a_2, \ldots$

$$\sum_{k=1}^{\infty} \frac{a_1 a_2 \ldots a_{k-1}}{(x+a_1)\ldots(x+a_k)} = \frac{1}{x}.$$

② $$\zeta(3) =: \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}. \tag{1}$$

③ Consider the recursion:

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}, \quad n \geq 2. \tag{2}$$

Let $\{b_n\}$ be the sequence defined by $b_0 = 1$, $b_1 = 5$, and $b_n = u_n$ for all n; then the b_n all are integers! Let $\{a_n\}$ be the sequence defined by $a_0 = 0$, $a_1 = 6$, and $a_n = u_n$ for all n; then the a_n are rational numbers with denominator dividing $2[1, 2, \ldots, n]^3$ (here $[1, 2, \ldots, n]$ is the lcm (lowest common multiple) of $1, \ldots, n$).

④ $a_n/b_n \to \zeta(3)$; indeed the convergence is so fast as to prove that $\zeta(3)$ cannot be rational. To be precise, for all integers p, q with q sufficiently large relative to $\epsilon > 0$,

$$\left|\zeta(3) - \frac{p}{q}\right| > \frac{1}{q^{\theta+\epsilon}}, \quad \theta = 13.41782\ldots.$$

Moreover, analogous claims were made for $\zeta(2)$:

Exercise *(continued)*

②' $$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} \tag{3}$$

③' Consider the recursion:

$$n^2 u_n - (n-1)^2 u_{n-2} = (11n^2 - 11n + 3)u_{n-1}, \quad n \geq 2. \tag{4}$$

Let $\{b_n'\}$ be the sequence defined by $b_0' = 1$, $b_1' = 3$ and the recursion; then the b_n' all are integers! Let $\{a_n'\}$ be the sequence defined by $a_0' = 0$, $a_1' = 5$ and the recursion; then the a_n' are rational numbers with denominator dividing $[1, 2, \ldots, n]^2$.

④' $a_n'/b_n' \to \zeta(2) = \pi^2/6$; indeed the convergence is so fast as to imply that for all integers p, q with q sufficiently large relative to $\epsilon > 0$

$$\left|\pi^2 - \frac{p}{q}\right| > \frac{1}{q^{\theta'+\epsilon}}, \quad \theta' = 11.85078\ldots.$$

I heard with some incredulity that, for one, Henri Cohen (Bordeaux, now Grenoble) believed that these claims might well be valid. Very much intrigued, I joined Hendrik Lenstra (Amsterdam) and Cohen in an evening's discussion in which Cohen explained and demonstrated most of the details of the proof. We came away convinced that Professeur Apéry had indeed found a quite miraculous and magnificent demonstration of the irrationality of $\zeta(3)$. But we remained unable to prove a critical step.

2. For the Nonexpert Reader

A number β is *irrational* if it is not of the form p_0/q_0: p_0, q_0 integers ($\in \mathbb{Z}$). A rational number b is characterised by the property that for $p, q \in \mathbb{Z}$ ($q > 0$) and $b \neq p/q$ there exists an integer q_0 (> 0, of course) such that

$$\left|b - \frac{p}{q}\right| \geq \frac{1}{qq_0}.$$

On the other hand for irrational β there are always infinitely many p/q (for instance, the *convergents* of the *continued fraction* expansion of β) such that

$$|\beta - \frac{p}{q}| < \frac{1}{q^2}.$$

Plainly this yields a criterion for irrationality.

If there is a $\delta > 0$ and a sequence $\{p_n/q_n\}$ of rational numbers such that $p_n/q_n \neq \beta$ and

$$|\beta - \frac{p_n}{q_n}| < \frac{1}{q_n^{1+\delta}} \quad n = 1, 2, \ldots$$

then β is irrational.

A successful application of the criterion may yield a measure of irrationality:

If $|\beta - (p_n/q_n)| < 1/q_n^{1+\delta}$, and the q_n are monotonic increasing with $q_n < q_{n-1}^{1+\kappa}$ (for n sufficiently large relative to $\kappa > 0$), then for all integers $p, q > 0$ sufficiently large relative to $\epsilon > 0$, $|\beta - (p/q)| > (1/q^{(1+\delta)(\delta-\kappa)+\mathrm{E}})$.

For example if the sequence $\{q_n\}$ increases geometrically we may take $\kappa > 0$ arbitrarily small so that $1 + (1/\delta)$ becomes an *irrationality degree* for β. To see the claim suppose that $|\beta - (p/q)| \leqslant 1/q^\tau$ and select n so that $q_{n-1}^{1+\delta} \leqslant q^\tau < q_n^{1+\delta}$. Then

$$\frac{1}{qq_n} \leqslant |\frac{p}{q} - \frac{p_n}{q_n}| \leqslant |\beta - \frac{p_n}{q_n}| + |\beta - \frac{p}{q}| \leqslant \frac{1}{q_n^{1+\delta}} + \frac{1}{q^\tau} < \frac{2}{q^\tau}.$$

Hence $\frac{1}{2} q^\tau \leqslant qq_n < qq_{n-1}^{1+\kappa} < q^{1+\tau(1+\kappa)/(1+\delta)}$ or $\tau < (1+\delta)/(\delta-\kappa) + \epsilon$ as claimed. This argument is effective (the "sufficiently large" requirements can be made explicit.)

It is well-known (the theorem of Thue–Siegel–Roth) that for β *algebraic* (a zero of a polynomial $a_0X^n + a_1X^{n-1} + \ldots + a_n, a_i \in \mathbb{Z}$) always: $|\beta - (p/q)| > 1/q^{2+\epsilon}$, for q sufficiently large relative to $\epsilon > 0$. So if β is *too well* approximable by rationals ($\delta > 1$ above) then β is *not* algebraic, but *transcendental.* Unfortunately only a set of measure zero of transcendental numbers can be detected in this way, whilst, since the set of algebraic numbers is countable, almost all numbers are transcendental.

It is notoriously difficult to prove that any given naturally occurring number is irrational, let alone transcendental. One may be fortunate: for example the usual series for e implies immediately (*easy exercise*) that e is irrational. In the case of the (Riemann) ζ-function: $\zeta(s) =: \sum_1^\infty n^{-s}$ (Re $s > 1$) there is the quite well-known fact that

$$\zeta(2k) =: \sum_1^\infty n^{-2k} = (-1)^{k-1} \frac{(2\pi)^{2k}}{2\cdot(2k)!} B_{2k},$$

$$k = 1, 2, \ldots \tag{5}$$

where the Bernoulli numbers, B_m, are rational ($\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945, \ldots$). There are some classical techniques[1] for detecting the irrationality of powers of π, but it is most useful to appeal to the theorem of Hermite–Lindemann (whereby e^α is transcendental for algebraic $\alpha \neq 0$) whence π is transcendental (because $e^{\pi i} = -1$) and so *a fortiori* its powers are irrational. So it has long been known that $\zeta(2k)$ is irrational, $k = 1, 2, \ldots$. On the other hand there are no useful analogous closed evaluations of ζ at odd arguments.[2] Incidentally, (5) is demonstrated quite easily.

The Bernoulli numbers are defined by the generating function (a nontrivial example of an even function!)

$$\frac{z}{e^z - 1} + \frac{1}{2} z = \sum_{m=0}^\infty \frac{B_{2m}}{(2m)!} z^{2m},$$

hence by the recursion

$$\binom{n+1}{0}B_0 + \binom{n+1}{1} B_1 + \ldots + \binom{n+1}{n}B_n = 0,$$

$$B_0 = 1, \quad B_1 = -\frac{1}{2}; \quad n = 2, 3, \ldots.$$

On the other hand it is well-known that

$$\sin \pi z = \pi z \prod_{n=1}^\infty \left(1 - \frac{z^2}{n^2}\right),$$

1 See for example I. Niven: *Irrational Numbers* (Carus Monographs #11, MAA-Wiley, 1967).

2 There is however a famous formula of Ramanujan: let α and β be positive numbers such that $\alpha\beta = \pi^2$. Then if n is any positive integer

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^\infty \frac{k^{-2n-1}}{e^{2\alpha k} - 1}\right\}$$

$$= (-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^\infty \frac{k^{-2n-1}}{e^{2\beta k} - 1}\right\}$$

$$- 2^{2n} \sum_{k=0}^{n+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2n+2-2k}}{(2n+2-2k)!} \alpha^{n+1-k}\beta^k.$$

Taking α a rational multiple of π one sees that $\zeta(2n+1)$ is given as a rational multiple of π^{2n+1} plus two very rapidly convergent series. See for example: Bruce C. Berndt: Modular transformations and generalizations of several formulae of Ramanujan: *Rocky Mountain J. of Maths.* 7 (1977) 147–189. Indeed the above formula is the natural analoque of Euler's formula (5). The cited paper gives many other formulas and detailed references.

so

$$\frac{\pi \cos \pi z}{\sin \pi z} = \pi \cot \pi z = \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{2z}{n^2 - z^2} \right).$$

But

$$\pi z \cot \pi z = \pi i z \, \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \frac{2\pi i z}{e^{2\pi i z} - 1} + \pi i z =$$
$$= \sum_{m=0}^{\infty} (-1)^m \frac{(2\pi)^{2m}}{(2m)!} B_{2m} z^{2m},$$

and on the other hand

$$\pi z \cot \pi z = 1 - 2 \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2m}} \right) z^{2m}.$$

Comparing coefficients one has (5). With a little ingenuity one can avoid a direct appeal to the infinite product for $\sin \pi z$ or to the expansion for $\pi \cot \pi z$.[3]

Indeed, proving the irrationality of $\zeta(2n+1)$, $n = 1, 2, \ldots$ constitutes one of the outstanding problems of the theory (ranking with the arithmetic nature of $\gamma =: \lim_{n\to\infty}(1 + \ldots + (1/n) - \log n)$, and of $e\pi$, $e + \pi \ldots$ which are yet undetermined). It is some measure of Apéry's achievement that these questions have been considered by mathematicians of the top rank over the past few centuries without much success being achieved.

3. Some Irrelevant Explanations

For much of the following details I am indebted to Henri Cohen. All this due to Apéry, of course. The identity

$$\sum_{k=1}^{K} \frac{a_1 a_2 \ldots a_{k-1}}{(x + a_1) \ldots (x + a_k)} = \frac{1}{x} - \frac{a_1 a_2 \ldots a_K}{x(x + a_1) \ldots (x + a_K)}$$

follows easily on writing the right-hand side as $A_0 - A_K$ and noting that each term on the left is $A_{k-1} - A_k$. This explains (1). Now put $x = n^2$, $a_k = -k^2$, and take $k \leqslant K \leqslant n - 1$, to obtain

$$\sum_{k=1}^{n-1} \frac{(-1)^{k-1}(k-1)!^2}{(n^2 - 1^2) \ldots (n^2 - k^2)} =$$
$$= \frac{1}{n^2} - \frac{(-1)^{n-1}(n-1)!^2}{n^2(n^2 - 1^2) \ldots (n^2 - (n-1)^2)} =$$
$$= \frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2 \binom{2n}{n}}.$$

3 For a detailed set of references, and some new proofs, see Bruce C. Berndt: Elementary evaluation of $\zeta(2n)$. *Maths Mag.* 48 (1975), 148–153.

Writing

$$\epsilon_{n,k} = \frac{1}{2} \frac{k!^2 (n-k)!}{k^3 (n+k)!},$$

because

$$(-1)^k n (\epsilon_{n,k} - \epsilon_{n-1,k}) = \frac{(-1)^{k-1}(k-1)!^2}{(n^2 - 1^2) \ldots (n^2 - k^2)},$$

we have

$$\sum_{n=1}^{N} \sum_{k=1}^{n-1} (-1)^k (\epsilon_{n,k} - \epsilon_{n-1,k}) =$$
$$= \sum_{n=1}^{N} \frac{1}{n^3} - 2 \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} =$$
$$= \sum_{k=1}^{N} (-1)^k (\epsilon_{N,k} - \epsilon_{k,k}) =$$
$$= \sum_{k=1}^{N} \frac{(-1)^k}{2k^3 \binom{N+k}{k} \binom{N}{k}} + \frac{1}{2} \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}$$

and on noting that as $N \to \infty$ the first term on the right vanishes, we have (2).[4]

4. Some Nearly Relevant Explanations

All this is quite irrelevant to the proof. It would suffice to introduce the quantities $(k \leqslant n)$

4 Actually the formula (2) is quite well known: it was observed some years ago by Raymond Ayoub (Penn State) and it in fact appears in print: Margrethe Munthe Hjortnaes: Overføring av rekken $\sum_{k=1}^{\infty} (1/k^3)$ til et bestemt integral *Proc. 12th Cong. Scand. Maths, Lund 10–15 Aug. 1953* (Lund 1954); independently again it was noticed by R. William Gosper, Jr. (Palo Alto), see Gosper's paper: A calculus of series rearrangements *Algorithms and Complexity, New Directions and Recent Results*, ed. J. Traub (Academic Press, 1976) 121–151, for relevant techniques.
Henri Cohen remarked that the formula is:

$$\zeta(3) = \frac{5}{4} \; \mathrm{Li}_3 \omega^{-2} + \frac{2\pi^2}{15} \log \omega - \frac{2}{3} \log^3 \omega$$

(with $\omega = \frac{1}{2}(1 + \sqrt{5})$ and $\mathrm{Li}_3(x) = \sum x^3/n^3$, the trilogarithm). Hjortnaes and Ayoub, and respectively Gosper note the integral representations (easily shown equivalent)

$$\zeta(3) = 10 \int_0^{\log \omega} t^2 \coth t \, dt$$
$$\zeta(3) = 10 \int_0^{1/2} \frac{(\operatorname{arcsinh} t)^2}{t} dt.$$

In the case $\zeta(2)$ the formula is even better known. It is, for example, referred to by Z. R. Melzak: *Introduction to Concrete Mathematics* (Wiley, 1973), p. 85 (but the suggested proof is not quite appropriate). (2') may be proved by slightly varying the argument in Section 3 – multiply by $(-1)^{n-1}$ instead of dividing by n. Many formulas similar to (2) and (2') appear in the literature and the folklore.

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}, \qquad (6)$$

and to remark that plainly $c_{n,k} \to \zeta(3)$ as $n \to \infty$, uniformly in k. One might hope that a sequence $c_{n,k}$ already implies the irrationality of $\zeta(3)$ (say, the diagonal, with $k = n$) but this is not quite so. To see this, it is useful to prove a lemma:

Lemma.

$$2c_{n,k}\binom{n+k}{k} \in \mathbf{Z} + \frac{\mathbf{Z}}{2^3} + \ldots + \frac{\mathbf{Z}}{n^3} = \frac{\mathbf{Z}}{[1, 2, \ldots, n]^3}$$

(equivalently: $2[1, 2, \ldots, n]^3\, c_{n,k}\binom{n+k}{k}$ is an integer).

Proof. We check the number of times that any given prime p divides the denominator. But

$$\binom{n+k}{k} \Big/ \binom{n+m}{m} = \binom{n+k}{k-m} \Big/ \binom{k}{m}$$

so, because

$$\operatorname{ord}_p\left(\binom{n}{m}\right) \leqslant \left[\frac{\log n}{\log p}\right] - \operatorname{ord}_p m = \operatorname{ord}_p[1, \ldots, n] - \\ - \operatorname{ord}_p m,$$

we have

$$\operatorname{ord}_p\left(m^3\binom{n}{m}\binom{n+m}{m}\Big/\binom{n+k}{k}\right) = \operatorname{ord}_p\left(m^3\binom{n}{m}\binom{k}{m}\Big/\binom{n+k}{k-m}\right)$$
$$\leqslant 3(\operatorname{ord}_p m) + \left[\frac{\log n}{\log p}\right] + \left[\frac{\log k}{\log p}\right] - 2(\operatorname{ord}_p m),$$

which yields the assertion, because $m \leqslant k \leqslant n$. □

We remark that those who know it well[5] know that for n sufficiently large relative to $\epsilon > 0$,

$$[1, 2, \ldots, n] \leqslant e^{n(1+\epsilon)}$$

(roughly: $[1, 2, \ldots, n] = \Pi_{p \leqslant n} p^{[\log n/\log p]} \leqslant \Pi_{p \leqslant n} n \simeq n^{n/\log n} = e^n$). It will turn out that the $c_{n,k}$ have too large a denominator relative to their closeness to $\zeta(3)$. Hence to apply the irrationality criterion we must somehow *accelerate* the convergence. Apéry described this process as follows:

Consider two triangular arrays (defined for $k \leqslant n$) with entries $d_{n,k}^{(0)} = c_{n,k}\binom{n+k}{k}$ and $\binom{n+k}{k}$ respectively. We recall that the arrays have the property that their "quotient" converges to $\zeta(3)$, in the sense that given *any* "diagonal" $\{n, k(n)\}$, the quotient of the corresponding elements of the two arrays converges to $\zeta(3)$. Now apply the following transformations to each array:

$$d_{n,k}^{(0)} \to d_{n,n-k}^{(0)} = d_{n,k}^{(1)}$$
$$d_{n,k}^{(1)} \to \binom{n}{k} d_{n,k}^{(1)} = d_{n,k}^{(2)}$$
$$d_{n,k}^{(2)} \to \sum_{k'=0}^{k} \binom{k}{k'} d_{n,k'}^{(2)} = d_{n,k}^{(3)}$$
$$d_{n,k}^{(3)} \to \binom{n}{k} d_{n,k}^{(3)} = d_{n,k}^{(4)}$$
$$d_{n,k}^{(4)} \to \sum_{k'=0}^{k} \binom{k}{k'} d_{n,k'}^{(4)} = d_{n,k}^{(5)}$$

$$\binom{n+k}{k} \to \binom{2n-k}{n}$$
$$\to \binom{n}{k}\binom{2n-k}{n}$$
$$\to \sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n}$$
$$\to \sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{n}{k}\binom{2n-k_1}{n}$$
$$\to \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \binom{k}{k_2}\binom{k_2}{k_1}\binom{n}{k_1}\binom{n}{k_2}\binom{2n-k_1}{n}.$$

Of course the arrays have retained the property that their "quotient" converges to $\zeta(3)$, and we still have $2[1, 2, \ldots, n]^3 d_{n,k} \in \mathbf{Z}$: We now take the *main diagonals* ($k = n$) of the arrays, calling them respectively $\{a_n\}$ and $\{b_n\}$ and make the fantastic assertions embodied in 3 ! That is, each sequence satisfies the recurrence (2)! This is plainly absurd since surely *inter alia* a solution $\{u_n\}$ of (2) (with integral *initial values* u_0, u_1) will have u_n with denominator more like $n!^3$ than like 1 (or even $2[1, 2, \ldots, n]^3$). In Marseille, our amazement was total when our HP-67s, calculating $\{b_n\}$ on the one hand from the definition above, and on the other hand by the recurrence (2), kept on producing the same values!

5. It Seems that Apéry Has Shown that $\zeta(3)$ Is Irrational

We were quite unable to prove that the sequences $\{a_n\}$ defined above did satisfy the recurrence (2) (Apéry rather tartly pointed out to me in Helsinki that he regarded this more a compliment than a criticism of his method). But empirically (numerically) the evidence in favour was utterly compelling. It seemed indeed that $\zeta(3)$ had been proved irrational, because the rest, thus 4, follows quite easily: Given (with $P(n-1) = 34n^3 - 51n^2 + 27n - 5$),

5 Those who know it really well write $\log[1, \ldots, n] = \sum_{m=1}^{\infty} \theta(n^{1/m}) = \psi(n)$ where $\theta(n) = \sum_{p \leqslant n} \log p$. Then it is known that $\psi(n)/n \leqslant 1.03883\ldots$ (with maximum at $n = 113$) and indeed $\psi(n) - n < (0.0242334\ldots)\, n/\log n$ for $n \geqslant 525\,752$; See J. Barkley Rosser and Lowell Schoenfeld: *Math. Comp.* 29 (1975), 243–269.

$$n^3 a_n - P(n-1)a_{n-1} + (n-1)^3 a_{n-2} = 0,$$
$$n^3 b_n - P(n-1)b_{n-1} + (n-1)^3 b_{n-2} = 0,$$

one multiplies the first equation by b_{n-1}, the second by a_{n-1}, to obtain

$$n^3(a_n b_{n-1} - a_{n-1}b_n) =$$
$$= (n-1)^3(a_{n-1}b_{n-2} - a_{n-2}b_{n-1}).$$

Recalling $a_1 b_0 - a_0 b_1 = 6 \times 1 - 0 \times 5 = 6$, this cleverly yields

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}. \tag{7}$$

Seeing that $\zeta(3) - a_0/b_0 = \zeta(3)$, it is easily induced[6] that

$$|\zeta(3) - \frac{a_n}{b_n}| = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}},$$

so

$$\zeta(3) - \frac{a_n}{b_n} = O(b_n^{-2}).$$

On the other hand the recurrence relation makes it easy to estimate b_n, at any rate asymptotically. We have

$$b_n - (34 - 51n^{-1} + 27n^{-2} - 5n^{-3})b_{n-1}$$
$$+ (1 - 3n^{-1} + 3n^{-2} - n^{-3})b_{n-2} = 0,$$

and since the polynomial $x^2 - 34x + 1$ has zeros $17 \pm 12\sqrt{2} = (1 \pm \sqrt{2})^4$ we readily conclude that $b_n = O(\alpha^n)$, $\alpha = (1 + \sqrt{2})^4$. In fact Cohen has, more precisely, calculated that

$$b_n =$$
$$\frac{(1+\sqrt{2})^2}{(2\pi\sqrt{2})^{3/2}} \frac{(1+\sqrt{2})^{4n}}{n^{3/2}} \left(1 - \frac{48 - 15\sqrt{2}}{64n} + O(n^{-2})\right). \tag{8}$$

We have to recall that the a_n are not integers. But writing

$$p_n = 2[1, 2, \ldots, n]^3 a_n, \quad q_n = 2[1, 2, \ldots, n]^3 b_n$$

we have $p_n, q_n \in \mathbb{Z}$ and

$$q_n = O(\alpha^n e^{3n}), \quad \zeta(3) - \frac{p_n}{q_n} = O(\alpha^{-2n}) = O(q_n^{-(1+\delta)}),$$

6 Write $\zeta(3) - a_n/b_n = x_n$ and note that we have $x_n - x_{n-1} = 6/n^3 b_n b_{n-1}$ and $x_\infty = 0$.

with

$$\delta = \frac{\log\alpha - 3}{\log\alpha + 3} = 0.080\,529\ldots > 0.$$

Hence, by the irrationality criterion, $\zeta(3)$ is indeed irrational, and moreover, because $\delta^{-1} = 12.417\,820\ldots$ we have:

> *For all integers $p, q > 0$ sufficiently large relative to $\epsilon > 0$:*
>
> $$|\zeta(3) - \frac{p}{q}| > q^{-(\theta+\epsilon)}, \quad \theta = 13.417\,820\ldots$$

6. Some Trivial Verifications

To convince ourselves of the validity of Apéry's proof we need only complete the following exercise.

Exercise
Prove the following identities:

(5) Let

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k},$$

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}.$$

Then $a_0 = 0, a_1 = 6; b_0 = 1, b_1 = 5$ and each sequence $\{a_n\}$ and $\{b_n\}$ satisfies the recurrence (2).

In the same spirit, the case of $\zeta(2)$ requires:

(5′) Let

$$b'_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \quad a'_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} c'_{n,k}$$

$$c'_{n,k} = 2\sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^2} + \sum_{m=1}^{k} \frac{(-1)^{n+m-1}}{m^2\binom{n}{m}\binom{n+m}{m}}.$$

Then $a'_0 = 0, a'_1 = 5; b'_0 = 1, b'_1 = 3$ and each sequence $\{a'_n\}$ and $\{b'_n\}$ satisfies the recurrence (4).

It is useful to notice that very little more than just proving these claims is required for Apéry's proof. After all, it is quite plain that $a_n/b_n \to \zeta(3)$; the b_n are integers, and the lemma of Section 4 shows that the a_n are "near-integers." In Section 5 we showed that given that the sequences satisfy the recursion (2) the irrationality of $\zeta(3)$ follows because from $\log \alpha > 3$ $(\alpha = (1+\sqrt{2})^4)$ we obtain $\delta > 0$. Thus, as implied in various asides, most of the earlier argument is quite irrelevant. Indeed I am indebted to John Conway for the remark that even (5) is irrelevant.

Exercise

Be the first in your block to prove by a 2-line argument that $\zeta(3)$ is irrational.[7]

(6) Given the definitions of (5) show that $a_n b_{n-1} - a_{n-1} b_n = b_n^{-3}$ and $b_n = O(\alpha^n)$ with $\alpha = (1+\sqrt{2})^4$. Conclude that $\zeta(3)$ is irrational because $\log \alpha > 3$.

Exercise

Astound your friends with an excellent irrationality measure for π^2.

(6′) Given the definitions of (5′) show that $a_n b_{n-1} - a_{n-1} b_n = 5(-1)^{n-1} n^{-2}$ and $b_n = O(\alpha^n)$ with $\alpha = (\frac{1}{2}(1+\sqrt{5}))^5$. Conclude that $|\pi^2 - (p/q)| > q^{-(\theta+\epsilon)}$ with $\theta = 11.850\,782\ldots$ for all integers p, $q > Q(\epsilon)$.[8]

7. ICM '78, Helsinki, August 1978

Neither Cohen nor I had been able to prove (5) or (5′) in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence $\{b'_n\}$ satisfies the recurrence (4). This more or less broke the dam and (5′) and (5) were quickly conquered. Henri Cohen addressed a very well-attended meeting at 17.00 on Friday, August 18 in the language of the majority, proving (5) and explaining how this implied the irrationality of $\zeta(3)$. Apéry then made some remarks on the status of the French language, and alluded to the underlying motivation (as mentioned in Section 3) for his astonishing proof.

Exercise

A red herring?

(7) Show that

$$\zeta(3) = \cfrac{6}{5 - \cfrac{1}{117 - \cfrac{64}{535 - \cfrac{729}{1436 - \cfrac{4096}{3105 - \ldots}}}}}$$

$$- \frac{n^6}{34n^3 + 51n^2 + 27n + 5 \ldots}$$

and deduce that $\zeta(3) = 1.202\,056\,903\ldots$ is irrational.

(7′) Show that

$$\zeta(2) = \frac{\pi^2}{6} = \cfrac{5}{3 + \cfrac{1}{25 + \cfrac{16}{69 + \cfrac{81}{135 + \cfrac{256}{223 + \ldots}}}}}$$

$$+ \frac{n^4}{11n^2 + 11n + 3 + \ldots}$$

and deduce that π^2 has irrationality degree at most $11.850\,782\ldots$.

8. Some Rather Complicated but Ingenious Explanations

According to a dictum of Littlewood any identity, once verified, is trivial. Surely (5) is very nearly a counterexample. The following is principally due to Zagier and Cohen. Incidentally, we first considered (5′) which appeared simpler, but this was because we had failed to notice that

$$\sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{2n-l}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n-k}{n}^2 .$$

Now writing $n - k$ for k links the arrays of Section 4 to (5).

7 The author does not pretend to be able to do this. Notice that in fact even less is needed: it is sufficient to show $a_n b_{n-1} - a_{n-1} b_n = O(\gamma^n)$ and $b_n = O(\beta^n)$, with $\log \beta - \log \gamma > 3$.

8 Though we have long known that $\zeta(2)$ is irrational, Apéry's result in this case is significant. The irrationality degree for π^2 is the best known; the irrationality degree implied for π is 23,701564 . . . . These results compare very favourably with those of Mahler: $|\pi - (p/q)| > q^{-42}$: On the approximation of π *Proc. K. Ned. Akad. Wet. Amsterdam A*, 56 (= *Indag. Math.* 15) (1953) 29–42, and an indication that $|\pi - (p/q)| > q^{-30}$; see also K. Mahler: Applications of some formulae by Hermite to the approximation of exponentials and logarithms. *Math. Annalen* 168 (1967) 200–227. Wirsing announced $|\pi - (p/q)| > q^{-21}$ and Mignotte proved that (for q sufficiently large) $|\pi - (p/q)| > q^{-20}$; this is the best known result. It should be noted that the cited results depend on deep techniques and complicated estimations in transcendence theory as contrasted with the essentially elementary methods in Apéry's proof. Mignotte (*op, cit.*) also shows that $|\pi^2 - (p/q)| > q^{-18}$, which is weaker than Apéry's result.

It is quite convenient to write:

$$b_{nk} = \binom{n}{k}^2\binom{n+k}{k}^2, \quad a_{nk} = b_{nk}c_{nk},$$

$$\left(b_n = \sum_{k=0}^{n} b_{nk}, \quad a_n = \sum_{k=0}^{n} b_{nk}c_{nk}\right).$$

Then we wish to show that

$$\sum_k \{(n+1)^3 b_{n+1,k} - (34n^3 + 51n^2 + 27n + 5)b_{n,k} + n^3 b_{n-1,k}\} = 0.$$

We cleverly construct

$$B_{n,k} = 4(2n+1)\,(k(2k+1) - (2n+1)^2)\binom{n}{k}^2\binom{n+k}{k}^2,$$

with the motive that

$$\begin{aligned} B_{n,k} - B_{n,k-1} &= (n+1)^3\binom{n+1}{k}^2\binom{n+1+k}{k}^2 - \\ &- (34n^3 + 51n^2 + 27n + 5)\binom{n}{k}^2\binom{n+k}{k}^2 + \\ &+ n^3\binom{n-1}{k}^2\binom{n-1+k}{k}^2, \end{aligned}$$

and, *O mirabile dictu*, the sequence $\{b_n\}$ does indeed satisfy the recurrence (2) by virtue of the method of *creative telescoping* (by the usual conventions: $B_{nk} = 0$ for $k < 0$ or $k > n$; note also that $P(n) = 34n^3 + 51n^2 + 27n + 5$ implies $P(n-1) = -P(-n)$.)

The rest is plain sailing (or is it plane sailing?). We notice that

$$\begin{aligned} (n+1)^3 b_{n+1,k}c_{n+1,k} - P(n)b_{n,k}c_{n,k} + n^3 b_{n-1,k}c_{n-1,k} &= \\ = (B_{n,k} - B_{n,k-1})c_{n,k} &+ \\ + (n+1)^3 b_{n+1,k}(c_{n+1,k} - c_{n,k}) &- \\ - n^3 b_{n-1,k}(c_{n,k} - c_{n-1,k}). & \qquad (9) \end{aligned}$$

Clearly

$$\begin{aligned} c_{n,k} - c_{n-1,k} &= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^m (m-1)!^2 (n-m-1)!}{(n+m)!} \\ &= \frac{1}{n^3} + \sum_{m=1}^{k}\left(\frac{(-1)^m m!^2(n-m-k)!}{n^2(n+m)!} - \frac{(-1)^{m-1}(m-1)!}{n^2(n+m+1)!}\right) \\ &= \frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!} \end{aligned}$$

whilst not even a minor miracle is required to write down $c_{n,k} - c_{n,k-1}$. After some massive reorganisation (9) becomes

$$A_{n,k} - A_{n,k-1} \quad \text{with} \quad A_{n,k} = B_{n,k}c_{n,k} + \frac{5(2n+1)(-1)^{k-1}k}{n(n+1)}\binom{n}{k}\binom{n+k}{k}$$

and we have completed ⑤, and, in passing, proved ③. This of course verifies Apéry's claim to have proved $\zeta(3)$ irrational.

9. The Case of $\zeta(2)$

The arguments required to deal with the exercises ⓧ are quite similar to those already described. It may however be a kindness to the reader to reveal that it would be wise to take

$$B_{n,k} = (k^2 + 3(2n+1)k - 11n^2 - 9n - 2)\binom{n}{k}^2\binom{n+k}{k},$$

$$A_{n,k} = B_{n,k}c_{n,k} + 3(-1)^{n+k-1}\frac{(n-1)!}{(k-1)!}.$$

Moreover

$$c_{n,k} - c_{n-1,k} = 2(-1)^{n+k-1}\frac{k!^2(n-k-1)!}{n(n+k)!},$$

and

$$b_n = \frac{(\frac{1}{2}(1+\sqrt{5}))^4}{2\pi\sqrt{5+2\sqrt{5}}}\;\frac{(\frac{1}{2}(1+\sqrt{5}))^{5n}}{n}\;(1 + O(n^{-1})); \quad (10)$$

(also note that if $Q(n) = 11n^2 + 11n + 3$ then $Q(n-1) = -Q(-n)$).

10. What on Earth is Going on Here?

Apéry's incredible proof appears to be a mixture of miracles and mysteries. The dominating question is how to generalise all this, down to the Euler constant γ and up to the general $\zeta(t)$? Here we have, apparently, the tip of an iceberg which relates $(1+\sqrt{2})^4$ to $\zeta(3)$ and $(\frac{1}{2}(1+\sqrt{5}))^5$ to $\zeta(2)$; we have surprising identities ② and ②', and startling continued fractions (produced by Cohen for his Helsinki talk), ⑦ and ⑦'. Does the complete berg look like this? For my part I incline to the view that much of what has been presented constitutes a mystification rather than an explanation. For example Richard Askey (Madison, Wisconsin) has pointed out to me that the sequences $\{b_n\}$ and $\{b'_n\}$ may be recognized as special values of certain hypergeometric polynomials; immediately the recurrences

(2) and (4) become identities relating hypergeometric functions and much of the magic fades away. Unfortunately the difficulties remain, because not all that much is known about the higher generalisations of the classical hypergeometric functions. For this, and other reasons, it is however likely that one should think about recurrences of order greater than 2. This, incidentally, means that the continued fractions constitute a red herring. In any event (7) obscures a fundamental miracle. Its convergents P_n/Q_n are of course such that the sequences $\{P_n\}$ and $\{Q_n\}$ both satisfy

$$U_{n+1} = (34n^3 + 51n^2 + 27n + 5)U_n - n^6 U_{n-1}.$$

The proof works (not because the continued fraction does not terminate; that only works for *regular* continued fractions, but) because if $U_0 = 1$, $U_1 = 5$ then it happens that $(n!)^3$ divides the integers U_n; more honestly: it is already enough (and is necessary) that for *any* initial *integer* values U_0, U_1, $(n!)^3$ always divides $2[1, \ldots, n]^3 U_n$. An analogous miracle makes the recurrence

$$U_{n+1} = (11n^2 + 11n + 3)U_n + n^4 U_{n-1}$$

useful in proving the irrationality of $\zeta(2)$.[9] These surprises generalise the following quite well known fact (to which I was alerted by Frits Beukers (Leiden)): the recurrence

$$U_{n+1} = (6n + 3)U_n - n^2 U_{n-1}$$

is such that $n!$ divides U_n if $U_0 = 1$, $U_1 = 3$; and $n!$ divides $[1, \ldots, n]U_n$ for all integer initial values U_0, U_1.

Exercise

What are the higher analogues?

(8) Show that if

$$B(z) = (1 - 6z + z^2)^{-1/2} = \sum_{n=0}^{\infty} b_n z^n,$$

then the $b_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$, and all are integers. Find an expression for the a_n in

$$A(z) = (1 - 6z + z^2)^{-1/2} \int_0^z (1 - 6t + t^2)^{-1/2}\, dt = \sum_{n=0}^{\infty} a_n z^n$$

and notice that the $[1, \ldots, n]a_n$ all are integers. Show that sequences $\{a_n\}$ ($a_0 = 0$, $a_1 = 1$) and $\{b_n\}$ ($b_0 = 1$, $b_1 = 3$) both satisfy

$$nu_n + (n-1)u_{n-2} = (6n-3)u_{n-1}.$$

Now prove that there is a constant λ such that $A(z) - \lambda B(z) = \sum_{n=0}^{\infty} c_n z^n$ has no singularity at $3 - 2\sqrt{2}$. Deduce that then $c_n = O(\alpha^{-n})$ with $\alpha = (1 + \sqrt{2})^2$ and conclude that it follows that log 2 has irrationality degree at most 4. 662 100 831 . . ..

Of course Exercise (6) should remind us that recurrences may be quite irrelevant to the proof. The vital thing then is suitable definition of the $c_{n,k}$, so one is brought back to looking for generalisations of (2). But, for the present, generalisation of Apéry's work remains, as they say, a mystery wrapped in an enigma.[10]

Most startling of all though should be the fact that Apéry's proof has no aspect that would not have been accessible to a mathematician of 200 years ago. The proof we have seen is one that many mathematicians could have found, but missed.

This note was written at Queen's University, Kingston, Ontario whilst the author was on study leave from the University of New South Wales, Sydney, Australia.

October, 1978

Postscripts. See L. Lewin *Dilogarithms and associated functions* (Macdonald, London, 1958) for many delightful facts, including the trilogarithm formula of[4] which is given at p. 139. At p. 89 of Louis Comtet *Advanced Combinatorics* (D. Reidel, Dordrecht, 1974) one is astonished to be asked to prove as an exercise that

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\pi\sqrt{3}}{27}\,; \quad \sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} = \frac{\pi\sqrt{3}}{9}\,;$$

---

9 Tom Cusick (Buffalo) has noticed that the following recurrences also yield continued fractions converging to $\pi^2/6$:

$$n^2 u_n = (7n^2 - 7n + 2)u_{n-1} + 8(n-1)^2 u_{n-2}$$

(one solution of which is $\Sigma\binom{n}{k}^3$), and

$$n^3 u_n = 2(2n-1)(3n^2 - 3n + 1)u_{n-1} + (4n-3)(4n-4)(4n-5)u_{n-2}$$

(A solution is $\Sigma\binom{n}{k}^4$).

On first impression the first yields a worse irrationality degree for π^2 than that obtained by Apéry, and the second does not yield irrationality at all. Apéry's results are indeed remarkable.

10 Well, not really. It is just that it is not at all clear where to go. A numerical test (suggested by Cohen) implies that $\zeta(4) = \pi^4/90 = (36/17)\sum_{n=1}^{\infty}(1/n^4\binom{2n}{n})$ (so this is *true for all practical purposes*) and it has been shown by Gosper that

$$\zeta(5) = \frac{5}{2}\sum_{n=1}^{\infty}\left(\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(n-1)^2} - \frac{4}{5n^2}\right)\frac{(-1)^n}{n^3\binom{2n}{n}}.$$

David Hawkins (Boulder) suggests similar formulas. Apparently such expressions can be generated virtually at will on using appropriate *series accelerator identities.*

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}; \quad \sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}} = \frac{17\pi^4}{3{,}240}.$$

Seeing that

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n^2\binom{2n}{n}} = 2\left(\sin^{-1}\frac{x}{2}\right)^2$$

(see for example Melzak *op cit* p. 108) the first three formulae (and the one with the trilogarithm) become quite accessible to proof, but I had not detected anyone able to prove the expression for $\zeta(4)$, until I proved it in March 1979 after noticing a remark of Lewin that also

$$2\int_0^{\pi/3} x\left(\log\left(2\sin\frac{x}{2}\right)\right)^2 dx = \frac{17\pi^4}{3{,}240}.$$

Sam Wagstaff (Illinois) and Andrew Odlyzko (Bell Labs) have mentioned to me that numerical evidence suggests that there are formulae of the shape (2) or (2') for $\zeta(t)$ only for $t = 2, 3, 4$ and this is verified by my studies in a current manuscript *Some wonderful formulae*... The recurrences[9] are long known, see Comtet *op cit* p. 90. One can recognise the b_n as $b_n = {}_4F_3\left({n+1,\ -n,\ n+1,\ -n \atop 1,\ 1,\ 1}; 1\right)$ and determine the recurrence (3) by way of three term relations with *contiguous balanced series*; see J. A. Wilson *Hypergeometric series, recurrence relations and some new orthogonal functions* (Ph. D. thesis; U. Wisconsin-Madison, 1978).

Frits Beukers (Leiden) *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$* (*J. Lond. Math. Soc.* to appear) has found an elegant approach to Apéry's proofs which entirely avoids explicit identities, recurrences and other magic. Instead just consider

$$I = -1/2 \int_0^1\int_0^1 \frac{P_n(x)P_n(y)\log xy}{1-xy}\,dx\,dy = b_n\zeta(3) - a_n$$

noticing that the b_n are integers and the a_n are rationals with the $2[1,\ldots,n]^3 a_n$ integral, whilst

$$|I| \leqslant \zeta(3)\,(1-\sqrt{2})^{4n}$$

here $P_n(z) = \dfrac{1}{n!}\dfrac{d^n}{dz^n}(z^n(1-z)^n)$ is the

Legendre polynomial. Again, there is no obvious way to generalise the proof.

In retrospect it seems clear that exercise (8) really is useful; implications are being considered by Bombieri *et al* (at Princeton). For example, one's intuition is just wrong in feeling incredulity at the facts of (3). All that these report is that the differential equation

$$\begin{aligned}\frac{d}{dX}\Big\{(X^4 - 34X^3 + X^2)\frac{d^3y}{dX^3} &+ \\ + (6X^3 - 103X^2 + 3X)\frac{d^2y}{dX^2} &+ \\ + (7X^2 - 112X + 1)\frac{dy}{dX} &+ \\ + (X-5)y - (u_1 - 5u_0)\Big\} &= 0\end{aligned}$$

has two *G-function* solutions, namely $a(X) = 6X + a_2X^2 + \ldots$; $b(X) = 1 + b_1X + b_2X^2 + \ldots$; and $a(X) - \zeta(3)b(X)$ is regular (in fact vanishes) at $\alpha' = (1-\sqrt{2})^4$. This is interesting, but no longer incredible; and it is readily generalisable ... All this too is an idea of Beukers. In keeping with the bizarre nature of the events reported here *La Recherche No. 97* (France's *Scientific American*) contained a report *Roger Apéry et l'irrationnel* by Michel Mendes-France; the report includes a lively description of the lecture at Marseille (politely suppressed here) although Mendes-France was in the U.S.A. at the time.

Some officious readers have been critical of my casual use of the O-symbol; the fault is mine, not Apéry's. No harm is done. Similarly it has been claimed that Apéry's proof was not missed by Euler – 'Euler did not know the prime number theorem'; to me it seems hypercritical to suggest that $[1,\ldots,n] = O((1+\sqrt{2})^{4n/3})$ could not have been noticed at the time, had it been needed. Anyhow, I considered it a racy title. It arose after Cohen's report at Helsinki, with someone sourly commenting 'A victory for the French peasant ...'; to this Nick Katz retorted: 'No ...! No! This is marvellous! It is something Euler could have done ...'

March 1979

School of Mathematics and Physics
Macquarie University
North Ryde, New South Wales
Australia 2113

MATHEMATICS OF COMPUTATION, VOLUME 34, NUMBER 149
JANUARY 1980, PAGES 305–312

Some New Algorithms for High-Precision Computation of Euler's Constant

By Richard P. Brent* and Edwin M. McMillan

Abstract. We describe several new algorithms for the high-precision computation of Euler's constant $\gamma = 0.577\ldots$. Using one of the algorithms, which is based on an identity involving Bessel functions, γ has been computed to 30,100 decimal places. By computing their regular continued fractions we show that, if γ or $\exp(\gamma)$ is of the form P/Q for integers P and Q, then $|Q| > 10^{15000}$.

1. Introduction. Euler's constant γ is defined by

$$\gamma = \lim_{m \to \infty} (H_m - \ln(m)), \tag{1}$$

where $H_m = \Sigma_{k=1}^{m} 1/k$.

Recently $\zeta(3)$ was proved irrational [17] with the aid of a rapidly converging continued fraction, and conceivably a similar method might be used to prove the irrationality of γ. Thus, there is some interest in finding rapidly converging expressions for γ. We give several such expressions below.

Early computations of γ used the Euler-Maclaurin expansion to accelerate convergence of (1): see Brent [7] and Glaisher [12]. Sweeney [16] suggested a method which avoids the need for computation of the Bernoulli numbers which appear in the Euler-Maclaurin expansion, and Brent [7], [8] used Sweeney's method to compute γ to 20,700 decimal places. In Section 3 we describe an algorithm which is about twice as fast as Sweeney's. The algorithm depends on some identities, given in Section 2, involving modified Bessel functions. To demonstrate the effectiveness of the algorithm we have used it to compute γ to 30,100 decimal places; see Section 5. Some other algorithms for the high-precision computation of γ are briefly described and compared in Section 4.

2. Some Bessel Function Identities. The modified Bessel functions $I_\nu(z)$ and $K_0(z)$ are defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu + k + 1)} \quad \text{and} \quad K_0(z) = -\,\partial I_\nu(z)/\partial\nu|_{\nu=0}.$$

Received January 22, 1979; revised May 15, 1979.

AMS (MOS) subject classifications (1970). Primary 10–04, 10A40; Secondary 10F20, 10F35, 65A05, 68A20.

Key words and phrases. Euler's constant, Mascheroni's constant, gamma, Bessel functions, rational approximation, regular continued fractions, multiple-precision arithmetic, Gauss-Kusmin law.

*The work of the first author was supported in part by National Science Foundation grant 1-442427-21164-2 at the University of California, Berkeley. This work was also supported by the U. S. Department of Energy under Contract W-7405-ENG-48.

It is easy to verify, by carrying out the differentiation indicated above, that

$$\gamma + \ln(z/2) = \frac{S_0(z) - K_0(z)}{I_0(z)}, \tag{2}$$

where

$$S_0(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2} H_k.$$

For real positive z, $K_0(z)$ and $I_0(z)$ have the asymptotic expansions

$$K_0(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{k=0}^{\infty} (-1)^k a_k(z) \tag{3}$$

and

$$I_0(z) \sim (2\pi z)^{-1/2} e^z \sum_{k=0}^{\infty} a_k(z), \tag{4}$$

where

$$a_k(z) = \frac{1^2 3^2 \cdots (2k-1)^2}{k!(8z)^k} = \frac{[(2k)!]^2}{(k!)^3 (32z)^k}.$$

A proof is given in Watson [19, Section 7.23].

For real $z \geqslant 1$, the first terms in the asymptotic expansions (3) and (4) give upper and lower bounds on $K_0(z)$ and $I_0(z)$, respectively:

$$0 < K_0(z) < \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \quad \text{and} \quad I_0(z) > (2\pi z)^{-1/2} e^z.$$

Thus, taking $z = 2n \geqslant 2$ in (2), we have

$$0 < U(n)/V(n) - \gamma = K_0(2n)/I_0(2n) < \pi e^{-4n}, \tag{5}$$

where

$$U(n) = S_0(2n) - I_0(2n)\ln(n) = \sum_{k=0}^{\infty} \left(\frac{n^k}{k!}\right)^2 (H_k - \ln(n)) \tag{6}$$

and

$$V(n) = I_0(2n) = \sum_{k=0}^{\infty} \left(\frac{n^k}{k!}\right)^2. \tag{7}$$

In the following section we describe an algorithm (B1) for computing γ using (5) to (7). It is interesting to note that the relations (2) to (4) were essentially given by Riemann [14] in 1855, but the possibility of using them to compute Euler's constant appears to have been overlooked.

3. The Algorithm B1. Suppose we wish to evaluate γ to d decimal places. If we choose

$$n = \lfloor c + \tfrac{1}{4}\ln(10)d \rfloor \tag{8}$$

for some suitable constant c then, from (5),

$$|\gamma - U(n)/V(n)| < \pi e^{4-4c} 10^{-d}.$$

Thus, we need only evaluate $U(n)$ and $V(n)$ sufficiently accurately, and then perform one high-precision division, to evaluate γ to the required accuracy.

Let

$$A_k = \left(\frac{n^k}{k!}\right)^2 (H_k - \ln(n)), \qquad U_k = \sum_{j=0}^{k} A_j,$$

$$B_k = \left(\frac{n^k}{k!}\right)^2, \qquad V_k = \sum_{j=0}^{k} B_j.$$

Then

$$A_0 = -\ln(n), \quad B_0 = 1, \quad U_0 = A_0, \quad V_0 = 1, \tag{9}$$

and for $k = 1, 2, \ldots$, we have

$$\begin{aligned} B_k &= B_{k-1} n^2/k^2, \qquad A_k = (A_{k-1} n^2/k + B_k)/k, \\ U_k &= U_{k-1} + A_k, \qquad V_k = V_{k-1} + B_k. \end{aligned} \tag{10}$$

For Algorithm B1, n is chosen according to (8), and working precision equivalent to slightly more than d (floating) decimal places is used. $\ln(n)$ is computed, e.g. by the $O(d^2)$ method of [9], and A_0, B_0, U_0 and V_0 are initialized as in (9). The iteration (10) is terminated when, to the working precision, $U_k = U_{k-1}$ and $V_k = V_{k-1}$. The storage required is $O(d)$ as B_k can overwrite B_{k-1}, etc.

For $j \geqslant 0$, let α_j be the real positive root of

$$\alpha_j \ln \alpha_j - \alpha_j = j. \tag{11}$$

Thus, $\alpha_0 = e \simeq 2.78$, $\alpha_1 \simeq 3.59$, $\alpha_2 \simeq 4.32$, $\alpha_3 \simeq 4.97$, etc.

Using Stirling's approximation, we see that the number of iterations of (10) required is

$$K = \alpha_1 n + O(\ln(n)) = \tfrac{1}{4}\alpha_1 \ln(10)d + O(\ln(d)) \simeq 2.07d. \tag{12}$$

In analyzing the time required by Algorithm B1 and other algorithms described in Section 4, we make the following simplifying assumptions.

(a) Only the time required for the inner loop(s) is considered. (The computation of $\ln(n)$ is common to all the algorithms considered, so the time required for this is neglected. The final division of U_K by V_K takes time $O(d^2)$ if done as in [9], but the constant factor is relatively small, and $o(d^2)$ methods exist [5].)

(b) Multiplication or division of a multiple-precision number (e.g. A_{k-1}) by a small integer (e.g. n^2 or k) takes time d units. In the analysis (though not in the implementation of the algorithm) the possibility of reducing the working precision (e.g. for A_k and B_k when $k \simeq K$) is neglected. Considering this possibility complicates the analysis but is unlikely to alter the ranking of the algorithms discussed below.

(c) Addition of two multiple-precision numbers takes time d units. (The constant is assumed to be the same as for (b). Again, this is unlikely to change the ranking of the algorithms.)

Using these assumptions, each iteration of (10) requires time $8d$ (for 3 multiple-precision additions, 2 multiplications and 3 divisions by small integers). Thus, from (12), the time required by method B1 is about $2\alpha_1 \ln(10)d^2 \simeq 16.5d^2$.

It is important to note that we avoid keeping H_k or $(H_k - \ln(n))$ as a multiple-precision number and multiplying by B_k in the inner loop. This would lead to a method with time $\Omega(d^3)$ if the classical multiplication algorithm were used as in [9]. The idea of using the Bessel function identities to compute γ was suggested by the second author, and the $O(d^2)$ implementation was discovered by the first author.

If terms in the sum (15) are grouped as in [5] and the Schönhage-Strassen fast multiplication algorithm [15] is used, it is possible to compute γ with error bounded by 10^{-d} in time $O\{d[\ln(d)]^3 \ln[\ln(d)]\}$, asymptotically faster than any of the $\Omega(d^2)$ algorithms considered here. However, such "fast" algorithms are very difficult to implement and are slower than Algorithm B1 unless d is very large. Thus, we do not consider them further.

4. Related Algorithms. In this section we briefly describe and compare several closely related algorithms for the computation of γ.

4.1. *Algorithm* B2. From (2) we have

$$\gamma + \ln(n) = \frac{S_0(2n) - K_0(2n)}{I_0(2n)},$$

and from (3)

$$K_0(2n) = \tfrac{1}{2}(\pi/n)^{1/2}e^{-2n} \sum_{k=0}^{4n} (-1)^k a_k(2n) + O(e^{-6n}/n). \tag{13}$$

Thus, we can find γ with error $O(n^{-1/2}e^{-8n})$ if $K_0(2n)$ is approximated using (13). If e^{2n} is computed using the Taylor series, and the time required to compute $(\pi/n)^{1/2}$ is neglected, the time required by this method (B2) is about

$$[(8\alpha_3 + 3\alpha_1 + 6)\ln(10)/8]d^2 \simeq 16.3d^2,$$

not appreciably less than for the simpler method B1.

4.2. *Algorithm* B3. To avoid the computation of $(\pi/n)^{1/2}e^{-2n}$ in (13), we may use the asymptotic series [1, Eq. (9.7.5)]

$$I_0(2n)K_0(2n) \sim \frac{1}{4n} \sum_{k=0}^{k'} \frac{[(2k)!]^3}{(k!)^4(16n)^{2k}} \tag{14}$$

with $k' \leqslant 2n$. Empirical evidence suggests that the relative error in (14) with $k' = 2n$ is $O(n^{-1/2}e^{-4n})$, but we have not been able to prove this. Assuming this error bound, the time required with $k' = 2n$ is about

$$(\alpha_3 + 3/8)\ln(10)d^2 \simeq 12.3d^2.$$

This is less than the estimate $16.5d^2$ for Algorithm B1, but we preferred to use B1 because of its simplicity and the difficulty in rigorously bounding the error in (14).

4.3. *Exponential Integral Methods.* Several algorithms are based on the identity

$$\gamma + \ln(n) = Q(n) - R(n),$$

where

$$Q(n) = \int_0^n \left(\frac{1 - e^{-x}}{x}\right) dx = \sum_{k=1}^{\infty} \frac{n^k(-1)^{k-1}}{k!k} \tag{15}$$

and

$$R(n) = \int_n^{\infty} \frac{e^{-x}}{x} dx = O(e^{-n}/n).$$

Beyer and Waterman [3], [4] took $n \simeq \ln(10)d$, worked to precision equivalent to $2d$ decimal places to compensate for cancellation in the sum for $Q(n)$, and neglected $R(n)$. The time required for this method is about $6\alpha_1 \ln(10)d^2 \simeq 49.6d^2$, or three times that for method B1.

Following the suggestion of Sweeney [16], Brent [7], [8] took $n \simeq \frac{1}{2}\ln(10)d$, summed the series for $Q(n)$ using the equivalent of $3d/2$ decimal places, and approximated $R(n)$ by its asymptotic expansion

$$R(n) = \frac{e^{-n}}{n} \sum_{k=0}^{n-2} \frac{k!}{(-n)^k} + O(e^{-2n}/n). \tag{16}$$

Assuming the power series is used to compute e^n, the time required for this method is about

$$\frac{3}{4}(3\alpha_2 + \alpha_0 + 1)\ln(10)d^2 \simeq 28.8d^2,$$

about twice as much as for method B1. Actual running times confirm this ratio.

Using the identity

$$e^n Q(n) = \sum_{k=0}^{\infty} H_k \frac{n^k}{k!},$$

we can evaluate $Q(n)$ by computing $\Sigma H_k n^k/k!$ and $\Sigma n^k/k!$ by recurrences similar to (10). Because all the terms in the two sums are positive, there is no need to increase the working precision to much more than the equivalent of d decimal places. If $n \simeq \ln(10)d$, and $R(n)$ is neglected, the time required is about $7\alpha_0 \ln(10)d^2 \simeq 43.8d^2$, slightly less than for Beyer and Waterman's method. If the asymptotic series (16) is used for $R(n)$, the time required is about

$$\tfrac{1}{4}(14\alpha_1 + 3)\ln(10)d^2 \simeq 30.7d^2,$$

slightly greater than for Sweeney's method.

Instead of using the asymptotic expansion (16) for $R(n)$, we could use Euler's continued fraction [18, p. 350]

$$e^n R(n) = 1/n + 1/1 + 1/n + 2/1 + 2/n + 3/1 + 3/n + \cdots$$

and the forward or backward recurrence relations. This has the advantage that $R(n)$ can be evaluated as accurately as desired, whereas with the asymptotic expansion (16) the error is $\Omega(e^{-2n}/n)$. The choice of the optimal n and the optimal number of terms in the continued fraction (evaluated by the backward recurrence relations) gives a method competitive with Algorithm B1, but much more complicated.

4.4. *A Generalization.* For fixed $p > 0$, it follows from (1) that

$$\gamma = \lim_{n\to\infty} \frac{\sum_{k=0}^{\infty} (n^k/k!)^p (H_k - \ln(n))}{\sum_{k=0}^{\infty} (n^k/k!)^p}. \tag{17}$$

With $p = 1$ we obtain essentially one of the exponential integral methods mentioned above, with error $O(e^{-n}/n)$. With $p = 2$ we obtain method B1, with error $O(e^{-4n})$.

We shall sketch how the error in (17) may be estimated for integer $p \geqslant 2$. Let $y(z)$ be a function of the real variable z, and L the operator defined by $Ly = z(dy/dz)$. Then

$$V_p(z) = \sum_{k=0}^{\infty} (z^k/k!)^p$$

and

$$U_p(z) = \sum_{k=0}^{\infty} (H_k - \ln(z) - \gamma)(z^k/k!)^p$$

are independent solutions of

$$(L^p - (pz)^p)y = 0. \tag{18}$$

{To verify this for $y = U_p$, let

$$V_{p,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^{kp+\nu}}{(k!)^{p-1}\Gamma(k+\nu+1)}.$$

Then it is easy to check that

$$[(L + (p-1)\nu)(L-\nu)^{p-1} - (pz)^p]\, V_{p,\nu}(z) = 0, \tag{19}$$

and the result follows by differentiating (19) with respect to ν, setting $\nu = 0$, and observing that $U_p(z) = -\partial V_{p,\nu}(z)/\partial\nu|_{\nu=0}$.} Now

$$V_p(z) \sim p^{-1/2}(2\pi z)^{(1-p)/2}\exp(pz)$$

is the dominant solution of (18) as $z \longrightarrow +\infty$. By analyzing the asymptotic behavior of the subdominant solutions we obtain

$$|U_p(z)/V_p(z)| = O(\exp(-c(p)z)) \quad \text{as } z \longrightarrow +\infty,$$

where $c(p) = p(1 - \cos(2\pi/p))$. Thus, the error in (17) for integer $p \geqslant 2$ is $O(\exp(-c(p)n))$ as $n \longrightarrow \infty$. Since $c(2) = 4$, $c(3) = 4.5$, and $c(p) \leqslant 4$ for $p \geqslant 4$, only the case $p = 3$ is worth considering as a computational alternative to method B1 (i.e. the case $p = 2$).

5. Computational Results. γ was computed to more than 30,100 decimal places using method B1 and a multiple-precision arithmetic package [9] on a Univac 1100/42. Three independent computations were performed, with $n = 17{,}332$ (using base 10,000 and 7,527 digits), $n = 17{,}357$ (base 65,535 and 6,260 digits), and $n = 17{,}387$ (base 65,536 and 6,271 digits). All three agreed to 30,100 decimal places, and the last two agreed to 30,141 decimal places. The computer time required for each computation was about 20 hours, much the same as for the 20,700 decimal place computation [7] using Sweeney's method on the same machine.

We also computed $G = \exp(\gamma)$ to more than 30,100 decimal places using the exponential routine in Brent's package [9] (with base 65,536 and 6,260 digits), and verified it by computing $\ln(G)$ by the Gauss-Salamin algorithm [6] (with base 10,000 and 7,550 digits). The rounded 30,100D values of γ and G are given in [10].

The first 29,200 partial quotients in the regular continued fractions for γ and G were computed and verified as in [7], [8]. Statistics on the distributions of the first 29,000 partial quotients are given in Table 1, with notation as in [7, Table 2]. A chi-squared test did not show any significant difference (at the 5% level) between the actual distributions and the distribution predicted by the Gauss-Kusmin theorem [13]. A table of the first 29,000 partial quotients for γ and G is given in [11].

TABLE 1

Distribution of first 29,000 partial quotients for γ and G

n	number of $q_i(\gamma) = n$	number of $q_i(G) = n$	expected number
1	12112	11992	12036.1
2	4809	4875	4927.8
3	2791	2760	2700.2
4	1727	1757	1707.9
5	1181	1168	1178.6
6	867	848	862.7
7	642	716	658.9
8	497	520	519.7
9	420	417	420.5
10	346	335	347.2
11–20	1624	1729	1694.1
21–50	1148	1103	1133.9
51–100	411	390	400.2
101–1000	378	349	370.4
>1000	47	41	41.8

From the continued fractions for γ and G we can improve the Theorem of [7, Section 7], where the lower bound on $|Q|$ was $10^{10{,}000}$.

THEOREM 1. *If γ or $G = P/Q$ for integers P and Q, then $|Q| > 10^{15{,}000}$.*

Acknowledgement. This work was initiated while the first author was visiting the Department of Electrical Engineering and Computer Sciences at the University of California, Berkeley. The computations were performed at the Australian National University.

Department of Computer Science
Australian National University
P. O. Box 4
Canberra, A. C. T. 2600, Australia

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

1. M. ABRAMOWITZ & I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Washington, D. C., 1964.

2. M. BEELER, R. W. GOSPER & R. SCHROEPPEL, "Hakmem," Memo No. 239, M.I.T. Artificial Intelligence Lab., Cambridge, Mass., 1972, pp. 70–71.

3. W. A. BEYER & M. S. WATERMAN, "Error analysis of a computation of Euler's constant," *Math. Comp.*, v. 28, 1974, pp. 599–604. MR **49** #6555.

4. W. A. BEYER & M. S. WATERMAN, "Decimals and partial quotients of Euler's constant and ln 2," UMT **19**, *Math. Comp.*, v. 28, 1974, p. 667. Errata: *Math. Comp.*, MTE **549**, v. 32, 1978, pp. 317–318.

5. R. P. BRENT, "The complexity of multiple-precision arithmetic," *Complexity of Computational Problem Solving* (R. S. Anderssen and R. P. Brent, Eds.), Univ. of Queensland Press, Brisbane, 1976, pp. 126–165.

6. R. P. BRENT, "Multiple-precision zero-finding methods and the complexity of elementary function evaluation," *Analytic Computational Complexity* (J. F. Traub, Ed.), Academic Press, New York, 1976, pp. 151–176. MR **52** #15938, **54** #11843.

7. R. P. BRENT, "Computation of the regular continued fraction for Euler's constant," *Math. Comp.*, v. 31, 1977, pp. 771–777. MR **55** #9490.

8. R. P. BRENT, "γ and $\exp(\gamma)$ to 20700D and their regular continued fractions to 20000 partial quotients," UMT 1, *Math. Comp.*, v. 32, 1978, p. 311.

9. R. P. BRENT, "A Fortran multiple-precision arithmetic package," *ACM Trans. Math. Software*, v. 4, 1978, pp. 57–70.

10. R. P. BRENT, "Euler's constant and its exponential to 30,100 decimals," Computing Research Group, Australian National University, Sept. 1978. Submitted to *Math. Comp.* UMT file.

11. R. P. BRENT & E. M. McMILLAN, "The first 29,000 partial quotients in the regular continued fractions for Euler's constant and its exponential," submitted to *Math. Comp.* UMT file.

12. J. W. L. GLAISHER, "History of Euler's constant," *Messenger of Math.*, v. 1, 1872, pp. 25–30.

13. A. YA. KHINTCHINE (A. JA. HINČIN), *Continued Fractions*, 3rd ed., (English transl. by P. Wynn), Noordhoff, Groningen, 1963. MR **28** #5038.

14. G. F. B. RIEMANN, "Zur Theorie der Nobili'schen Farbenringe," *Poggendorff's Annalen der Physik und Chemie*, Bd. 95, 1855, pp. 130–139. (Reprinted in *Bernhard Riemann's Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass*, Teubner, Leipzig, 1876, pp. 54–61.)

15. A. SCHÖNHAGE & V. STRASSEN, "Schnelle Multiplikation grosser Zahlen," *Computing*, v. 7, 1971, pp. 281–292.

16. D. W. SWEENEY, "On the computation of Euler's constant," *Math. Comp.*, v. 17, 1963, pp. 170–178. MR **28** #3522.

17. A. VAN DER POORTEN, "A proof that Euler missed–Apéry's proof of the irrationality of $\zeta(3)$," *Mathematical Intelligence*, v. 1, 1979, pp. 196–203.

18. H. WALL, *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948.

19. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, London, 1944.

A Proof that Euler Missed: Evaluating $\zeta(2)$ the Easy Way

Tom M. Apostol

R. Apéry [1] was the first to prove the irrationality of

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Motivated by Apéry's proof, F. Beukers [2] has given a shorter proof which uses multiple integrals to establish the irrationality of both $\zeta(2)$ and $\zeta(3)$. In this note we show that one of the double integrals considered by Beukers,

$$I = \int_0^1 \int_0^1 \frac{1}{1-xy}\,dxdy,$$

can be used to establish directly that $\zeta(2) = \pi^2/6$. This evaluation has been presented by the author for a number of years in elementary calculus courses, but does not seem to be recorded in the literature.

The relation between the foregoing integral and $\zeta(2)$ is obtained by expanding the integrand in a geometric series and integrating term by term. Thus, we have

$$I = \int_0^1 \int_0^1 \sum_{n=0}^{\infty} x^n y^n \,dxdy =$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{y^n}{n+1}\,dy = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \zeta(2).$$

Now we evaluate the integral another way and show that $I = \pi^2/6$. We simply rotate the coordinate axes clockwise through an angle of $\pi/4$ radians by introducing the change of variables

$$x = \frac{u-v}{\sqrt{2}},\ y = \frac{u+v}{\sqrt{2}}$$

so that $1 - xy = (2 - u^2 + v^2)/2$. The new region of integration in the uv-plane is a square with two opposite vertices at $(0,0)$ and $(\sqrt{2},0)$. Making use of the symmetry of this square about the u-axis we find

$$I = 4\int_0^{1/\sqrt{2}} \left(\int_0^u \frac{dv}{2-u^2+v^2} \right) du$$

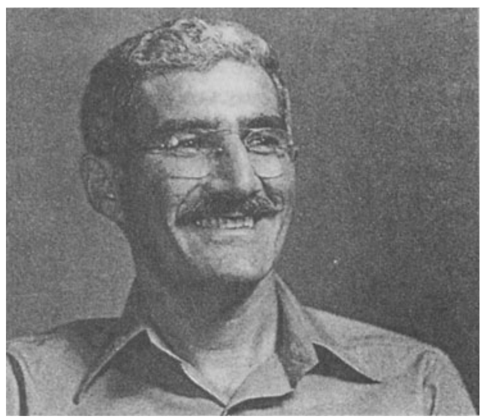
$$+ 4\int_{1/\sqrt{2}}^{\sqrt{2}} \left(\int_0^{\sqrt{2}-u} \frac{dv}{2-u^2+v^2} \right) du.$$

Since

$$\int_0^x \frac{dt}{a^2+t^2} = \frac{1}{a}\arctan\frac{x}{a}$$

we have

$$\int_0^u \frac{dv}{2-u^2+v^2} = \frac{1}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}}$$



Tom M. Apostol

and

$$\int_0^{\sqrt{2}-u} \frac{dv}{2-u^2+v^2} = \frac{1}{\sqrt{2-u^2}} \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}}$$

hence

$$I = 4 \int_0^{1/\sqrt{2}} \arctan \frac{u}{\sqrt{2-u^2}} \frac{du}{\sqrt{2-u^2}}$$

$$+ 4 \int_{1/\sqrt{2}}^{\sqrt{2}} \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}} \frac{du}{\sqrt{2-u^2}} = I_1 + I_2,$$

say. Put $u = \sqrt{2} \sin \theta$ in I_1 so that $du = \sqrt{2} \cos \theta \, d\theta = \sqrt{2-u^2}\, d\theta$, and $\tan \theta = u/\sqrt{2-u^2}$. This gives us

$$I_1 = 4 \int_0^{\pi/6} \theta \, d\theta = 2\left(\frac{\pi}{6}\right)^2.$$

In I_2 we put $u = \sqrt{2} \cos 2\theta$ so that

$$du = -2\sqrt{2} \sin 2\theta \, d\theta = -2\sqrt{2}\sqrt{1-\cos^2 2\theta}\, d\theta = -2\sqrt{2}\sqrt{1-u^2/2}\, d\theta = -2\sqrt{2-u^2}\, d\theta,$$

and

$$\frac{\sqrt{2}-u}{\sqrt{2-u^2}} = \frac{\sqrt{2}(1-\cos 2\theta)}{\sqrt{2-2\cos^2 2\theta}}$$

$$= \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \tan\theta,$$

hence

$$I_2 = 8 \int_0^{\pi/6} \theta \, d\theta = 4\left(\frac{\pi}{6}\right)^2.$$

Therefore $I = I_1 + I_2 = 6\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{6}$.

Note. Another evaluation of $\zeta(2)$ using double integrals in a less straightforward manner was given by F. Goldscheider [3] in response to a problem proposed by P. Stäckel. He considers the two double integrals

$$P = \int_0^1 \int_0^1 \frac{dx\,dy}{1-xy} \text{ and } Q = \int_0^1 \int_0^1 \frac{dx\,dy}{1+xy}$$

and shows first that $P - Q = \frac{1}{2}P$ so $P = 2Q$. On the other hand,

$$P + Q = \int_{-1}^{1} dy \int_0^1 \frac{dx}{1+xy}.$$

The substitution $u = y + \frac{1}{2}x(y^2-1)$ in the integral with respect to y converts this to

$$P + Q = \int_{-1}^{1} du \int_0^1 \frac{dx}{1+2ux+x^2}.$$

Now put $u = \cos\varphi$ so that $(\sin\varphi)/(1+2ux+x^2) = \frac{d}{dx}\left(\arctan\frac{x+\cos\varphi}{\sin\varphi}\right)$, hence

$$P + Q = \frac{1}{2}\int_0^{\pi} \varphi \, d\varphi = \frac{\pi^2}{4},$$

which, together with $P = 2Q$, implies $P = \pi^2/6$.

References

1. R. Apéry (1979) Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque 61:*11–13. Paris: Societé Mathématique de France.
2. F. Beukers (1979) A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. Lon. Math. Soc. 11:*268–272.
3. F. Goldscheider (1913) *Arch. Math. Phys. 20:*323–324.

California Institute of Technology
Pasadena, California 91125

Putting God Back In Math*

by Lynn O'Shaughnessy

While creationists' beliefs are being weighed by an Arkansas judge, a sister organization has evolved, if you will, hundreds of miles away in the hallowed halls of Emporia State University.

A bold collection of free-thinking Kansas heretics has decided to continue the work started by the creationists, who want the biblical explanation of the beginning of man taught in the schools. About 100 professors, students and a few publicity-shy Emporia ministers have formed the Institute of Pi Research.

Quite simply, the institute wants to put God back into mathematics. Or at least back into pi.

Pi is the symbol for 3.14159265 and on to infinity, that impossibly awkward number that is used to multiply the diameter of a circle to obtain the circle's circumference. The contemporary pi was discovered by Anaxagoras, an ancient Greek, who was sentenced to die for his efforts. But members of the institute are clamoring for the return of the pi used by architects of King Solomon's Temple. According to the Bible, the builders used a pi of 3 to construct parts of the majestic structure. Measurements revealing the use of pi equaling three are mentioned in 1 Kings 7:23 and 2 Chronicles 4:2.

There is absolutely no mention of 3.14 and its subsequent, non-repeating digits anywhere in the Bible, the professors note.

If a pi of 3 is good enough for the Bible, it is good enough for modern man, concludes the institute's founder, Samuel Dicks, a professor of medieval history. Mr. Dicks attributes a great deal of the modern malaise to the Godless pi, which he contends is "an atheistic concept promoted by secular humanists."

"To think that God in his infinite wisdom would create something as messy as this (3.14 and on) is a monstrous thought," Mr. Dicks concludes.

Are these pi patrons really on the level?

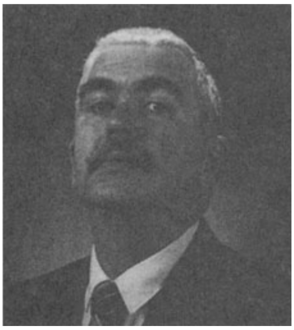
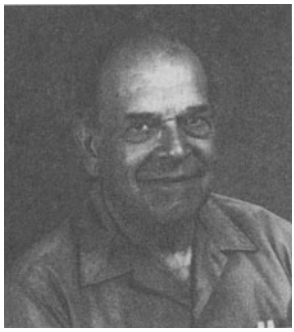
"I think we deserve to be taken as seriously as the creationists," Mr. Dicks replies bluntly, appearing to be not the least bit amused.

Only with a great deal of reluctance will members of the institute, with the motto, "Pi is 3 any way you slice it," admit they are full of baloney. It is their way of poking fun at the creationists by pointing out that even the Bible makes mistakes—or at least seems to.

In jest, the members say they want public schools to give the ancient pi equal time in the classrooms.

"If the Bible is right in biology, it's right in math," states Loren Pennington, an economic historian.

To lend some credibility to its crusade, the institute charmed an Emporia State mathematics professor, Marion Emerson, into its fold. Mr. Emerson, who realizes he is being exploited for the good of the institute,

Samuel Dicks (left) and **Marion Emerson** (right)

says he has paid a price among his colleagues for his conversion.

"Some of them think it's awful," he says.

Not content to stay within the ivory tower, the institute recently turned to the university's cable channel to spread the word. The show featured a string of professors explaining why their theory should not be discounted as "pi in the sky."

A physicist maintained that the glow of the constellation Perseus proves that pi 3 is correct. A historian blamed Arabs using oil barrels made with the lesser pi for the energy crisis, and Mr. Emerson, the token mathematician, produced a wheel he made using pi 3. The wheel, however, had six sides.

"Of course it makes for a bumpy wheel," he conceded.

The institute's convoluted theories have won converts at several universities, including the University of Kansas, Iowa State University and the Texas Tech math department. The group even sent a letter to President Reagan asking for his support. The reply is long overdue, but Mr. Dicks believes the leader is a closet believer because he said in a recent speech: "The pi(e) isn't as big as we think."

The group now is trying to lend its taped show to any sympathetic television station, but the response so far has been underwhelming. Refusing defeat, pi enthusiasts are sending letters to Midwestern legislatures asking them to give pi equal time. If they succeed they have plenty of other targets.

Says Mr. Pennington, the historian: "Our next step is to replace the insidious secular humanistic meter with the biblical cubit."

Editor's Note: *Those interested in joining the Institute for Pi Research can do so by writing to The Institute for Pi Research, 617 Exchange Street, Emporia, Kansas 66801, USA. There is no membership fee. The Institute encourages individuals to form their own local chapter of research associates.*

Readers are also invited to borrow a videotape on the Institute and its activities which is available on loan at no cost (except return postage). The videotape is suitable for mathematics clubs or departmental viewing. Write to Instructional Media Center, Emporia State University, Emporia, Kansas 66801 and specify either 1/2" VHS or 3/4" U-matic.

69.30 A remarkable approximation to π

In the first book of Kings, we find a description of the temple built by King Solomon in which the measurements of the various parts are stated. In Chapter 7 *v.* 23 the 'molten sea', a large basin containing water in which the priests washed their hands and feet before performing the rites, is described. The verse states (in the A.V.)

> "And he (Solomon) made a molten sea, ten cubits from one brim to the other: it was round all about, and its height was five cubits; and a *line* of thirty cubits did compass it round about."

From this verse it appears that π is taken to be 3, which is clearly inaccurate. Various explanations have been advanced by commentators; for example that the Bible is not interested in giving exact constructional details and so rounds to the nearest integer, or that the verse has taken into account the thickness of the material i.e. the ten cubits is an external diameter whereas the 30 cubits is the internal circumference. In this note I shall take a different approach, based on the Masoretic text and certain peculiarities that one finds in the original Hebrew of this verse.

Before discussing our problem it is necessary to note two points. Firstly, in the Masoretic text there are certain peculiarities of spelling found occasionally (*qethib*) which are read differently (*qere*). Both of these are found in Hebrew Bibles, the former in the text and the latter either in the margin or at the foot of the page. In our verse there is one of these; the word translated *line* is written (in transliteration) *qwh* but read *qw*. (Vowels are not considered as proper letters in Hebrew and so have not been included, the word in fact is read *qaw*.)

The second point concerns the numerical notation of the ancients. Both the ancient Greeks and Jews used letters to denote numbers, the first nine letters representing the units 1 to 9, the next nine 10, 20 up to 90, and the following letters 100, 200 etc. Numbers were represented by a combination of letters in no particular order, and so every word had a numerical equivalent. This is the basis of numerological interpretations of Scripture which seem very forced to those of us who do not use letters to represent numbers. However, to the ancients, a word was a number and so equating words of the same numerical value seemed quite natural.

If it is assumed that Scripture is Holy Writ, i.e. that it contains no meaningless items, the problem of the discrepancy between *qere* (*qw*) and *qethib* (*qwh*) is apparent. Why does our verse have an extra *h*? The following numerical calculation may prove interesting to the reader.

In Hebrew, the letters *q*, *w* and *h* have numerical values, 100, 6 and 5 respectively. Thus the word translated *line* in its written form has numerical value 111 whereas as read the value is 106. If we take the ratio of these numbers as a correcting factor for the apparent value of π as 3 and calculate

$$3 \times \frac{111}{106}$$

we obtain $3{\cdot}141509$ to 7 significant figures. This differs from the true value of π by less than 10^{-4} which is remarkable. In view of this, it might be suggested that this peculiar spelling is of more significance than a cursory reading might have suggested.

M. D. STERN

John Dalton Faculty of Technology, Manchester Polytechnic, Chester Street, Manchester M1 5GD

MATHEMATICS OF COMPUTATION
VOLUME 42, NUMBER 165
JANUARY 1984, PAGES 199-217

On a Sequence Arising in Series for π

By Morris Newman* and Daniel Shanks

Abstract. In a recent investigation of dihedral quartic fields [6] a rational sequence $\langle a_n \rangle$ was encountered. We show that these a_n are positive integers and that they satisfy surprising congruences modulo a prime p. They generate unknown p-adic numbers and may therefore be compared with the cubic recurrences in [1], where the corresponding p-adic numbers are known completely [2]. Other unsolved problems are presented. The growth of the a_n is examined and a new algorithm for computing a_n is given. An appendix by D. Zagier, which carries the investigation further, is added.

1. Introduction. The sequence $\{a_n\}$ that begins with

$$a_1 = 1, \quad a_2 = 47, \quad a_3 = 2488, \quad a_4 = 138799, \quad a_5 = 7976456, \quad a_6 = 467232200, \tag{1}$$

and which is defined below, is encountered in a set of remarkable convergent series for π. These are (see [6]):

$$\pi = \frac{1}{\sqrt{N}}\left(-\log|U| - 24\sum_{n=1}^{\infty}(-1)^n \frac{a_n}{n} U^n\right), \tag{2}$$

where N is a positive integer and $U = U(N)$ is a real algebraic number determined by N. Some of these series are remarkable because of their almost unbelievably rapid rates of convergence.

For example, for $N = 3502$, (2) converges at 79 decimals per term and its leading term, namely

$$-\frac{1}{\sqrt{3502}} \log U,$$

differs from π by less than $7.37 \cdot 10^{-82}$. In this case,

$$U = U(3502) = (2defg)^{-6}, \tag{3}$$

where

$$d = D + \sqrt{D^2 - 1}, \qquad e = E + \sqrt{E^2 - 1}, \qquad f = F + \sqrt{F^2 - 1}, \qquad g = G + \sqrt{G^2 - 1}, \tag{4}$$

for the quadratic surds

$$D = \tfrac{1}{2}(1071 + 184\sqrt{34}), \quad E = \tfrac{1}{2}(1553 + 266\sqrt{34}), \quad F = 429 + 304\sqrt{2}, \quad G = \tfrac{1}{2}(627 + 442\sqrt{2}). \tag{5}$$

In this example, the six a_n in (1) already give π correctly to over 500 decimals.

Received September 9, 1982.

1980 *Mathematics Subject Classification*. Primary 10D05, 10A35, 10B40.

*The work of the first author was supported by the National Science Foundation.

For $N = 2737$, and the more general

$$U = (-1)^N (2defg)^{-6}, \tag{6}$$

the quadratic surds

$$\begin{aligned} D &= \tfrac{1}{2}(621 + 49\sqrt{161}), & E &= \tfrac{1}{4}(321 + 25\sqrt{161}), \\ F &= \tfrac{1}{4}(393 + 31\sqrt{161}), & G &= \tfrac{1}{4}(2529 + 199\sqrt{161}), \end{aligned} \tag{7}$$

and (4) unchanged, define its negative value of $U(2737)$. Now (2) converges at only 69 decimals per term. See [6] for other examples of even and odd N, and the corresponding positive and negative values of U, where (2) also converges very rapidly.

The definition given in [6] of a_n is rather complicated. We have a relation

$$U = V \prod_{n=1}^{\infty} (1 + V^n)^{24} \tag{8}$$

between our $U = U(N)$ and the number

$$V = V(N) = (-1)^N e^{-\pi\sqrt{N}}. \tag{9}$$

The inversion of (8) gives V as a power series in U:

$$V = \sum_{n=1}^{\infty} (-1)^{n-1} c_n U^n \tag{10}$$

that begins with $c_1 = 1$, $c_2 = 24$, $c_3 = 852, \ldots$. Now, in the power series for

$$\log\left\{ \prod_{n=1}^{\infty} (1 + V^n) \right\} = V + \frac{V^2}{2} + \cdots, \tag{11}$$

substitute (10), and thereby define a_n recursively by

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{a_n}{n} U^n = \log\left\{ \prod_{n=1}^{\infty} (1 + V^n) \right\}. \tag{12}$$

Then, the logarithm of (8) gives us (2).

In [6], only the six coefficients in (1) were given, since they were computed by hand, a tedious operation. (The original a_n so computed contained an error which was discovered when R. Brent kindly attempted to verify (2) for $N = 3502$ to the aforementioned 500 decimals.) Clearly, the a_n are best calculated using a digital computer. The first 100 values of a_n and c_n were so computed in about 8 minutes. The first 50 values of a_n and c_n are given in Tables 1 and 2.

2. Properties of a_n. A. We observe that all a_n in Table 1 are positive integers. It was obvious from the recursion above that the a_n are rational but not that they are positive and integral. However, we prove below that

$$24a_n \text{ is the coefficient of } x^n \text{ in } \prod_{k=1}^{\infty} (1 + x^{2k-1})^{24n}, \tag{13}$$

which implies that a_n is a positive integer.

B. We observe that all a_n in Table 1 satisfy

$$a_n \text{ is odd if and only if } n \text{ is a power of 2.} \tag{14}$$

This unexpected result is reminiscent of C. R. Johnson's conjecture for the parity of the number of subgroups of the classical modular group of a given index N, see [7]. That conjecture was proved by Stothers and, independently, by A. O. L. Atkin. The present observation (14) is proved below.

C. A striking paradox about this proven (14) for the parity of a_n is this: As presented above, the c_n in (10) would appear to constitute a simpler sequence than our a_n in (12), since its definition is much more direct. Nonetheless, we have been unable to determine the parity of c_n. In Table 2 one readily observes that

(14a) $\quad c_n$ is odd only when $n = 8k + 1$ and is odd if $k = 0, 1, 2, 4, 6$.

But what are these k? We do not know, and do not even have a conjecture for the parity of c_n.

It is easy to prove (14a) and to compute c_n modulo 2. The parity of c_n appears to be random with increasing k just as is the parity of the unrestricted partition function $p(n)$. (See [8] for the latter.) As for the claim above that we have a paradox here, see Zagier's comment in the appendix.

D. A second, more important paradox concerns a_n modulo 3. We conjectured

$$a_n \not\equiv 0 \bmod 3 \tag{15}$$

for all n. While (15) appears simpler than (14), we did not prove it. Every positive integer n has a unique representation

$$n = 3^k(3m \pm 1) \tag{16}$$

with nonnegative k, m. A stronger conjecture than (15) is

$$a_{3^k(3m \pm 1)} \equiv \pm 1 \bmod 3. \tag{17}$$

For greater clarity, let us rewrite (17) as follows:

$$a_{3m+1} \equiv 1 \bmod 3, \tag{18a}$$

$$a_{3m-1} \equiv -1 \bmod 3, \tag{18b}$$

$$a_{3m} \equiv a_m \bmod 3. \tag{18c}$$

These are clearly equivalent to (17). We did not prove the *simple-looking* (18a) and (18b). The more *subtle-looking* (18c) we did prove; it is a simple corollary of a much more general congruence given in E below.

We did verify (17) up to $a_{143} \equiv -1 \bmod 3$ by computer, and we both believed it to be true. After we finished the first version of this paper, we showed the conjecture to D. Zagier, and, as we expected, he proved it. See the appendix.

E. The important general congruence alluded to above, and proved below, is

$$a_{mp^k} \equiv a_{mp^{k-1}} \bmod p^k, \tag{19}$$

valid for every prime p and all positive integers m and k. For $k = 1$ this gives us

$$a_{mp} \equiv a_m \bmod p \tag{20}$$

and (18c) is obviously the case $p = 3$.

Congruence (20) is computationally useful. For example, what is a_{94} modulo 94? Since

$$a_{2\cdot 47} \equiv a_2 = 47 \bmod 47,$$

we have $a_{94} \equiv 0 \bmod 47$. But also $a_{94} \equiv 0 \bmod 2$, by (14). Therefore $a_{94} \equiv 0 \bmod 94$. Similarly, we can evaluate a_{2p} modulo $2p$ for any prime p, and in particular we see that, for any prime p,

$$a_{2p} \not\equiv 1 \bmod 2p. \tag{21}$$

F. The choice $m = 1$ in (20) gives us

$$a_p \equiv a_1 \equiv 1 \bmod p, \tag{22}$$

which we call the *Fermat Property*. It is a necessary condition for primality. Of course, we ask: Is

$$a_n \equiv 1 \bmod n, \qquad n > 1, \tag{23}$$

a *sufficient* condition for primality?

We have just seen in (21) that $n = 2p$ can never satisfy (23). But consider

$$a_3 = 2488 = 3 \cdot 829 + 1.$$

Since 829 is prime, we have by (20) that

$$a_{2487} \equiv a_3 \equiv 1 \bmod 829,$$

and similarly

$$a_{2487} \equiv a_{829} \bmod 3.$$

But $829 = 3m + 1$, and since (18a) is now true, we also have

$$a_{2487} \equiv 1 \bmod 3. \tag{24}$$

Then (23) holds for the composite $2487 = 3 \cdot 829$. So (23) is not a sufficient condition for primality. Even if it were, it would not be a *practical* test for primality. The calculation of a_n modulo n requires at least $O(n)$ operations by any algorithm known to us.

G. We return to (19) and specialize in a different direction; $m = 1$ gives us

$$a_{p^k} \equiv a_{p^{k-1}} \bmod p^k. \tag{25}$$

Fix p and consider the sequence

$$\{a_{p^k} \text{ modulo } p^k\}, \qquad k = 1, 2, 3, \ldots. \tag{26}$$

If we write these numbers to the base p, (25) guarantees that each time k is increased by 1, and we add one more p-adic digit on the left, *all the earlier p-adic digits on the right remain unchanged*. Thus, for each p, the sequence (26) defines a p-adic number.

For example, for $p = 2$, (26) begins (in decimal) as 1, 3, 7, 15, 15, 47, ..., and so we have the 2-adic number (reading from right to left)

$$\ldots 1\,0\,0\,0\,1\,0\,1\,1\,1\,1\,. \quad \text{(base 2)}.$$

Similarly, for $p = 3$ and 5, we have

$$\ldots 0\,1\,1\,1\,. \quad \text{(base 3)}$$
$$\ldots 4\,1\,1\,. \quad \text{(base 5)}.$$

But what are these p-adic numbers? We do not know. Are they algebraic or transcendental? We do not know. Contrast this ignorance with the situation in I below.

We do have, for every p,

$$a_{p^2} \equiv 1 + p \bmod p^2, \tag{27}$$

so the first two p-adic digits on the right are both 1. The first 1 follows from the Fermat Property (22) but the second 1 does *not* follow from the general congruence (19), and again contrasts with the situation in I below. This (27) was first proved by our colleague L. Washington. Our proof below is different.

Perhaps we should note that the sequence

$$\{a_{p^k}\}, \qquad k = 1, 2, 3, \ldots \tag{28}$$

defines the same p-adic number that (26) does. The latter looks a little simpler since it adds exactly one p-adic digit each time.

H. After we discovered (18c), we were inspired to generalize it to (19) because of a recent paper [1] concerning some entirely different sequences; namely, a doubly infinite set of cubic recurrences. It suffices for our discussion here to examine only one of these recurrences. Let

$$A(1) = 1, \quad A(2) = 1, \quad A(3) = 4, \quad A(n+3) = A(n+2) + A(n). \tag{29}$$

We have [1]

$$A(mp^k) \equiv A(mp^{k-1}) \bmod p^k \tag{30}$$

just as before. So we also have the Fermat Property and p-adic numbers defined by

$$\{A(p^k) \text{ modulo } p^k\}. \tag{31}$$

I. But the $A(n)$ are nonetheless quite different than the a_n. First, since

$$A(4) \equiv 1 \bmod 4, \qquad A(9) \equiv 4 \bmod 9,$$

(27) does not hold, and the second p-adic digit is not invariant. Second, we can identify the p-adic numbers (31). For example, for $p = 2$, we now have

$$\ldots 1\,0\,0\,1\,0\,1\,. = x \quad (\text{base } 2).$$

Squaring this, it is easy to show that

$$x^2 + x + 2 = 0,$$

and so x is one of the 2-adic numbers

$$\tfrac{1}{2}(-1 \pm \sqrt{-7}).$$

In fact, for every p, (31) is an abelian algebraic integer; see [1], [2].

The evaluation of these algebraic integers is of much algorithmic interest and is also of much mathematical interest since, e.g., it leads to new ideas in cyclotomy; see [5]. But more to the present investigation, this p-adic approach enables one to solve problems about $A(n)$ that were previously intractable, as in [2].

One might hope that the determination of the p-adic numbers in (26) would be equally valuable for a_n. Presumably, the distinctive property (27) plays a role in their arithmetic characterization. We commend these problems to the reader.

J. If we generalize (31) to

$$\{A(mp^k) \text{ modulo } p^k\} \tag{32}$$

for p fixed, and m any integer, we define a set of p-adic numbers. This set is finite, and each of these numbers is either an algebraic conjugate of that for $m = 1$, or is a related abelian integer of a lower degree.

Similarly, in the present investigation,

$$\{a_{mp^k} \text{ modulo } p^k\}, \tag{33}$$

with m a fixed positive integer, defines a p-adic number for each m generalizing (26). But we have not seriously examined this set of p-adic numbers and know little about it.

K. Let us note some other differences between $A(n)$ and a_n. The former sequence is periodic modulo p for every p, but the latter is not. The former is a reversible recurrence, and so we have

$$A(0) = 3, \quad A(-1) = 0, \quad A(-2) = -2, \ldots,$$

while a_n is not defined for $n < 1$. The value of $A(n)$ modulo n can be computed in $O(\log n)$ operations. We know of no algorithm that is that efficient for our a_n modulo n. We have

$$A(n) = \alpha^n + \beta^n + \gamma^n$$

for known values of α, β, γ while we know of no explicit formula for a_n.

Since a_n and $A(n)$ are so very different, it is all the more surprising that they have, in (19) and (30), an elaborate, important property in common. We call this property the *generalized p-adic law*.

Naturally, one asks: Can one characterize all sequences $\alpha(n)$ that satisfy this law? This may already be known.

Zagier also comments on the comparison of a_n and $A(n)$.

L. We now turn to the growth of the a_n. In the analytic function $V(U)$ in (10) the closest singularity to the point $U = 0$, $V = 0$ is the branch point at $U = -\frac{1}{64}$, $V = -e^{-\pi}$; see [6, Appendix B]. Therefore, the radius of convergence of (10) is $\frac{1}{64}$, and it follows that

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 64. \tag{34}$$

In the substitution of (10) into (11), the growth of the a_n is dominated by the growth of the c_n, and it may be shown that also

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 64. \tag{35}$$

M. We therefore have the asymptotic formula

$$\log a_n \sim n \log 64, \tag{36}$$

but an asymptotic formula for a_n itself was lacking. We expected that

$$a_n \sim \frac{C}{n^\beta}(64)^n, \qquad C, \beta \text{ constants}, \tag{37}$$

but we did not prove it.

In the Appendix, Zagier determines that $\beta = \frac{1}{2}$ (as we expected), and that

$$C = \frac{\sqrt{\pi}}{12}\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^2.$$

Further, he gives two more terms in the asymptotic series, and thereby enables one to estimate a_n very accurately.

Prior to this work we had already found the inequalities (38) below, and since these are of some interest, we include the derivation.

$$\frac{1}{3\sqrt{n}}(63.87)^n < 24a_n < (64)^n. \tag{38}$$

N. Zagier's evaluation of C suggests the following sequel. This C is closely related to the famous lemniscate constant, and, in retrospect, some such result should have been expected. In [6], the group $C(4)$ was basic, and therefore our sequence a_n is intimately connected with this group. But the lemniscate constant often arises with $C(4)$; for example, $Q(\sqrt{-14})$ has $C(4)$ as its class group, and, in counting numbers of the form $u^2 + 14v^2$, the lemniscate constant enters via the constant β_{14} referred to in [9, Eq. (5)].

Now, in the modular group, one encounters $\rho = \sqrt[3]{1}$ as well as $i = \sqrt[4]{1}$, and therefore $C(3)$ as well as $C(4)$, and [6, p. 405] specifically refers to analogous theories for $C(3)$ and $C(6)$. So, there may well be other sequences analogous to a_n that would arise in this way. We have not yet studied this.

In the quadratic form $4u^2 + 2uv + 7v^2$ we do have class number 3, and in counting numbers of *this* form one does indeed encounter a constant which contains $\Gamma(1/6)$ instead of $\Gamma(1/4)$; see [10, Eq. (5)]. If there are such sequences, one would expect Zagier's calculations to have analogues here.

3. Proofs of the Theorems. The function

$$y = x\prod_{k=1}^{\infty}(1 + x^k)^{24}$$

defined in (8) (the variable names have been changed) is of importance in the theory of the elliptic modular functions. y is a Hauptmodul for the congruence subgroup $\Gamma_0(2)$ of the classical modular group Γ, considered as a function of the complex variable τ, where $x = \exp(2\pi i\tau)$, $\operatorname{im}\tau > 0$. (See [4] for a good general reference on this topic.) However, all that is required here is a formal study of the coefficients of y^n, where n is an integer. In this connection certain complex integral formulas associated with the inversion of a function of the form $y = x + b_2x^2 + \cdots$ (or the reversion of a power series of this form) will be used freely. These are classical, and may be found for example in the book by Behnke and Sommer [3].

The numbers a_n are defined by the relationship (12), rewritten as

$$\log y - \log x = 24\sum_{n=1}^{\infty}(-1)^{n-1}\frac{a_n}{n}y^n. \tag{39}$$

Differentiating (39) with respect to y, and then multiplying by y, we have that

$$1 - \frac{y}{x}\frac{dx}{dy} = 24\sum_{n=1}^{\infty}(-1)^{n-1}a_n y^n. \tag{40}$$

Hence for some suitable positive number r, we have that

$$(-1)^{n-1}24a_n = \frac{1}{2\pi i}\int_{|y|=r}\left(1 - \frac{y}{x}\frac{dx}{dy}\right)y^{-n-1}\,dy,$$

so that, for $n \geqslant 1$,

$$(-1)^{n-1} 24 a_n = \frac{1}{2\pi i} \int_{|y|=r} \left(\frac{1}{x} \frac{dx}{dy} \right) y^{-n} \, dy.$$

This implies that, for some suitable positive number r',

$$\begin{aligned}(-1)^{n-1} 24 a_n &= \frac{1}{2\pi i} \int_{|x|=r'} \frac{1}{x} y^{-n} \, dx \\ &= -\frac{1}{2\pi i} \int_{|x|=r'} x^{-n-1} \prod_{k=1}^{\infty} (1 + x^k)^{-24n} \, dx.\end{aligned}$$

It follows that, for $n \geqslant 1$, $(-1)^n \cdot 24 a_n$ is the coefficient of x^n in the power series expansion of $\prod_{k=1}^{\infty}(1 + x^k)^{-24n}$. If we use the fact that

$$\prod_{k=1}^{\infty} (1 + x^k)^{-1} = \prod_{k=1}^{\infty} (1 - x^{2k-1}),$$

and replace x by $-x$, we obtain (13) and write

THEOREM 1. *The number* $24 a_n$ *defined by* (39) *is the coefficient of* x^n *in the infinite product* $\prod_{k=1}^{\infty}(1 + x^{2k-1})^{24n}$.

This proves immediately that these numbers are positive, but a small additional discussion is required to prove that a_n is an integer (because of the factor 24).

We set

$$\prod_{k=1}^{\infty} (1 + x^{2k-1})^{24n} = \sum_{k=0}^{\infty} C_n(k) x^k, \tag{41}$$

so that

$$24 a_n = C_n(n). \tag{42}$$

We find by logarithmic differentiation of (41) and known properties of Lambert series that the integers $C_n(k)$ satisfy the recurrence formula

$$k C_n(k) = 24n \sum_{s=1}^{k} (-1)^{s-1} \sigma^*(s) C_n(k-s), \qquad k \geqslant 1, \tag{43}$$

where $C_n(0) = 1$, and

$$\sigma^*(s) = \sum_{\substack{d \mid s \\ d \text{ odd}}} d. \tag{44}$$

For the choice $k = n$, (42) and (43) imply that

$$a_n = \sum_{s=1}^{n} (-1)^{s-1} \sigma^*(s) C_n(n-s), \tag{45}$$

which shows at once that a_n is an integer. That is, we have proved

THEOREM 2. *The numbers* a_n *defined by* (39) *are positive integers.*

Our next objective is to prove (14), which states the remarkable fact that a_n is odd if and only if n is a power of 2. For this purpose we need to know the parity of the function $\sigma^*(s)$, defined by (44). We have the following simple lemma, whose proof we omit:

LEMMA 1. *The function $\sigma^*(s)$ is odd if and only if s is a square, or twice a square.*

This lemma and formula (45) imply that

$$a_n \equiv \sum C_n(n - s^2) + \sum C_n(n - 2s^2) \bmod 2. \tag{46}$$

In the first summation, s runs over all positive integers such that $s^2 \leqslant n$, and, in the second summation, s runs over all positive integers such that $2s^2 \leqslant n$.

First note that

$$(1 + u)^{16} \equiv (1 + u^2)^8 \bmod 16,$$

where the congruence means that coefficients of corresponding powers of u are congruent. This readily implies that

$$\prod_{k=1}^{\infty} (1 + x^{2k-1})^{48n} \equiv \prod_{k=1}^{\infty} (1 + x^{4k-2})^{24n} \bmod 16,$$

which in turn implies that

$$24a_{2n} \equiv 24a_n \bmod 16,$$

$$a_{2n} \equiv a_n \bmod 2. \tag{47}$$

Congruence (47) is the special case $p = 2$ of the general congruence (20), to be proved later.

Thus, in order to determine the parity of a_n, it is only necessary to choose n odd, which we now do. If we note that

$$\prod_{k=1}^{\infty} (1 + x^{2k-1})^{24n} \equiv \prod_{k=1}^{\infty} (1 + x^{16k-8})^{3n} \bmod 2,$$

we see that $C_n(k)$ is even except possibly when $k \equiv 0 \bmod 8$. Then (46) implies that

$$a_n \equiv \sum_{n - s^2 \equiv 0 \bmod 8} C_n(n - s^2) + \sum_{n - 2s^2 \equiv 0 \bmod 8} C_n(n - 2s^2) \bmod 2. \tag{48}$$

But n is odd. Thus the second sum in (48) is empty, and in the first sum s must be odd, implying that $n \equiv 1 \bmod 8$. Put $n = 8t + 1$. Then

$$a_{8t+1} \equiv \sum_{s \text{ odd}} C_{8t+1}(8t + 1 - s^2) \equiv \sum C_{8t+1}\left(8\left(t - \frac{r^2 + r}{2}\right)\right) \bmod 2, \tag{49}$$

where r runs over all nonnegative integers such that $\frac{1}{2}(r^2 + r) \leqslant t$.

We have

$$\sum_{k=0}^{\infty} C_{8t+1}(k)x^k = \prod_{k=1}^{\infty} (1 + x^{2k-1})^{24(8t+1)}$$
$$\equiv \prod_{k=1}^{\infty} (1 + x^{8k-16})^{3(8t+1)} \bmod 2,$$

so that

$$\sum_{k=0}^{\infty} C_{8t+1}(8k)x^k \equiv \prod_{k=1}^{\infty} (1 + x^{2k-1})^{24t+3} \bmod 2.$$

Thus

$$\prod_{k=1}^{\infty}(1+x^{2k-1})^{-3}\cdot\sum_{k=0}^{\infty}C_{8t+1}(8k)x^k\equiv\prod_{k=1}^{\infty}(1+x^{2k-1})^{24t}\bmod 2.$$

Now use the Jacobi identity

$$\prod_{k=1}^{\infty}(1-x^k)^3=\sum_{k=0}^{\infty}(-1)^k(2k+1)x^{(k^2+k)/2}$$

and the fact that

$$\prod_{k=1}^{\infty}(1+x^{2k-1})^{-3}\equiv\prod_{k=1}^{\infty}(1-x^k)^3\bmod 2.$$

Then

$$\sum_{k=0}^{\infty}x^{(k^2+k)/2}\sum_{k=0}^{\infty}C_{8t+1}(8k)x^k\equiv\prod_{k=1}^{\infty}(1+x^{2k-1})^{24t}\bmod 2.$$

It follows that

$$\sum C_{8t+1}\left(8\left(t-\tfrac{1}{2}(r^2+r)\right)\right)$$

is congruent modulo 2 to the coefficient of x^t in $\prod_{k=1}^{\infty}(1+x^{2k-1})^{24t}$. But this coefficient is odd if and only if $t=0$ (it is divisible by 24 otherwise, since then the coefficient is $24a_t$). It follows from (49) that a_{8t+1} is odd if and only if $t=0$.

Summarizing, we have proved

THEOREM 3. *The number a_n is odd if and only if n is a power of* 2.

Our next objective is to prove (19). If p is a prime and k a positive integer, then

$$(1+u)^{p^k}\equiv(1+u^p)^{p^{k-1}}\bmod p^k,$$

where once again the congruence is understood to hold for corresponding powers of u. It follows that if m is any positive integer,

$$(1+u)^{mp^k}\equiv(1+u^p)^{mp^{k-1}}\bmod p^k. \tag{50}$$

Formula (50) now implies that

$$\prod_{s=1}^{\infty}(1+x^{2s-1})^{24mp^k}\equiv\prod_{s=1}^{\infty}(1+x^{2ps-p})^{24mp^{k-1}}\bmod p^{k+\delta}, \tag{51}$$

where

$$\delta=\begin{cases}3, & p=2,\\ 1, & p=3,\\ 0, & p>3.\end{cases}$$

Comparing coefficients of x^{mp^k} on both sides of (51), we find that

$$24a_{mp^k}\equiv 24a_{mp^{k-1}}\bmod p^{k+\delta},$$

so that, for all primes p,

$$a_{mp^k}\equiv a_{mp^{k-1}}\bmod p^k.$$

That is, we have proved

THEOREM 4. *Let p be a prime, m, k positive integers. Then*

$$a_{mp^k} \equiv a_{mp^{k-1}} \bmod p^k. \tag{52}$$

We now go on to formula (27), which reads

$$a_{p^2} \equiv 1 + p \bmod p^2, \qquad p \text{ prime}.$$

Since (52) implies that

$$a_{p^2} \equiv a_p \bmod p^2,$$

it is sufficient to prove that

$$a_p \equiv 1 + p \bmod p^2, \qquad p \text{ prime}.$$

We may assume that $p > 3$, since the cases $p = 2, 3$ may be verified directly. We have

$$(1+u)^p = 1 + u^p + \sum_{r=1}^{p-1} \binom{p}{r} u^r \equiv 1 + u^p + p \sum_{r=1}^{p-1} \frac{(-1)^{r-1}}{r} u^r \bmod p^2,$$

so that

$$\frac{(1+u)^p}{1+u^p} \equiv 1 + p \sum_{r=1}^{p-1} \frac{(-1)^{r-1}}{r} \frac{u^r}{1+u^p} \bmod p^2.$$

Now choose $u = x^{2k-1}$, product for $k = 1, 2, 3, \ldots$, and raise both sides to the 24th power. We get

$$\prod_{k=1}^{\infty} \frac{(1+x^{2k-1})^{24p}}{(1+x^{2kp-p})^{24}} \equiv 1 + 24p \sum_{\substack{1 \leq r \leq p-1 \\ k \geq 1}} \frac{(-1)^{r-1}}{r} \frac{x^{r(2k-1)}}{1+x^{p(2k-1)}} \bmod p^2,$$

$$\prod_{k=1}^{\infty} (1+x^{2k-1})^{24p} \equiv \prod_{k=1}^{\infty} (1+x^{2kp-p})^{24} \cdot S \bmod p^2,$$

where

$$S = 1 + 24p \sum_{\substack{1 \leq r \leq p-1 \\ k \geq 1}} \frac{(-1)^{r-1}}{r} \frac{x^{r(2k-1)}}{1+x^{p(2k-1)}}.$$

Comparing coefficients of x^p, we find that

$$24a_p \equiv 24 + 24p \bmod p^2,$$

so that

$$a_p \equiv 1 + p \bmod p^2.$$

We state this result as L. Washington's

THEOREM 5. *Let p be a prime. Then*

$$a_{p^2} \equiv a_p \equiv 1 + p \bmod p^2.$$

We note that these congruences may be strengthened, if desired. A slightly more involved proof along the same lines will show for example that

$$a_{p^k} \equiv a_{p^{k-1}} + p^k \bmod p^{k+1}. \tag{53}$$

However, it does not seem possible to determine a_{p^k} modulo p^k precisely, except for small values of k.

We now turn to the inequalities of (38). Theorem 1 implies that $24a_n$ is equal to

$$\sum \binom{24n}{n_1}\binom{24n}{n_3}\binom{24n}{n_5}\cdots \tag{54}$$
$$n_1 + 3n_3 + 5n_5 + \cdots = n, \qquad n_i \geq 0.$$

Since $n_1 = n$, $n_3 = n_5 = \cdots = 0$ is a permissible choice, we find that

$$24a_n \geq \binom{24n}{n}. \tag{55}$$

A simple application of Stirling's formula gives

$$24a_n > \frac{1}{3\sqrt{n}}\left(\frac{24^{24}}{23^{23}}\right)^n > \frac{1}{3\sqrt{n}}(63.87)^n,$$

proving the lower bound.

For the upper bound, we have that if r is any number such that $0 < r < 1$, then

$$24a_n = \frac{1}{2\pi i}\int_{|x|=r} g(x)^n \frac{dx}{x},$$

where

$$g(x) = \frac{1}{x}\prod_{k=1}^{\infty}(1 + x^{2k-1})^{24}.$$

It follows that

$$24a_n \leq g(r)^n. \tag{56}$$

Now the function $g(x)$ is an entire modular function on the congruence subgroup $\Gamma_0(4)$ of Γ, considered as a function of the complex variable τ, where $x = \exp(2\pi i\tau)$, and $\operatorname{im}\tau > 0$. It is easy to show by the transformation formulae for $g(x)$ that

$$g(e^{-\pi}) = 64.$$

Choosing $r = e^{-\pi}$ in (56) gives

$$24a_n < 64^n,$$

which is the desired upper bound.

Summarizing, we have proved

THEOREM 6. *The number a_n satisfies the inequalities*

$$\frac{1}{3\sqrt{n}}(63.87)^n < 24a_n < 64^n.$$

4. Computation. The first dozen or so coefficients a_n were initially computed using the complicated formula (40). After Theorem 1 was discovered, recurrence formula (43) was used. The coefficients $\sigma^*(s)$ are small and easily computed, and (43) is convenient and simple to implement. The practical programming problems that arise are consequences of the fact that the a_n become large. This is best handled by computing them modulo a sufficient number of large primes, and then using the Chinese Remainder Theorem to recover their exact values.

The coefficients c_n were computed by means of a general program that reverts a power series $y = x + \cdots$. This program computes the coefficients of the powers of y and then solves a triangular system of equations to determine the desired coefficients in the reverted power series $x = y + \cdots$. Once again, residue arithmetic must be used, since the coefficients c_n also become large.

The computation of a_n modulo m, where some prime factors of m are small, is awkward (if not impossible) using formula (43), because of the necessity of the division there. The alternative here is to generate $u = \prod_{k=1}^{\infty}(1 + x^{2k-1})$ modulo $24m$ and then to form u^{24n} by successive squarings modulo $24m$. This is time-consuming and becomes impractical if n is only moderately large; say $n = 1000$.

We note that multiprecision computation (rather than modular computation) would be even more time-consuming. In any case there is very little point in calculating the exact value of a_{1000}, say, since it is a number of some 1800 decimal digits.

TABLE 1. a_n, $n = 1(1)50$

1	1
2	47
3	2488
4	138799
5	7976456
6	467232200
7	27736348480
8	1662803271215
9	100442427373480
10	6103747246289272
11	372725876150863808
12	22852464771010647496
13	1405886026610765892544
14	86741060172969340021952
15	5365190340823180439326208
16	332577246704242939511725615
17	20655377769544663820919905000
18	1285027807539621869480480977880
19	80066610886753513409821525593280
20	4995543732566526565060187887772024
21	312067903389730540416319245145039936
22	19516459352109724206910675815791735872
23	1221787478073080268912138739833447254528
24	76558881238278398609546573647116818306504
25	4801399849802188285872546222298724299377856
26	301358552889212442951924121355286655092791360
27	18928524108186660537926825906927824486973500672 0
28	1189719542605042010945455887482239233732751142080
29	74824958481405101799295401923145498080031496317440
30	4708731584940969251488540213411242070133095720768000
31	296483323638911778793802123013217365155428610625064960
32	18677571039055424502042574350078071038555962934810664495
33	1177200955467256907707767829606512556434525730284672082280
34	74229820742983998523807878655148660941364964757170232076440
35	4682657672641000613276353688819373189604961982881761635174080
36	295516785862704112676947743865736338547152307208873658542187480
37	18656838683258040776726836797753969443154060448210951169536087360
38	1178287550937265649491805466460363896744099593833261406542090821440
39	74441259433548426510664621182339422182178689134172479673100078686720
40	4704546876230537649051669928635037299315044055233418643313504347890040
41	297410696380227510473584821926459754598587577997951261584830786025989440
42	18807176292551896455842616399574167855948518855982280636468413444438841280
43	1189632505858785415664268185396568316810012962868095237190924015678644805120
44	75269436592700558660145646818728077669744495747378078929068335671082935790496 0
45	4763606735739477078702262301306618196904330454342036172567804617626114845601280
46	301550219357655322958904198748139655940272138707157414253352878909612335524237056 0
47	1909349110538243794796143059549600905192746979460012460737459486229780997349742592 0
48	120922942183312821453216523190439802408845653257918467337476570220452538689270958228 0
49	765994622221714882174695628075554448403298203759366456284285039675998425364037483926 40
50	48532494762795849430187525441355182058358236525693281040718085970999763022067776723822 72

TABLE 2. c_n, $n = 1(1)50$

1.	1
2.	24
3.	852
4.	35744
5.	1645794
6.	80415216
7.	4094489992
8.	214888573248
9.	11542515402255
10.	631467591949480
11.	35063515239394764
12.	1971043639046131296
13.	111949770626330347638
14.	6414671157989386260432
15.	370360217892318010055832
16.	21525284426246779936288192
17.	1258348271935918462435403307
18.	73942189694396970582980105352
19.	4364976407960556546884928368476
20.	258741036471764253091461517733856
21.	15394586990299636314282137771674830
22.	919051542126841276042022053610468752
23.	55036467624031911199129205093854619064
24.	3305113970018146870837951018822929583296
25.	198997564644299363614619190584670328932936
26.	12010095419986698523773417250172646465263808
27.	726447806449307612142334095641037351570840864
28.	44030338964408484455732048896063797435000101120
29.	2673788167993641289448328163141757626940496197160
30.	162657220544413978163790054177951326622909359275200
31.	9911527685383195721813290296878399721821791890405024
32.	604899283848988432022069057045272028344035971329679616
33.	36970837629844039304385084970877592615837024206916373053
34.	2262723529649336738110964266117808613673092565887151549624
35.	138664468558308431577618908119374772575631693607388403107204
36.	8508025994367861890277592274660883399661217762484511042274592
37.	522628821564568754438041506364388503224274143202783433146082586
38.	32138985548624371564064047392187046675586611595448962068083978800
39.	1978429759430649446757266681537394592324196828947816361679884306280
40.	121909076104562854936147780364667494353737124539846206817532045147200
41.	7518952236423651538428481416024822280758718735041665624856781401845142
42.	464157063121846868150595275179448760027913195093138271111529615837395088
43.	28677467647508968049978935619470366659282071479283246492919997795984278904
44.	1773241664402616710570230882425007538906213421415490637996700519568471249856
45.	109731314877402045883363217526258373371802193645670427761282465837822892310196
46.	6795384565685668272289146836919987952721991497880544929801024614700081667049312
47.	421118690078289455115442968174088626001358532117276172625513521520959714092751440
48.	26114944381531477954478272273365362544699925144997518688874107744442010809229803648
49.	1620524841254019270695075088632356841408000251247290974011208956749850387668408953895
50.	100621989558697666940849746551782896264800698167286014343658307743170090611911363941160

APPENDIX

By D. Zagier

Asymptotics and Congruence Properties of the a_n

In this appendix we prove an asymptotic formula and a congruence modulo 3 for the numbers a_n, assuming various more or less well-known facts from the theory of modular forms whose proofs can be found in standard textbooks on modular and elliptic functions (e.g. Lang's or Weil's).

Let τ denote a variable in the upper half-plane, $q = e^{2\pi i\tau}$, and $U(\tau) = q\prod(1 + q^n)^{24}$ (q and U were denoted by V and U in Section 1 and by x and y in Section 3). Then $U(\tau) = \Delta(2\tau)/\Delta(\tau)$, where $\Delta(\tau) = q\prod(1 - q^n)^{24}$ is the usual discriminant function, so U is a nowhere vanishing modular function on $\Gamma_0(2)$ and its logarithmic derivative

$$(1)\qquad f(\tau)=\frac{1}{2\pi i}\frac{U'(\tau)}{U(\tau)}=1+24\sum_{n=1}^{\infty}\sigma^*(n)q^n \qquad (\sigma^* \text{ as in } (44))$$

is a modular form of weight 2 on $\Gamma_0(2)$. The definition of a_n can be expressed as

$$(2)\qquad \frac{1}{f(\tau)}=1+24\sum_{n=1}^{\infty}(-1)^n a_n U(\tau)^n,$$

an identity valid in a neighborhood of $\tau=i\infty$ (it cannot be valid for all τ for which the series converges, since U is $\Gamma_0(2)$-invariant and f is not). From the formula for the number of zeros of a modular form, we see that $f(\tau)$ vanishes only at points τ which are $\Gamma_0(2)$-equivalent to $\tau_0=(1+i)/2$ (that f does vanish at τ_0 can be seen by applying the transformation equation of f to $\left(\begin{smallmatrix}1 & -1\\ 2 & -1\end{smallmatrix}\right)\in\Gamma_0(2)$), and (1) then shows that $\tau\to U(\tau)$ is locally biholomorphic except at these points. Hence the only singularity in (2) occurs at $U=U(\tau_0)=-1/64$, so to obtain the asymptotics of the a_n we must look at the Taylor series expansions of f and U near τ_0. In view of (1) and the equation $f(\tau_0)=0$, it will suffice for this to compute the derivatives $f^{(\nu)}(\tau_0)$ for $\nu\geqslant 1$.

Now the derivative of a modular form is not a modular form, but, if F is a modular form of weight k on a subgroup Γ of $SL(2, Z)$, then $F'-(\pi ik/6)E_2F$ is a modular form of weight $k+2$ on Γ, where $E_2=1-24\Sigma_{n\geqslant 1}(\Sigma_{d\mid n}d)q^n$ is the usual "Eisenstein series of weight 2 on $SL(2, Z)$" (not actually a modular form), related to f by $f(\tau)=2E_2(2\tau)-E_2(\tau)$. Applying this fact ν times and using the identity $E_2'=(\pi i/6)(E_2^2-E_4)$, where $E_4=1+240\Sigma_{n\geqslant 1}(\Sigma_{d\mid n}d^3)q^n$ is the Eisenstein series of weight 4 on $SL(2, Z)$, we find by induction that the function

$$(3)\qquad \sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{\Gamma(k+\nu)}{\Gamma(k+\mu)}\left(-\frac{\pi i}{6}E_2\right)^{\nu-\mu}F^{(\mu)}$$

is a modular form of weight $k+2\nu$ on Γ. We apply this to $F=f$, $\Gamma=\Gamma_0(2)$, $k=2$. All modular forms on $\Gamma_0(2)$ are polynomials in f and E_4 (this follows easily from the formulas for the dimensions of the spaces of modular forms of given weight), so we can identify (3) by computing the first few terms of its q-expansion; we find

$$f'-\frac{\pi i}{3}E_2f=-\frac{\pi i}{3}\left(2f^2-E_4\right),$$

$$f''-\pi iE_2f'-\frac{\pi^2}{6}E_2^2f=-\frac{\pi^2}{6}fE_4,$$

$$f'''-2\pi iE_2f''-\pi^2E_2^2f'+\frac{\pi^3 i}{9}E_2^3f=\frac{\pi^3 i}{9}f^2\left(4f^2-3E_4\right),$$

etc. At $\tau=\tau_0=(1+i)/2$ we have $f=0$, $E_2=6/\pi$ and $E_4=-12\alpha^4$, where

$$\alpha=\frac{1}{2\sqrt{\pi}}\frac{\Gamma(1/4)}{\Gamma(3/4)}=0.834626841678\cdots$$

(this follows from the well-known $E_2(i) = 3/\pi$ and $E_4(i) = 3\alpha^4$ together with the transformation properties of E_2 and E_4 under $SL(2, Z)$). Hence we find inductively from the above formulas the values

$$f'(\tau_0) = -4\pi i\alpha^4, \quad f''(\tau_0) = 24\pi\alpha^4, \quad f'''(\tau_0) = 144\pi i\alpha^4$$

and, continuing in the same way,

$$f^{(\mathrm{iv})}(\tau_0) = -960\pi\alpha^4, \quad f^{(\mathrm{v})}(\tau_0) = -7200\pi i\alpha^4 - 96\pi^5 i\alpha^{12}.$$

Using (1), we obtain the Taylor expansions

$$f(\tau_0 + i\varepsilon) = 4\pi\alpha^4\left(\varepsilon - 3\varepsilon^2 + 6\varepsilon^3 - 10\varepsilon^4 + \left(15 + \pi^4\alpha^8/5\right)\varepsilon^5 + \cdots\right)$$

and

$$U(\tau_0 + i\varepsilon) = -\frac{1}{64}e^{-4\pi^2\alpha^4(\varepsilon^2 - 2\varepsilon^3 + 3\varepsilon^4 - 4\varepsilon^5 + (5 + \pi^4\alpha^8/3)\varepsilon^6 + \cdots)}.$$

The second of these expresses $\sqrt{1 + 64U}$ as a power series in ε with leading term $2\pi\alpha^2\varepsilon$; inverting this power series and substituting the result into the Taylor expansion of f, we can write $1/f$ as a Laurent series in $(1 + 64U)^{1/2}$:

$$\begin{aligned}\frac{1}{f(\tau)} &= \frac{1}{2\alpha^2}(1 + 64U)^{-1/2} + \frac{1}{2\pi\alpha^4} + \frac{3 - \pi^2\alpha^4}{8\pi^2\alpha^6}(1 + 64U)^{1/2} \\ &\quad + \frac{1}{4\pi^3\alpha^8}(1 + 64U) + \frac{15 + 9\pi^2\alpha^4 - 4\pi^4\alpha^8}{96\pi^4\alpha^{10}}(1 + 64U)^{3/2} + \cdots.\end{aligned}$$

Comparing this with (2) gives

$$\begin{aligned}a_n &= \frac{64^n}{24}\cdot 2^{-2n}\binom{2n}{n}\left(\frac{1}{2\alpha^2} - \frac{3 - \pi^2\alpha^4}{8\pi^2\alpha^6}\frac{1}{2n - 1}\right. \\ &\qquad\qquad \left. + \frac{15 + 9\pi^2\alpha^4 - 4\pi^4\alpha^8}{96\pi^4\alpha^{10}}\frac{3}{(2n - 1)(2n - 3)} + \cdots\right) \\ &= \frac{64^n}{48\alpha^2\sqrt{\pi n}}\left(1 - \frac{3}{8\pi^2\alpha^4}n^{-1} + \left(\frac{15}{64\pi^4\alpha^8} - \frac{1}{128}\right)n^{-2} + \cdots\right).\end{aligned}$$

We have proved

THEOREM. *The sequence a_n has an asymptotic expansion of the form*

$$a_n = C\frac{64^n}{\sqrt{n}}\left(1 - \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots\right),$$

with

$$C = \frac{\sqrt{\pi}}{12}\frac{\Gamma(3/4)^2}{\Gamma(1/4)^2} = 0.0168732651505\cdots,$$

$$\alpha_1 = 6\frac{\Gamma(3/4)^4}{\Gamma(1/4)^4} = 0.07830067\cdots, \quad \alpha_2 = 60\frac{\Gamma(3/4)^8}{\Gamma(1/4)^8} - \frac{1}{128} = 0.002405668\cdots.$$

We give two numerical examples.

n	a_n	$C\frac{64^n}{\sqrt{n}}(1 - \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2})$
50	$4.853249476 \times 10^{87}$	$4.853249382 \times 10^{87}$
100	$6.996107097 \times 10^{177}$	$6.996107081 \times 10^{177}$

As a second application of the modular form description of the a_n, we prove the congruence properties (18a, b) of the numbers a_n (mod 3). These can be written in the form

$$na_n \equiv \begin{cases} 0 \pmod 3 & \text{if } 3 \mid n, \\ 1 \pmod 3 & \text{if } 3 \nmid n, \end{cases}$$

or

$$\sum_{n=1}^{\infty} (-1)^{n-1} n\, a_n U^n \equiv \frac{U(1-U)}{1+U^3} \pmod 3.$$

On the other hand, differentiating (2) and substituting (1), we see that

$$f(\tau)^3 \sum_{n=1}^{\infty} (-1)^{n-1} n\, a_n U(\tau)^n = \frac{1}{48\pi i} f'(\tau) = \sum_{n=1}^{\infty} n\sigma^*(n) q^n.$$

Since $f \equiv 1 \pmod 3$, we have to prove that

$$\frac{U(1-U)}{1+U^3} \equiv \sum_{n=1}^{\infty} n\sigma^*(n) q^n \pmod 3.$$

From the description of modular forms on $\Gamma_0(2)$ as polynomials in f and E_4 it follows that the modular function U must be related to E_4/f^2 by a fractional linear transformation; comparing the first few Fourier coefficients we find

$$\frac{E_4}{f^2} = \frac{1 + 256U}{1 + 64U}, \qquad U = \frac{1}{64} \frac{E_4 - f^2}{4f^2 - E_4} = \frac{\phi}{f^2 - 64\phi},$$

where

$$\phi = \frac{1}{192}\left(E_4 - f^2\right) = q + 8q^2 + 28q^3 + \cdots = \sum_{n \geq 1} b(n) q^n, \quad \text{say},$$

a modular form of weight 4 on $\Gamma_0(2)$. Since E_4 and f^2 are congruent to 1 (mod 48), it is clear that 4ϕ has integral coefficients, so that the numbers $b(n)$ are 3-integral, which is all we will need; actually, the $b(n)$ themselves are integral, as one can see from the identity $\phi = U(f^2 - 64\phi)$ or from the formula

$$\phi = \left(\sum_{\substack{n>0 \\ n \text{ odd}}} q^{n^2/8} \right)^8.$$

From $U = \phi/(f^2 - 64\phi)$ we obtain

$$\frac{U(1-U)}{1+U^3} = \frac{\phi(f^2 - 64\phi)(f^2 - 65\phi)}{(f^2 - 64\phi)^3 + \phi^3}$$

$$\equiv \frac{\phi(f^2 - \phi)(f^2 + \phi)}{f^6} = \frac{\phi}{f^2} - \left(\frac{\phi}{f^2}\right)^3 \pmod 3.$$

Since $f \equiv 1 \pmod 3$, the q-expansion of the right-hand side of this is congruent to $\phi - \phi^3$ or $\sum (b(n) - b(n/3))q^n$ modulo 3 (with the usual convention $b(n/3) = 0$ if $3 \nmid n$), so the congruence we have to prove is

$$(4) \qquad n\sigma^*(n) \equiv b(n) - b(n/3) \pmod 3.$$

The form $E_4(2\tau) = 1 + 240\sum_{n \geq 1} \sigma_3(n)q^{2n}$ is a modular form of weight 4 on $\Gamma_0(2)$ and hence a linear combination of f^2 and E_4 or of E_4 and ϕ. Comparing two Fourier coefficients gives $E_4(2\tau) = E_4 - 240\phi$ or

$$\phi(\tau) = \frac{1}{240}(E_4(\tau) - E_4(2\tau)), \qquad b(n) = \sigma_3(n) - \sigma_3(n/2).$$

Clearly $\sigma_3(n) \equiv \sigma_3(n/3) \pmod 3$ if $3 \mid n$, so (4) is true in this case. On the other hand, $\sigma_3(n) \equiv \sigma_1(n) = \sum_{d \mid n} d \pmod 3$ since d^3 and d are congruent, and, combining the divisors d and n/d, we see that $\sigma_1(n) \equiv 0 \pmod 3$ if $n \equiv -1 \pmod 3$ or equivalently $\sigma_1(n) \equiv n\sigma_1(n) \pmod 3$ if $n \not\equiv 0 \pmod 3$. Hence for $3 \nmid n$ we have

$$\sigma_3(n) - \sigma_3(n/2) \equiv n(\sigma_1(n) - 2\sigma_1(n/2)) = n\sigma^*(n) \pmod 3$$

as required.

Having proved the formula for $a_n \pmod 3$ we offer a conjectural formula for $a_n \pmod 5$:

$$a_n \equiv \begin{cases} a_{n/5} & \text{if } 5 \mid n, \\ 0 & \text{if } n = 5k + \delta, 0 < \delta < 5, k \text{ odd}, \\ \delta\binom{2r}{r}^3 & \text{if } n = 10r + \delta, 0 < \delta < 5. \end{cases}$$

It is true up to $n = 100$.

Finally, we make a remark about the nature of the numbers a_n. Equation (2) suggests that the natural generalization of this sequence is the sequence $\{\alpha_n\}$ defined by a generating function of the form $F = \sum \alpha_n u^n$, where u is a Hauptmodul for some group Γ of genus 0 (e.g. $\Gamma = SL_2(Z)$, $u = j^{-1}$, $\Gamma = \Gamma_0(2)$, $u = U$, or $\Gamma = \Gamma_0(2) \cup \Gamma_0(2)\begin{pmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$, $u = 1/(U + 2^{12}/U)$) and F a meromorphic modular form of some weight k on Γ. This definition includes both the a_n (with $k = -2$) and the sequence $\{A(n)\}$ mentioned several times in the paper (since these satisfy a recursion with constant coefficients and hence $\sum A(n)U^n$ is a rational function of U and therefore a modular form of weight $k = 0$), which may explain their parallel properties. The sequence $\{c_n\}$ defined by (10) of the paper has no such interpretation, which may explain why it apparently does not have such nice arithmetic properties.

Department of Mathematics
University of California
Santa Barbara, California 93106

Department of Mathematics
University of Maryland
College Park, Maryland 20742

1. WILLIAM ADAMS & DANIEL SHANKS, "Strong primality tests that are not sufficient," *Math. Comp.*, v. 39, 1982, pp. 255–300.

2. WILLIAM ADAMS & DANIEL SHANKS, "Strong primality tests. II—Algebraic identification of the p-adic limits and their implications." (To appear.)

3. H. BEHNKE & F. SOMMER, *Theorie der analytischen Funktionen einer complexen Veränderlichen*, Springer, Berlin, 1965, viii + 603 pp.

4. MARVIN I. KNOPP, *Modular Functions in Analytic Number Theory*, Markham, Chicago, Ill., 1970, x + 150 pp.

5. DERRICK H. LEHMER & EMMA LEHMER, "Cyclotomy with short periods," *Math. Comp.*, v. 41, 1983, pp. 743–758.

6. DANIEL SHANKS, "Dihedral quartic approximations and series for π," *J. Number Theory*, v. 14, 1982, pp. 397–423.

7. DANIEL SHANKS, "Review of A. O. L. Atkin's table," *Math. Comp.*, v. 32, 1978, p. 315.

8. THOMAS R. PARKIN & DANIEL SHANKS, "On the distribution of parity in the partition function," *Math. Comp.*, v. 21, 1967, pp. 446–480.

9. DANIEL SHANKS & LARRY P. SCHMID, "Variations on a theorem of Landau," *Math. Comp.*, v. 20, 1966, pp. 551–569.

10. DANIEL SHANKS, "Review 112", *Math. Comp.*, v. 19, 1965, pp. 684–686.

L'Enseignement Mathématique, t. 30 (1984), p. 275-330

THE ARITHMETIC-GEOMETRIC MEAN OF GAUSS

by David A. Cox

Introduction

The arithmetic-geometric mean of two numbers a and b is defined to be the common limit of the two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ determined by the algorithm

$$
\begin{aligned}
&a_0 = a\,, && b_0 = b\,, \\
&a_{n+1} = (a_n+b_n)/2\,, && b_{n+1} = (a_n b_n)^{1/2}\,, \quad n = 0, 1, 2, \dots\,.
\end{aligned}
\tag{0.1}
$$

Note that a_1 and b_1 are the respective arithmetic and geometric means of a and b, a_2 and b_2 the corresponding means of a_1 and b_1, etc. Thus the limit

$$M(a, b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n \tag{0.2}$$

really does deserve to be called the arithmetic-geometric mean of a and b. This algorithm first appeared in a paper of Lagrange, but it was Gauss who really discovered the amazing depth of this subject. Unfortunately, Gauss published little on the agM (his abbreviation for the arithmetic-geometric mean) during his lifetime. It was only with the publication of his collected works [12] between 1868 and 1927 that the full extent of his work became apparent. Immediately after the last volume appeared, several papers (see [15] and [35]) were written to bring this material to a wider mathematical audience. Since then, little has been done, and only the more elementary properties of the agM are widely known today.

In § 1 we review these elementary properties, where a and b are positive real numbers and the square root in (0.1) is also positive. The convergence of the algorithm is easy to see, though less obvious is the connection between the agM and certain elliptic integrals. As an application, we use $M(\sqrt{2}, 1)$ to determine the arc length of the lemniscate. In § 2, we allow a and b to be complex numbers, and the level of difficulty changes dramatically.

The convergence of the algorithm is no longer obvious, and as might be expected, the square root in (0.1) causes trouble. In fact, $M(a, b)$ becomes a multiple valued function, and in order to determine the relation between the various values, we will need to "uniformize" the agM using quotients of the classical Jacobian theta functions, which are modular functions for certain congruence subgroups of level four in $SL(2, \mathbf{Z})$. The amazing fact is that Gauss knew all of this! Hence in § 3 we explore some of the history of these ideas. The topics encountered will range from Bernoulli's study of elastic rods (the origin of the lemniscate) to Gauss' famous mathematical diary and his work on secular perturbations (the only article on the agM published in his lifetime).

I would like to thank my colleagues David Armacost and Robert Breusch for providing translations of numerous passages originally in Latin or German. Thanks also go to Don O'Shea for suggesting the wonderfully quick proof of (2.2) given in § 2.

1. The arithmetic-geometric mean of real numbers

When a and b are positive real numbers, the properties of the agM $M(a, b)$ are well known (see, for example, [5] and [26]). We will still give complete proofs of these properties so that the reader can fully appreciate the difficulties we encounter in § 2.

We will assume that $a \geqslant b > 0$, and we let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be as in (0.1), where b_{n+1} is always the positive square root of $a_n b_n$. The usual inequality between the arithmetic and geometric means,

$$(a+b)/2 \geqslant (ab)^{1/2},$$

immediately implies that $a_n \geqslant b_n$ for all $n \geqslant 0$. Actually, much more is true: we have

$$a \geqslant a_1 \geqslant \dots \geqslant a_n \geqslant a_{n+1} \geqslant \dots \geqslant b_{n+1} \geqslant b_n \geqslant \dots \geqslant b_1 \geqslant b \tag{1.1}$$

$$0 \leqslant a_n - b_n \leqslant 2^{-n}(a-b). \tag{1.2}$$

To prove (1.1), note that $a_n \geqslant b_n$ and $a_{n+1} \geqslant b_{n+1}$ imply

$$a_n \geqslant (a_n+b_n)/2 = a_{n+1} \geqslant b_{n+1} = (a_n b_n)^{1/2} \geqslant b_n,$$

and (1.1) follows. From $b_{n+1} \geqslant b_n$ we obtain

$$a_{n+1} - b_{n+1} \leqslant a_{n+1} - b_n = 2^{-1}(a_n - b_n),$$

and (1.2) follows by induction. From (1.1) we see immediately that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, and (1.2) implies that the limits are equal. Thus, we can use (0.2) to define the arithmetic-geometric mean $M(a, b)$ of a and b.

Let us work out two examples.

Example 1. $M(a, a) = a$.

This is obvious because $a = b$ implies $a_n = b_n = a$ for all $n \geqslant 0$.

Example 2. $M(\sqrt{2}, 1) = 1.1981402347355922074...$

The accuracy is to 19 decimal places. To compute this, we use the fact that $a_n \geqslant M(a, b) \geqslant b_n$ for all $n \geqslant 0$ and the following table (all entries are rounded off to 21 decimal places).

n	a_n	b_n
0	1.414213562373905048802	1.000000000000000000000
1	1.207106781186547524401	1.189207115002721066717
2	1.198156948094634295559	1.198123521493120122607
3	1.198140234793877209083	1.198140234677307205798
4	1.198140234735592207441	1.198140234735592207439

Such computations are not too difficult these days, though some extra programming was required since we went beyond the usual 16 digits of double-precision. The surprising fact is that these calculations were done not by computer but rather by Gauss himself. The above table is one of four examples given in the manuscript "De origine proprietatibusque generalibus numerorum mediorum arithmetico-geometricorum" which Gauss wrote in 1800 (see [12, III, pp. 361-371]). As we shall see later, this is an especially important example.

Let us note two obvious properties of the agM:

$$M(a, b) = M(a_1, b_1) = M(a_2, b_2) = \dots \tag{1.3}$$
$$M(\lambda a, \lambda b) = \lambda M(a, b).$$

Both of these follow easily from the definition of $M(a, b)$.

Our next result shows that the agM is not as simple as indicated by what we have done so far. We now get our first glimpse of the depth of this subject.

THEOREM 1.1. *If* $a \geqslant b > 0$, *then*

$$M(a, b) \cdot \int_0^{\pi/2} (a^2\cos^2\phi + b^2\sin^2\phi)^{-1/2} d\phi = \pi/2 .$$

Proof. Let $I(a, b)$ denote the above integral, and set $\mu = M(a, b)$. Thus we need to prove $I(a, b) = (\pi/2)\mu^{-1}$. The key step is to show that

$$I(a, b) = I(a_1, b_1) . \tag{1.4}$$

The shortest proof of (1.4) is due to Gauss. He introduces a new variable ϕ' such that

$$\sin\phi = \frac{2a\sin\phi'}{a + b + (a-b)\sin^2\phi'} . \tag{1.5}$$

Note that $0 \leqslant \phi' \leqslant \pi/2$ corresponds to $0 \leqslant \phi \leqslant \pi/2$. Gauss then asserts "after the development has been made correctly, it will be seen" that

$$(a^2\cos^2\phi + b^2\sin^2\phi)^{-1/2} d\phi = (a_1^2\cos^2\phi' + b_1^2\sin^2\phi')^{-1/2} d\phi' \tag{1.6}$$

(see [12, III, p. 352]). Given this, (1.4) follows easily. In "Fundamenta nova theoriae functionum ellipticorum," Jacobi fills in some of the details Gauss left out (see [20, I, p. 152]). Specifically, one first proves that

$$\cos\phi = \frac{2\cos\phi'(a_1^2\cos^2\phi' + b_1^2\sin^2\phi')^{1/2}}{a + b + (a-b)\sin^2\phi'}$$

$$(a^2\cos^2\phi + b^2\sin^2\phi)^{1/2} = a\frac{a + b - (a-b)\sin^2\phi'}{a + b + (a-b)\sin^2\phi'}$$

(these are straightforward manipulations), and then (1.6) follows from these formulas by taking the differential of (1.5).

Iterating (1.4) gives us

$$I(a, b) = I(a_1, b_1) = I(a_2, b_2) = \dots ,$$

so that $I(a, b) = \lim_{n\to\infty} I(a_n, b_n) = \pi/2\mu$ since the functions

$$(a_n^2\cos^2\phi + b_n^2\sin^2\phi)^{-1/2}$$

converge uniformly to the constant function μ^{-1}. QED

This theorem relates very nicely to the classical theory of complete elliptic integrals of the first kind, i.e., integrals of the form

$$F(k, \pi/2) = \int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2}d\phi = \int_0^1 ((1-z^2)(1-k^2z^2))^{-1/2}dz\,.$$

To see this, we set $k = \dfrac{a-b}{a+b}$. Then one easily obtains

$$I(a, b) = a^{-1} F\left(\frac{2\sqrt{k}}{1+k}, \pi/2\right), \qquad I(a_1, b_1) = a_1^{-1} F(k, \pi/2),$$

so that (1.4) is equivalent to the well-known formula

$$F\left(\frac{2\sqrt{k}}{1+k}, \pi/2\right) = (1+\mathrm{k})\, F(k, \pi/2)$$

(see [16, p. 250] or [17, p. 908]). Also, the substitution (1.5) can be written as

$$\sin\phi = \frac{(1+k)\sin\phi'}{1 + k\sin^2\phi'},$$

which is now called the Gauss transformation (see [32, p. 206]).

For someone well versed in these formulas, the derivation of (1.4) would not be difficult. In fact, a problem on the 1895 Mathematical Tripos was to prove (1.4), and the same problem appears as an exercise in Whittaker and Watson's *Modern Analysis* (see [36, p. 533]), though the agM is not mentioned. Some books on complex analysis do define $M(a, b)$ and state Theorem 1.1 (see, for example, [7, p. 417]).

There are several other ways to express Theorem 1.1. For example, if $0 \leqslant k < 1$, then one can restate the theorem as

$$(1.7) \qquad \frac{1}{M(1+k, 1-k)} = \frac{2}{\pi}\int_0^{\pi/2} (1-k^2\sin^2\gamma)^{-1/2}d\gamma = \frac{2}{\pi} F(k, \pi/2)\,.$$

Furthermore, using the well-known power series expansion for $F(k, \pi/2)$ (see [16, p. 905]), we obtain

$$(1.8) \qquad \frac{1}{M(1+k, 1-k)} = \sum_{n=0}^{\infty} \left[\frac{1\cdot 3\cdot \ldots \cdot(2n-1)}{2^n n!}\right]^2 k^{2n}\,.$$

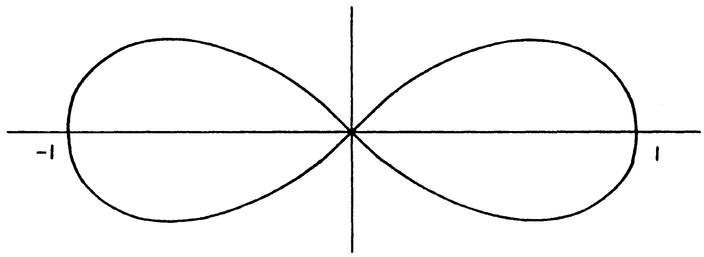
Finally, it is customary to set $k' = \sqrt{1-k^2}$. Then, using (1.3), we can rewrite (1.7) as

$$(1.9) \qquad \frac{1}{M(1, k')} = \frac{2}{\pi}\int_0^{\pi/2} (1-k^2\sin^2\gamma)^{-1/2}d\gamma\,.$$

This last equation shows that the average value of the function $(1-k^2\sin^2\gamma)^{-1/2}$ on the interval $[0, \pi/2]$ is the reciprocal of the agM of the reciprocals of the minimum and maximum values of the function, a lovely interpretation due to Gauss — see [12, III, p. 371].

One application of Theorem 1.1, in the guise of (1.7), is that the algorithm for the agM now provides a very efficient method for approximating the elliptic integral $F(k, \pi/2)$. As we will see in § 3, it was just this problem that led Lagrange to independently discover the algorithm for the agM.

Another application of Theorem 1.1 concerns the arc length of the lemniscate $r^2 = \cos 2\theta$:



Using the formula for arc length in polar coordinates, we see that the total arc length is

$$4\int_0^{\pi/4} (r^2+(dr/d\theta)^2)^{1/2}\,d\theta = 4\int_0^{\pi/4} (\cos 2\theta)^{-1/2}\,d\theta\,.$$

The substitution $\cos 2\theta = \cos^2\phi$ transforms this to the integral

$$4\int_0^{\pi/2} (1+\cos^2\phi)^{-1/2}\,d\phi = 4\int_0^{\pi/2} (2\cos^2\phi+\sin^2\phi)^{-1/2}\,d\phi\,.$$

Using Theorem 1.1 to interpret this last integral in terms of $M(\sqrt{2}, 1)$, we see that the arc length of the lemniscate $r^2 = \cos 2\theta$ is $2\pi/M(\sqrt{2}, 1)$.

From Example 2 it follows that the arc length is approximately 5.244, and much better approximations can be easily obtained. (For more on the computation of the arc length of the lemniscate, the reader should consult [33].)

On the surface, this arc length computation seems rather harmless. However, from an historical point of view, it is of fundamental importance. If we set $z = \cos\phi$, then we obtain

$$\int_0^{\pi/2} (2\cos^2\phi+\sin^2\phi)^{-1/2}\,d\phi = \int_0^1 (1-z^4)^{-1/2}\,dz\,.$$

The integral on the right appeared in 1691 in a paper of Jacob Bernoulli and was well known throughout the 18th century. Gauss even had a special notation for this integral, writing

$$\varpi = 2\int_0^1 (1-z^4)^{-1/2}\,dz\,.$$

Then the relation between the arc length of the lemniscate and $M(\sqrt{2}, 1)$ can be written

$$M(\sqrt{2}, 1) = \frac{\pi}{\varpi}\,.$$

To see the significance of this equation, we turn to Gauss' mathematical diary. The 98th entry, dated May 30, 1799, reads as follows:

> We have established that the arithmetic-geometric mean between 1 and $\sqrt{2}$ is π/ϖ to the eleventh decimal place; the demonstration of this fact will surely open an entirely new field of analysis.

(See [12, X.1, p. 542].) The genesis of this entire subject lies in Gauss' observation that these two numbers are the same. It was in trying to understand the real meaning of this equality that several streams of Gauss' thought came together and produced the exceptionally rich mathematics which we will explore in § 2.

Let us first examine how Gauss actually showed that $M(\sqrt{2}, 1) = \pi/\varpi$. The proof of Theorem 1.1 given above appeared in 1818 in a paper on secular perturbations (see [12, III, pp. 331-355]), which is the only article on the agM Gauss published in his lifetime (though as we've seen, Jacobi knew this paper well). It is more difficult to tell precisely when he first proved Theorem 1.1, although his notes do reveal that he had two proofs by December 23, 1799.

Both proofs derive the power series version (1.8) of Theorem 1.1. Thus the goal is to show that $M(1+k, 1-k)^{-1}$ equals the function

$$y = \sum_{n=0}^{\infty} \left(\frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n n!}\right)^2 k^{2n}\,. \tag{1.10}$$

The first proof, very much in the spirit of Euler, proceeds as follows. Using (1.3), Gauss derives the identity

$$M\left(1 + \frac{2t}{1+t^2},\ 1 - \frac{2t}{1+t^2}\right) = \frac{1}{1+t^2}\,M(1+t^2, 1-t^2)\,. \tag{1.11}$$

He then assumes that there is a power series expansion of the form

$$\frac{1}{M(1+k,1-k)} = 1 + A\,k^2 + B\,k^4 + C\,k^6 + \ldots\ .$$

By letting $k = t^2$ and $2t/(1+t^2)$ in this series and using (1.11), Gauss obtains

$$\begin{aligned} 1 + A\left(\frac{2t}{1+t^2}\right)^2 + B\left(\frac{2t}{1+t^2}\right)^4 + C\left(\frac{2t}{1+t^2}\right)^6 + \ldots \\ = (1+t^2)\,(1+At^4+Bt^8+Ct^{12}+\ldots)\,. \end{aligned}$$

Multiplying by $2t/(1+t^2)$, this becomes

$$\frac{2t}{1+t^2} + A\left(\frac{2t}{1+t^2}\right)^3 + B\left(\frac{2t}{1+t^2}\right)^5 + \ldots = 2t(1+At^4+Bt^8+\ldots)\,.$$

A comparison of the coefficients of powers of t gives an infinite system of equations in $A, B, C, \ldots$. Gauss showed that this system is equivalent to the equations $0 = 1 - 4A = 9A - 16B = 25B - 36C = \ldots$, and (1.8) follows easily (see [12, III, pp. 367-369] for details). Gauss' second proof also uses the identity (1.11), but in a different way. Here, he first shows that the series y of (1.10) is a solution of the hypergeometric differential equation

$$(k^3-k)y'' + (3k^2-1)y' + ky = 0\,. \tag{1.12}$$

This enables him to show that y satisfies the identity

$$y\left(\frac{2t}{1+t^2}\right) = (1+t^2)y(t^2)\,,$$

so that by (1.11), $F(k) = M(1+k, 1-k)y(k)$ has the property that

$$F\left(\frac{2t}{1+t^2}\right) = F(t^2)\,.$$

Gauss then asserts that $F(k)$ is clearly constant. Since $F(0) = 1$, we obtain a second proof of (1.8) (see [12, X.1, pp. 181-183]). It is interesting to note that neither proof is rigorous from the modern point of view: the first assumes without proof that $M(1+k, 1-k)^{-1}$ has a power series expansion, and the second assumes without proof that $M(1+k, 1-k)$ is continuous (this is needed in order to show that $F(k)$ is constant).

We can be certain that Gauss knew both of these proofs by December 23, 1799. The evidence for this is the 102nd entry in Gauss' mathematical diary. Dated as above, it states that "the arithmetic-geometric mean is itself an integral quantity" (see [12, X.1, p. 544]). However, this statement is not so easy to interpret. If we turn to Gauss' unpublished manuscript of 1800 (where we got the example $M(\sqrt{2}, 1)$), we find (1.7) and (1.8) as expected, but also the observation that a complete solution of the differential equation (1.12) is given by

$$\frac{A}{M(1+k, 1-k)} + \frac{B}{M(1, k)}, \quad A, B \in \mathbf{C} \tag{1.13}$$

(see [12, III, p. 370]). In eighteenth century terminology, this is the "complete integral" of (1.12) and thus may be the "integral quantity" that Gauss was referring to (see [12, X.1, pp. 544-545]). Even if this is so, the second proof must predate December 23, 1799 since it uses the same differential equation.

In § 3 we will study Gauss' early work on the agM in more detail. But one thing should be already clear: none of the three proofs of Theorem 1.1 discussed so far live up to Gauss' May 30, 1799 prediction of "an entirely new field of analysis." In order to see that his claim was justified, we will need to study his work on the agM of complex numbers.

2. The arithmetic-geometric mean of complex numbers

The arithmetic-geometric mean of two complex numbers a and b is not easy to define. The immediate problem is that in our algorithm

$$\begin{aligned} a_0 &= a, & b_0 &= b, \\ a_{n+1} &= (a_n+b_n)/2, & b_{n+1} &= (a_nb_n)^{1/2}, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{2.1}$$

there is no longer an obvious choice for b_{n+1}. In fact, since we are presented with two choices for b_{n+1} for all $n \geqslant 0$, there are uncountably many sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ for given a and b. Nor is it clear that any of these converge!

We will see below (Proposition 2.1) that in fact all of these sequences converge, but only countably many have a non-zero limit. The limits of these particular sequences then allow us to define $M(a, b)$ as a multiple valued function of a and b. Our main result (Theorem 2.2) gives the relationship between the various values of $M(a, b)$. This theorem was discovered by Gauss in 1800, and we will follow his proof, which makes extensive use of theta functions and modular functions of level four.

We first restrict ourselves to consider only those a's and b's such that $a \neq 0$, $b \neq 0$ and $a \neq \pm b$. (If $a=0$, $b=0$ or $a=\pm b$, one easily sees that the sequences (2.1) converge to either 0 or a, and hence are not very interesting.) An easy induction argument shows that if a and b satisfy these restrictions, so do a_n and b_n for all $n \geqslant 0$ in (2.1).

We next give a way of distinguishing between the two possible choices for each b_{n+1}.

Definition. Let $a, b \in \mathbf{C}^*$ satisfy $a \neq \pm b$. Then a square root b_1 of ab is called the *right choice* if $|a_1 - b_1| \leqslant |a_1 + b_1|$ and, when $|a_1 - b_1| = |a_1 + b_1|$, we also have $\operatorname{Im}(b_1/a_1) > 0$.

To see that this definition makes sense, suppose that $\operatorname{Im}(b_1/a_1) = 0$. Then $b_1/a_1 = r \in \mathbf{R}$, and thus

$$|a_1 - b_1| = |a_1|\,|1 - r| \neq |a_1|\,|1 + r| = |a_1 + b_1|$$

since $r \neq 0$. Notice also that the right choice is unchanged if we switch a and b, and that if a and b are as in § 1, then the right choice for $(ab)^{1/2}$ is the positive one.

It thus seems natural that we should define the agM using (2.1) with b_{n+1} always the right choice for $(a_nb_n)^{1/2}$. However, this is not the only possibility: one can make some wrong choices for b_{n+1} and still get an interesting answer. For instance, in Gauss' notebooks, we find the following example:

n	a_n	b_n
0	3.0000000	1.0000000
1	2.0000000	−1.7320508
2	.1339746	1.8612098i
3	.0669873 + .9306049i	.3530969 + .3530969i
4	.2100421 + .6418509i	.2836903 + .6208239i
5	.2468676 + .6313374i	.2470649 + .6324002i
6	.2469962 + .6318688i	.2469962 + .6318685i

(see [12, III, p. 379]). Note that b_1 is the wrong choice but b_n is the right choice for $n \geqslant 2$. The algorithm appears to converge nicely.

Let us make this idea more precise with a definition.

Definition. Let $a, b \in \mathbf{C}^*$ satisfy $a \neq \pm b$. A pair of sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ as in (2.1) is called *good* if b_{n+1} is the right choice for $(a_n b_n)^{1/2}$ for all but finitely many $n \geqslant 0$.

The following proposition shows the special role played by good sequences.

PROPOSITION 2.1. *If $a, b \in \mathbf{C}^*$ satisfy $a \neq \pm b$, then any pair of sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ as in (2.1) converge to a common limit, and this common limit is non-zero if and only if $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are good sequences.*

Proof. We first study the properties of the right choice b_1 of $(ab)^{1/2}$ in more detail. Let $0 \leqslant \operatorname{ang}(a, b) \leqslant \pi$ denote the unoriented angle between a and b.

Then we have:

$$|a_1 - b_1| \leqslant (1/2)|a - b| \tag{2.2}$$

$$\operatorname{ang}(a_1, b_1) \leqslant (1/2)\operatorname{ang}(a, b). \tag{2.3}$$

To prove (2.2), note that

$$|a_1 - b_1|\,|a_1 + b_1| = (1/4)|a - b|^2.$$

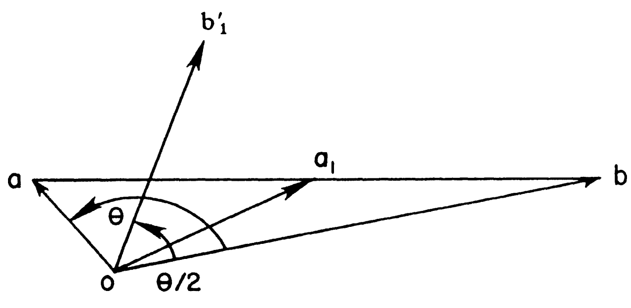
Since $|a_1 - b_1| \leqslant |a_1 + b_1|$, (2.2) follows immediately. To prove (2.3), let $\theta_1 = \operatorname{ang}(a_1, b_1)$ and $\theta = \operatorname{ang}(a, b)$. From the law of cosines

$$|a_1 \pm b_1|^2 = |a_1|^2 + |b_1|^2 \pm 2|a_1|\,|b_1|\cos\theta_1,$$

we see that $\theta_1 \leqslant \pi/2$ because $|a_1 - b_1| \leqslant |a_1 + b_1|$. Thus

$$\operatorname{ang}(a_1, b_1) = \theta_1 \leqslant \pi - \theta_1 = \operatorname{ang}(a_1, -b_1).$$

To compare this to θ, note that one of $\pm b_1$, say b'_1, satisfies $\operatorname{ang}(a, b'_1) = \operatorname{ang}(b'_1, b) = \theta/2$. Then the following picture



shows that $\operatorname{ang}(a_1, b'_1) \leqslant \theta/2$. Since $b'_1 = \pm b_1$, the above inequalities imply that

$$\operatorname{ang}(a_1, b_1) \leqslant \operatorname{ang}(a_1, b'_1) \leqslant (1/2)\operatorname{ang}(a, b),$$

proving (2.3).

Now, suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are not good sequences. We set $M_n = \max\{|a_n|, |b_n|\}$, and it suffices to show that $\lim_{n\to\infty} M_n = 0$. Note that $M_{n+1} \leqslant M_n$ for $n \geqslant 0$. Suppose that for some n, b_{n+1} is not the right choice for $(a_n b_n)^{1/2}$. Then $-b_{n+1}$ is the right choice, and thus (2.2), applied to a_n and b_n, implies that

$$|a_{n+2}| = (1/2)|a_{n+1} - b_{n+1}| \leqslant (1/4)|a_n - b_n| \leqslant (1/2)M_n .$$

However, we also have $|b_{n+2}| \leqslant M_n$. It follows easily that

$$M_{n+3} \leqslant (3/4)M_n . \tag{2.4}$$

Since $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are not good sequences, (2.4) must occur infinitely often, proving that $\lim_{n\to\infty} M_n = 0$.

Next, suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are good sequences. By neglecting the first N terms for N sufficiently large, we may assume that b_{n+1} is the right choice for all $n \geqslant 0$ and that $\operatorname{ang}(a, b) < \pi$ (this is possible by (2.3)). We also set $\theta_n = \operatorname{ang}(a_n, b_n)$. From (2.2) and (2.3) we obtain

$$|a_n - b_n| \leqslant 2^{-n}|a - b|, \qquad \theta_n \leqslant 2^{-n}\theta_0 . \tag{2.5}$$

Note that $a_n - a_{n+1} = (1/2)(a_n - b_n)$, so that by (2.5),

$$|a_n - a_{n+1}| \leqslant 2^{-(n+1)}|a - b|$$

Hence, if $m > n$, we see that

$$|a_n - a_m| \leqslant \sum_{k=n}^{m-1} |a_k - a_{k+1}| \leqslant \left(\sum_{k=n}^{m-1} 2^{-(k+1)}\right)|a - b| < 2^{-n}|a - b| .$$

Thus $\{a_n\}_{n=0}^{\infty}$ converges because it is a Cauchy sequence, and then (2.5) implies that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

It remains to show that this common limit is nonzero. Let

$$m_n = \min\{|a_n|, |b_n|\} .$$

Clearly $|b_{n+1}| \geqslant m_n$. To relate $|a_{n+1}|$ and m_n, we use the law of cosines:

$$\begin{aligned}(2|a_{n+1}|)^2 &= |a_n|^2 + |b_n|^2 + 2|a_n||b_n|\cos\theta_n \\ &\geq 2m_n^2(1+\cos\theta_n) = 4m_n^2\cos^2(\theta_n/2).\end{aligned}$$

It follows that $m_{n+1} \geq \cos(\theta_n/2)m_n$ since $0 \leq \theta_n < \pi$ (this uses (2.5) and the fact that $\theta_0 = \mathrm{ang}(a, b) < \pi$). Using (2.5) again, we obtain

$$m_n \geq \left(\prod_{k=1}^{n} \cos(\theta_0/2^k)\right) m_0 .$$

However, it is well known that

$$\prod_{k=1}^{\infty} \cos(\theta_0/2^k) = \frac{\sin\theta_0}{\theta_0}.$$

(See [16, p. 38]. When $\theta_0 = 0$, the right hand side is interpreted to be 1.) We thus have

$$m_n \geq \left(\frac{\sin\theta_0}{\theta_0}\right) m_0$$

for all $n \geq 1$. Since $0 \leq \theta_0 < \pi$, it follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \neq 0$. QED

We now define the agM of two complex numbers.

Definition. Let $a, b \in \mathbf{C}^*$ satisfy $a \neq \pm b$. A nonzero complex number μ is a value of the *arithmetic-geometric mean* $M(a, b)$ of a and b if there are good sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ as in (2.1) such that

$$\mu = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n .$$

Thus $M(a, b)$ is a multiple valued function of a and b and there are a countable number of values. Note, however, that there is a distinguished value of $M(a, b)$, namely the common limit of $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ where b_{n+1} is the right choice for $(a_nb_n)^{1/2}$ for *all* $n \geq 0$. We will call this the *simplest value* of $M(a, b)$. When a and b are positive real numbers, this simplest value is just the agM as defined in § 1.

We now come to the major result of this paper, which determines how the various values of $M(a, b)$ are related for fixed a and b.

THEOREM 2.2. *Fix $a, b \in \mathbf{C}^*$ which satisfy $a \neq \pm b$ and $|a| \geq |b|$, and let μ and λ denote the simplest values of $M(a, b)$ and $M(a+b, a-b)$ respectively. Then all values μ' of $M(a, b)$ are given by the formula*

$$\frac{1}{\mu'} = \frac{d}{\mu} + \frac{ic}{\lambda},$$

where d and c are arbitrary relatively prime integers satisfying $d \equiv 1 \bmod 4$ and $c \equiv 0 \bmod 4$.

Proof. Our treatment of the agM of complex numbers thus far has been fairly elementary. The proof of this theorem, however, will be quite different; we will finally discover the "entirely new field of analysis" predicted by Gauss in the diary entry quoted in § 1. In the proof we will follow Gauss' ideas and even some of his notations, though sometimes translating them to a modern setting and of course filling in the details he omitted (Gauss' notes are extremely sketchy and incomplete — see [12, III, pp. 467-468 and 477-478]).

The proof will be broken up into four steps. In order to avoid writing a treatise on modular functions, we will quote certain classical facts without proof.

Step 1. Theta Functions

Let $\mathfrak{H} = \{\tau \in \mathbf{C} : \operatorname{Im}\tau > 0\}$ and set $\mathrm{q} = e^{\pi i \tau}$. The Jacobi theta functions are defined as follows:

$$p(\tau) = 1 + 2\sum_{n=1}^{\infty} \mathrm{q}^{n^2} = \Theta_3(\tau, 0),$$

$$q(\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n \mathrm{q}^{n^2} = \Theta_4(\tau, 0),$$

$$r(\tau) = 2\sum_{n=1}^{\infty} \mathrm{q}^{(2n-1)^2/4} = \Theta_2(\tau, 0).$$

Since $|\mathrm{q}| < 1$ for $\tau \in \mathfrak{H}$, these are holomorphic functions of τ. The notation p, q and r is due to Gauss, though he wrote them as power series in $e^{-\pi t}$, $\operatorname{Re} t > 0$ (thus he used the right half plane rather than the upper half plane $\mathfrak{H}$ — see [12, III, pp. 383-386]). The more common notation Θ_3, Θ_4 and Θ_2 is from [36, p. 464] and [32, p. 27].

A wealth of formulas are associated with these functions, including the product expansions:

$$p(\tau) = \prod_{n=1}^{\infty} (1-\mathrm{q}^{2n})(1+\mathrm{q}^{2n-1})^2,$$

$$q(\tau) = \prod_{n=1}^{\infty} (1-\mathrm{q}^{2n})(1-\mathrm{q}^{2n-1})^2, \tag{2.6}$$

$$r(\tau) = 2q^{1/4} \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n})^2 ,$$

(which show that $p(\tau)$, $q(\tau)$ and $r(\tau)$ are nonvanishing on $\mathfrak{H}$), the transformations:

$$\begin{aligned} p(\tau+1) &= q(\tau), & p(-1/\tau) &= (-i\tau)^{1/2}p(\tau), \\ q(\tau+1) &= p(\tau), & q(-1/\tau) &= (-i\tau)^{1/2}r(\tau), \\ r(\tau+1) &= e^{\pi i/4}r(\tau), & r(-1/\tau) &= (-i\tau)^{1/2}q(\tau), \end{aligned} \tag{2.7}$$

(where we assume that $\operatorname{Re}(-i\tau)^{1/2} > 0$), and finally the identities

$$\begin{aligned} p(\tau)^2 + q(\tau)^2 &= 2p(2\tau)^2 , \\ p(\tau)^2 - q(\tau)^2 &= 2r(2\tau)^2 , \\ p(\tau)q(\tau) &= q(2\tau)^2 , \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} p(2\tau)^2 + r(2\tau)^2 &= p(\tau)^2 , \\ p(2\tau)^2 - r(2\tau)^2 &= q(\tau)^2 , \\ q(\tau)^4 + r(\tau)^4 &= p(\tau)^4 . \end{aligned} \tag{2.9}$$

Proofs of (2.6) and (2.7) can be found in [36, p. 469 and p. 475], while one must turn to more complete works like [32, pp. 118-119] for proofs of (2.8). (For a modern proof of (2.8), consult [34].) Finally, (2.9) follows easily from (2.8). Of course, Gauss knew all of these formulas (see [12, III, pp. 386 and 466-467]).

What do these formulas have to do with the agM? The key lies in (2.8): one sees that $p(2\tau)^2$ and $q(2\tau)^2$ are the respective arithmetic and geometric means of $p(\tau)^2$ and $q(\tau)^2$! To make the best use of this observation, we need to introduce the function $k'(\tau) = q(\tau)^2/p(\tau)^2$.

Then we have:

LEMMA 2.3. *Let $a, b \in \mathbf{C}^*$ satisfy $a \neq \pm b$, and suppose there is $\tau \in \mathfrak{H}$ such that $k'(\tau) = b/a$. Set $\mu = a/p(\tau)^2$ and, for $n \geqslant 0$, $a_n = \mu\, p(2^n\tau)^2$ and $b_n = \mu\, q(2^n\tau)^2$. Then*

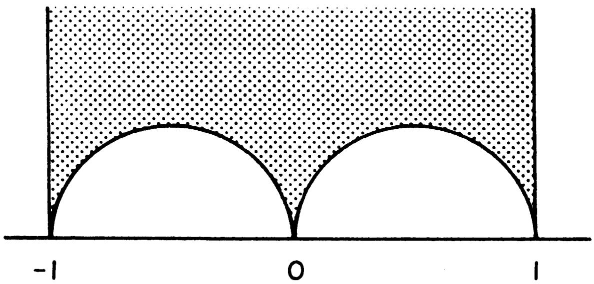
(i) *$\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are good sequences satisfying (2.1),*

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \mu$.

Proof. We have $a_0 = a$ by definition, and $b_0 = b$ follows easily from $k'(\tau) = b/a$. As we observed above, the other conditions of (2.1) are clearly satisfied. Finally, note that $\exp(\pi i 2^n \tau) \to 0$ as $n \to \infty$, so that $\lim_{n\to\infty} p(2^n\tau)^2 = \lim_{n\to\infty} q(2^n\tau)^2 = 1$, and (ii) follows. Since $\mu \neq 0$, Proposition 2.1 shows that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are good sequences. QED

Thus every solution τ of $k'(\tau) = b/a$ gives us a value $\mu = a/p(\tau)^2$ of $M(a, b)$. As a first step toward understanding all solutions of $k'(\tau) = b/a$, we introduce the region $F_1 \subseteq \mathfrak{H}$:

$$F_1 = \{\tau \in \mathfrak{H} : |\operatorname{Re}\tau| \leqslant 1\,,\ |\operatorname{Re}(1/\tau)| \leqslant 1\}$$

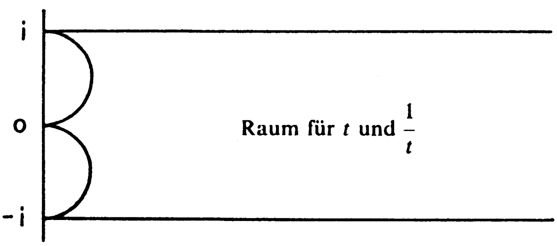


F_1 shaded

The following result is well known.

LEMMA 2.4. *k'^2 assumes every value in* $\mathbf{C} - \{0, 1\}$ *exactly once in* $F_1' = F_1 - (\partial F_1 \cap \{\tau \in \mathfrak{H} : \operatorname{Re}\tau < 0\})$.

A proof can be found in [36, pp. 481-484]. Gauss was aware of similar results which we will discuss below. He drew F_1 as follows (see [12, III, p. 478]).



Note that our restrictions on a and b ensure that $(b/a)^2 \in \mathbf{C} - \{0, 1\}$. Thus, by Lemma 2.4, we can always solve $k'(\tau)^2 = (b/a)^2$, i.e., $k'(\tau) = \pm b/a$. We will prove below that

$$k'\left(\frac{\tau}{2\tau+1}\right) = -\,k'(\tau)\,, \tag{2.10}$$

which shows that we can always solve $k'(\tau) = b/a$. Thus, for every a and b as above, $M(a, b)$ has at least one value of the form $a/p(\tau)^2$, where $k'(\tau) = b/a$.

Three tasks now remain. We need to find *all* solutions τ of $k'(\tau) = b/a$, we need to see how the values $a/p(\tau)^2$ are related for these τ's, and we need to prove that *all* values of $M(a, b)$ arise in this way. To accomplish these goals, we must first recast the properties of $k'(\tau)$ and $p(\tau)^2$ into more modern terms.

Step 2. Modular Forms of Weight One.

The four lemmas proved here are well known to experts, but we include their proofs in order to show how easily one can move from the classical facts of Step 1 to their modern interpretations. We will also discuss what Gauss had to say about these facts.

We will use the transformation properties (2.7) by way of the group

$$SL(2, \mathbf{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Z},\ ad-bc = 1\right\}$$

which acts on $\mathfrak{H}$ by linear fractional transformations as follows: if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ and $\tau \in \mathfrak{H}$, then $\gamma\tau = \dfrac{a\tau + b}{c\tau + d}$.

For example, if

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad\text{and}\quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad\text{then}\quad S\tau = \frac{-1}{\tau}, \qquad T\tau = \tau + 1,$$

which are the transformations in (2.7). It can be shown that S and T generate $SL(2, \mathbf{Z})$ (see [29, Ch. VII, Thm. 2]), a fact we do not need here.

We will consider several subgroups of $SL(2, \mathbf{Z})$. The first of these is $\Gamma(2)$, the principal congruence subgroup of level 2:

$$\Gamma(2) = \left\{\gamma \in SL(2, \mathbf{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2\right\}.$$

Note that $-1 \in \Gamma(2)$ and that $\Gamma(2)/\{\pm 1\}$ acts on $\mathfrak{H}$.

LEMMA 2.5.

(i) $\Gamma(2)/\{\pm 1\}$ *acts freely on* $\mathfrak{H}$.

(ii) $\Gamma(2)$ *is generated by* -1, $U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ *and* $V = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

(iii) *Given* $\tau \in \mathfrak{H}$, *there is* $\gamma \in \Gamma(2)$ *such that* $\gamma\tau \in F_1$.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\Gamma(2)$.

(i) If $\tau \in \mathfrak{H}$ and $\gamma\tau = \tau$, then we obtain $c\tau^2 + (d-a)\tau - b = 0$. If $c = 0$, then $\gamma = \pm 1$ follows immediately. If $c \neq 0$, then $(d-a)^2 + 4\,bc < 0$ because $\tau \in \mathfrak{H}$. Using $ad - bc = 1$, this becomes $(a+d)^2 < 4$, and thus $a + d = 0$ since a and d are odd. However, b and c are even so that

$$1 \equiv ad - bc \equiv ad \equiv -a^2 \bmod 4$$

This contradiction proves (i).

(ii) We start with a variation of the Euclidean algorithm. Given γ as above, let $r_1 = a - 2a_1c$, where $a_1 \in \mathbf{Z}$ is chosen so that $|r_1|$ is minimal. Then $|r_1| \leqslant |c|$, and hence $|r_1| < |c|$ since a and c have different parity. Thus

$$a = 2a_1c + r_1, \quad a_1, r_1 \in \mathbf{Z}, \quad |r_1| < |c|.$$

Note that c and r_1 also have different parity. Continuing this process, we obtain

$$\begin{aligned} c &= 2a_2 r_1 + r_2, \quad |r_2| < |r_1|, \\ r_1 &= 2a_3 r_2 + r_3, \quad |r_3| < |r_2|, \\ &\vdots \\ r_{2n-1} &= 2a_{2n+1} r_{2n} + r_{2n+1}, \quad r_{2n+1} = \pm 1, \\ r_{2n} &= 2a_{2n+2} r_{2n+1} + 0, \end{aligned}$$

since $\mathrm{GCD}(a, c) = 1$. Then one easily computes that

$$V^{-a_{2n+2}} U^{-a_{2n+1}} \ldots V^{-a_2} U^{-a_1} \gamma = \begin{pmatrix} \pm 1 & * \\ 0 & * \end{pmatrix}.$$

Since the left-hand side is in $\Gamma(2)$, the right-hand side must be of the form $\pm U^m$, and we thus obtain

$$\gamma = \pm U^{a_1} V^{a_2} \ldots U^{a_{2n+1}} V^{a_{2n+2}} U^m.$$

(iii) Fix $\tau \in \mathfrak{H}$. The quadratic form $|x\tau + y|^2$ is positive definite for $x, y \in \mathbf{R}$, so that for any $S \subseteq \mathbf{Z}^2$, $|x\tau + y|^2$ assumes a minimum value at some $(x, y) \in S$. In particular, $|c\tau + d|^2$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$, assumes a minimum value at some $\gamma_0 \in \Gamma(2)$. Since $\operatorname{Im} \gamma\tau = \operatorname{Im} \tau\, |c\tau + d|^{-2}$, we see that $\tau' = \gamma_0\tau$ has maximal imaginary part, i.e., $\operatorname{Im}\tau' \geqslant \operatorname{Im}\gamma\tau'$ for $\gamma \in \Gamma(2)$. Since $\operatorname{Im}\tau' = \operatorname{Im} U\tau'$, we may assume that $|\operatorname{Re}\tau'| \leqslant 1$. Applying the above inequality to $V^{\pm 1} \in \Gamma(2)$, we obtain

$$\operatorname{Im}\tau' \geqslant \operatorname{Im} V^{\pm 1}\tau' = \operatorname{Im}\tau' \,|\, 2\tau' \pm 1 \,|^{-2}.$$

Thus $|2\tau \pm 1| \geqslant 1$, or $|\tau \pm (1/2)| \geqslant 1/2$. This is equivalent to $|\operatorname{Re} 1/\tau'| \leqslant 1$, and hence $\tau' \in F_1$. QED

We next study how $p(\tau)$ and $q(\tau)$ transform under elements of $\Gamma(2)$.

LEMMA 2.6. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$, *and assume that* $a \equiv d \equiv 1 \bmod 4$. *Then*

(i) $p(\gamma\tau)^2 = (c\tau+d)\, p(\tau)^2$,

(ii) $q(\gamma\tau)^2 = i^c(c\tau+d)\, q(\tau)^2$.

Proof. From (2.7) and $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we obtain

$$\begin{aligned} p(U\tau)^2 &= p(\tau)^2, \quad & p(V\tau)^2 &= (2\tau+1)\, p(\tau)^2, \\ q(U\tau)^2 &= q(\tau)^2, \quad & q(V\tau)^2 &= -(2\tau+1)\, q(\tau)^2. \end{aligned} \tag{2.11}$$

Thus (i) and (ii) hold for U and V. The proof of the previous lemma shows that γ is in the subgroup of $\Gamma(2)$ generated by U and V. We now proceed by induction on the length of γ as a word in U and V.

(i) If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $p(\gamma\tau)^2 = (c\tau+d)\, p(\tau)^2$ then (2.11) implies that

$$\begin{aligned} p(U\gamma\tau)^2 &= p(\gamma\tau)^2 = (c\tau+d)\, p(\tau)^2, \\ p(V\gamma\tau)^2 &= (2\gamma\tau+1)\, p(\gamma\tau)^2 = (2\gamma\tau+1)(c\tau+d)\, p(\tau)^2 \\ &= ((2a+c)\tau+(2b+d))\, p(\tau)^2. \end{aligned}$$

However $U\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$, $V\gamma = \begin{pmatrix} * & * \\ 2a+c & 2b+d \end{pmatrix}$, so that (i) now holds for $U\gamma$ and $V\gamma$.

(ii) Using (2.11) and arguing as above, we see that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = U^{a_1} V^{b_1} \dots U^{a_n} V^{b_n}$, then

$$q(\gamma\tau)^2 = (-1)^{\Sigma b_i}(c\tau+d)\, q(\tau)^2.$$

However, U and V commute modulo 4, so that

$$\gamma \equiv \begin{pmatrix} 1 & 2\Sigma a_i \\ 2\Sigma b_i & 1 \end{pmatrix} \bmod 4 .$$

Thus $c \equiv 2\Sigma b_i \bmod 4$, and (ii) follows. QED

Note that (2.10) is an immediate consequence of Lemma 2.6.

In order to fully exploit this lemma, we introduce the following subgroups of $\Gamma(2)$:

$$\Gamma(2)_0 = \{\gamma \in \Gamma(2) : a \equiv d \equiv 1 \bmod 4\} ,$$

$$\Gamma_2(4) = \{\gamma \in \Gamma(2)_0 : c \equiv 0 \bmod 4\}$$

Note that $\Gamma(2) = \{\pm 1\} \cdot \Gamma(2)_0$ and that $\Gamma_2(4)$ has index 2 in $\Gamma(2)_0$. From Lemma 2.6 we obtain

$$(2.12) \qquad \begin{aligned} p(\gamma\tau)^2 &= (c\tau+d)\, p(\tau)^2 , \qquad \gamma \in \Gamma(2)_0 , \\ q(\gamma\tau)^2 &= (c\tau+d)\, q(\tau)^2 , \qquad \gamma \in \Gamma_2(4) . \end{aligned}$$

Since these functions are holomorphic on $\mathfrak{H}$, one says that $p(\tau)^2$ and $q(\tau)^2$ are weak modular forms of weight one for $\Gamma(2)_0$ and $\Gamma_2(4)$ respectively. The term more commonly used is modular form, which requires that the functions be holomorphic at the cusps (see [30, pp. 28-29] for a precise definition). Because $\Gamma(2)_0$ and $\Gamma_2(4)$ are congruence subgroups of level $N = 4$, this condition reduces to proving that

$$(2.13) \qquad (c\tau+d)^{-1}\, p(\gamma\tau)^2 , \quad (c\tau+d)^{-1}\, q(\gamma\tau)^2 ,$$

are holomorphic functions of $\mathrm{q}^{1/2} = \exp(2\pi i\tau/4)$ for all $\gamma \in SL(2, \mathbf{Z})$. This will be shown later.

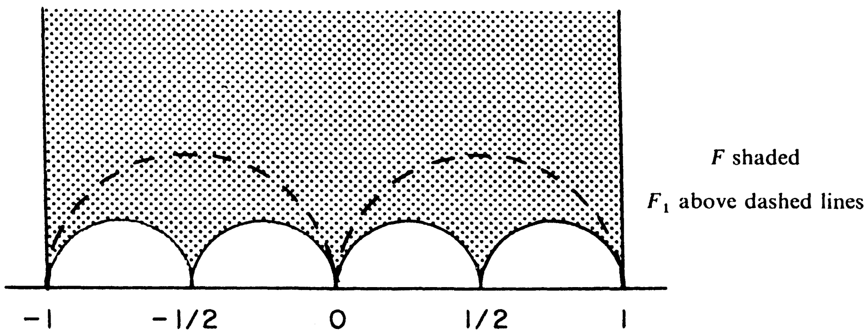
In general, it is well known that the square of a theta function is a modular form of weight one (see [27, Ch. I, § 9]), although the general theory only says that our functions are modular forms for the group

$$\Gamma(4) = \{\gamma \in SL(2, \mathbf{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 4\}$$

(see [27, Ch. I, Prop. 9.2]). We will need the more precise information given by (2.12).

We next study the quotients of $\mathfrak{H}$ by $\Gamma(2)$ and $\Gamma_2(4)$. From Step 1, recall the region $F_1 \subseteq \mathfrak{H}$. We now define a larger region F:

$$F = \{\tau \in \mathfrak{H} : |\operatorname{Re}\tau| \leqslant 1,\ |\tau \pm 1/4| \geqslant 1/4,\ |\tau \pm 3/4| \geqslant 1/4\} .$$



We also set

$$F'_1 = F_1 - (\partial F_1 \cap \{\tau \in \mathfrak{H} : \operatorname{Re}\tau < 0\})$$
$$F' = F - (\partial F \cap \{\tau \in \mathfrak{H} : \operatorname{Re}\tau < 0\}) .$$

LEMMA 2.7. *F'_1 and F' are fundamental domains for $\Gamma(2)$ and $\Gamma_2(4)$ respectively, and the functions k'^2 and k' induce biholomorphic maps*

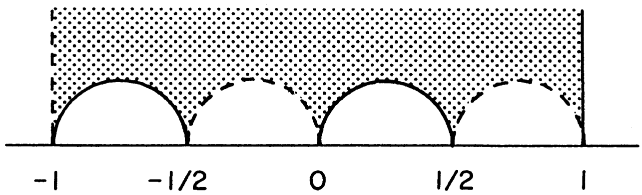
$$\overline{k'^2} : \mathfrak{H}/\Gamma(2) \xrightarrow{\sim} \mathbf{C} - \{0, 1\}$$
$$\overline{k'} \ : \mathfrak{H}/\Gamma_2(4) \xrightarrow{\sim} \mathbf{C} - \{0, \pm 1\} .$$

Proof. A simple modification of the proof of Lemma 2.6 shows that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$, then $p(\gamma\tau)^4 = (c\tau + d)^2 p(\tau)^4$, $q(\gamma\tau)^4 = (c\tau + d)^2 q(\tau)^4$. Thus k'^2 is invariant under $\Gamma(2)$.

Given $\tau \in \mathfrak{H}$, Lemma 2.5 shows that $\gamma\tau \in F_1$ for some $\gamma \in \Gamma(2)$. Since $U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ maps the left vertical line in ∂F_1 to the right one and $V = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ maps the left semicircle in ∂F_1 to the right one, we may assume that $\gamma\tau \in F'_1$. If we also had $\sigma\tau \in F'_1$ for $\sigma \in \Gamma(2)$, then $k'(\sigma\tau)^2 = k'(\tau)^2 = k'(\gamma\tau)^2$, so that $\sigma\tau = \gamma\tau$ by Lemma 2.4. This shows that F'_1 is a fundamental domain for $\Gamma(2)$.

Since $\Gamma(2)_0 \simeq \Gamma(2)/\{\pm 1\}$, F'_1 is also a fundamental domain for $\Gamma(2)_0$. Since $\Gamma_2(4)$ has index 2 in $\Gamma(2)_0$ with 1 and V as coset representatives, it follows that

$$F^* = F'_1 \cup V(F'_1 \cap \{\tau \in \mathfrak{H} : \operatorname{Re}\tau \leqslant 0\}) \cup V^{-1}(F'_1 \cap \{\tau \in \mathfrak{H} : \operatorname{Re}\tau > 0\})$$



F^* shaded

is a fundamental domain for $\Gamma_2(4)$. Since $\begin{pmatrix} -3 & -2 \\ -4 & -3 \end{pmatrix} \in \Gamma_2(4)$ takes the far left semicircle in ∂F to the far right one, it follows that F' is a fundamental domain for $\Gamma_2(4)$.

It now follows easily from Lemma 2.4 that k'^2 induces a bijection $\overline{k'^2}: \mathfrak{H}/\Gamma(2) \to \mathbf{C} - \{0, 1\}$. Since $\Gamma(2)/\{\pm 1\}$ acts freely on $\mathfrak{H}$ by Lemma 2.5, $\mathfrak{H}/\Gamma(2)$ is a complex manifold and $\overline{k'^2}$ is holomorphic. A straightforward argument then shows that $\overline{k'^2}$ is biholomorphic.

Next note that k' is invariant under $\Gamma_2(4)$ by (2.12), and thus induces a map $\overline{k'}: \mathfrak{H}/\Gamma_2(4) \to \mathbf{C} - \{0, \pm 1\}$. Since $\mathfrak{H}/\Gamma(2) = \mathfrak{H}/\Gamma(2)_0$, we obtain a commutative diagram:

$$\begin{array}{ccc} \mathfrak{H}/\Gamma_2(4) & \xrightarrow{\overline{k'}} & \mathbf{C} - \{0, 1\} \\ f \downarrow & & \downarrow g \\ \mathfrak{H}/\Gamma(2)_0 & \xrightarrow{\overline{k'^2}} & \mathbf{C} - \{0, 1\} \end{array}$$

where f is induced by $\Gamma_2(4) \subseteq \Gamma(2)_0$ and g is just $g(z) = z^2$. Note that g is a covering space of degree 2, and the same holds for f since $[\Gamma(2)_0 : \Gamma_2(4)] = 2$ and $\Gamma(2)_0$ acts freely on $\mathfrak{H}$. We know that $\overline{k'^2}$ is a biholomorphism, and it now follows easily that $\overline{k'}$ is also. QED

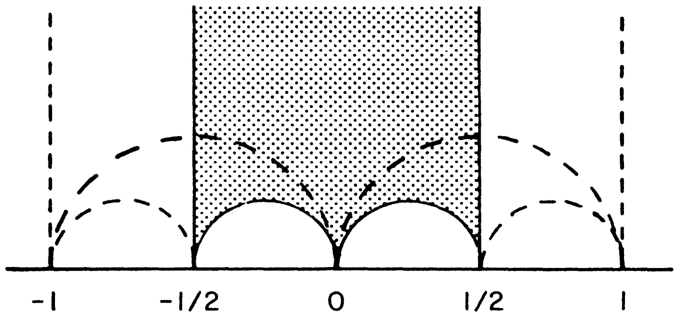
We should point out that $r(\tau)^2$ has properties similar to $p(\tau)^2$ and $q(\tau)^2$. Specifically, $r(\tau)^2$ is a modular form of weight one for the group

$$\Gamma_2(4)^t = \left\{\gamma \in \Gamma(2) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \bmod 4\right\},$$

which is a conjugate of $\Gamma_2(4)$. Furthermore, if we set $k(\tau) = r(\tau)^2/p(\tau)^2$, then k is invariant under $\Gamma_2(4)^t$ and induces a biholomorphism $\overline{k}: \mathfrak{H}/\Gamma_2(4)^t$

$\to \mathbf{C} - \{0, \pm 1\}$. We leave the proofs to the reader. Note also that $k(\tau)^2 + k'(\tau)^2 = 1$ by (2.9).

Our final lemma will be useful in studying the agM. Let F_2 be the region $(1/2)F_1$, pictured below. Note that $F_2 \subseteq F$.



F_2 shaded

F, F_1 indicated by dashed lines

LEMMA 2.8.

$$k'(F_1) = \{z \in \mathbf{C} - \{0, \pm 1\} : \mathrm{Re} z \geqslant 0\},$$

$$k'(F_2) = \{z \in \mathbf{C} - \{0, \pm 1\} : |z| \leqslant 1\}.$$

Proof. We will only treat $k'(F_2)$, the proof for $k'(F_1)$ being quite similar.

We first claim that $\{k'(\tau) : \mathrm{Re}\tau = \pm 1/2\} = S^1 - \{\pm 1\}$. To see this, note that $\mathrm{Re}\tau = \pm 1/2$ and the product expansions (2.6) easily imply that $\overline{k'(\tau)} = k'(\tau)^{-1}$, i.e., $|k'(\tau)| = 1$. How much of the circle is covered? It is easy to see that $k'(\pm 1/2 + it) \to 1$ as $t \to +\infty$. To study the limit as $t \to 0$, note that by (2.10) we have

$$k'(\pm 1/2 + it) = -k'\left(\pm 1/2 + \frac{i}{4t}\right).$$

As $t \to 0$, the right-hand side clearly approaches -1. Then connectivity arguments easily show that all of $S^1 - \{\pm 1\}$ is covered.

Since k' is injective on F' by Lemma 2.7, it follows that $k'(F_2) - S^1$ is connected. Since $|k'(it)| < 1$ for $t > 0$ by (2.6), we conclude that

$$k'(F_2) \subseteq \{z \in \mathbf{C} - \{0, \pm 1\} : |z| \leqslant 1\}.$$

Similar arguments show that

$$k'(F - F_2) \subseteq \{z \in \mathbf{C} : |z| > 1\}.$$

Since $k'(F) = \mathbf{C} - \{0, \pm 1\}$ by Lemma 2.7, both inclusions must be equalities. QED

Gauss' collected works show that he was familiar with most of this material, though it's hard to tell precisely what he knew. For example, he basically has two things to say about $k'(\tau)$:

(i) $k'(\tau)$ has positive real part for $\tau \in F_1$,

(ii) the equation $k'(\tau) = A$ has one and only one solution $\tau \in F_2$.

(See [12, III, pp. 477-478].) Neither statement is correct as written. Modifications have to be made regarding boundary behavior, and Lemma 2.8 shows that we must require $|A| \leqslant 1$ in (ii). Nevertheless, these statements show that Gauss essentially knew Lemma 2.8, and it becomes clear that he would not have been greatly surprised by Lemmas 2.4 and 2.7.

Let us see what Gauss had to say about other matters we've discussed. He was quite aware of linear fractional transformations. Since he used the right half plane, he wrote

$$t' = \frac{at - bi}{cti + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbf{Z}, \quad \operatorname{Re} t > 0$$

(see [12, III, p. 386]). To prevent confusion, we will always translate formulas into ones involving $\tau \in \mathfrak{H}$.

Gauss decomposed an element $\gamma \in SL(2, \mathbf{Z})$ into simpler ones by means of continued fractions. For example, Gauss considers those transformations $\tau^* = \gamma\tau$ which can be written as

$$\begin{aligned} \tau' &= \frac{-1}{\tau} + 2a_1 \\ \tau'' &= \frac{-1}{\tau'} + 2a_2 \\ &\vdots \\ \tau^* = \tau^{(n)} &= \frac{-1}{\tau^{(n-1)}} + 2a_n \end{aligned} \tag{2.14}$$

(see [12, X.1, p. 223]). If $U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, then $\tau'' = U^{a_2}V^{-a_1}\tau$, so that for n even we see a similarity to the proof of Lemma 2.5 (ii). The similarity becomes deeper once we realize that the algorithm used in the proof gives a continued fraction expansion for a/c, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

However, since n can be odd in (2.14), we are dealing with more than just elements of $\Gamma(2)$.

Gauss' real concern becomes apparent when we see him using (2.14) together with the transformation properties of $p(\tau)$. From (2.7) he obtains

$$p(\tau^*) = \sqrt{(-i\tau)(-i\tau')\cdots(-i\tau^{(n-1)})}\, p(\tau)$$

(see [12, X.1, p. 223]). The crucial thing to note is that if $\tau^* = \gamma\tau$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $(-i\tau)\cdots(-i\tau^{(n-1)})$ is just $c\tau + d$ up to a power of i. This tells us how $p(\tau)$ transforms under those γ's described by (2.14). In general, Gauss used similar methods to determine how $p(\tau)$, $q(\tau)$ and $r(\tau)$ transform under arbitrary elements γ of $SL(2, \mathbf{Z})$. The answer depends in part on how $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ reduces modulo 2. Gauss labeled the possible reductions as follows:

a	1	1	1	0	1	0
b	0	1	0	1	1	1
c	0	0	1	1	1	1
d	1	1	1	1	0	0
	1	2	3	4	5	6

(see [12, X.1, p. 224]). We recognize this as the isomorphism $SL(2, \mathbf{Z})/\Gamma(2) \simeq SL(2, \mathbf{F}_2)$, and note that (2.14) corresponds to cases 1 and 6. Then the transformations of $p(\tau)$, $q(\tau)$ and $r(\tau)$ under $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ are given by

(2.15)

	1	2	3	4	5	6
$h^{-1}p(\gamma\tau) =$	$p(\tau)$	$q(\tau)$	$r(\tau)$	$q(\tau)$	$r(\tau)$	$p(\tau)$
$h^{-1}q(\gamma\tau) =$	$q(\tau)$	$p(\tau)$	$p(\tau)$	$r(\tau)$	$p(\tau)$	$r(\tau)$
$h^{-1}r(\gamma\tau) =$	$r(\tau)$	$r(\tau)$	$q(\tau)$	$p(\tau)$	$q(\tau)$	$q(\tau)$

where $h = (i^{\lambda}(c\tau+d))^{1/2}$ and λ is an integer depending on both γ and which one of $p(\tau)$. $q(\tau)$ or $r(\tau)$ is being transformed (see [12, X.1, p. 224]). Note that Lemma 2.6 can be regarded as giving a careful analysis of λ in case 1. An analysis of the other cases may be found in [13, pp. 117-123]. One consequence of this table is that the functions (2.13) are holomorphic functions of $q^{1/2}$, which proves that $p(\tau)^2$, $q(\tau)^2$ and $r(\tau)^2$ are modular forms, as claimed earlier.

Gauss did not make explicit use of congruence subgroups, although they appear implicitly in several places. For example, the table (2.15) shows Gauss using $\Gamma(2)$. As for $\Gamma(2)_0$, we find Gauss writing

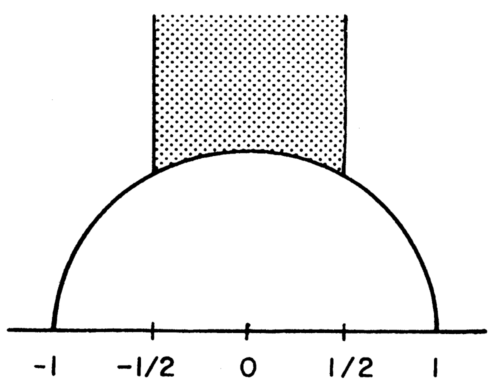
$$k'(\gamma\tau) = i^c k'(\tau)$$

where $\gamma = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ and, as he carefully stipulates, "$ad - bc = 1$, $a \equiv d \equiv 1 \bmod 4$, b, c even" (see [12, III, p. 478]). Also, if we ask which of these γ's leave k' unchanged, then the above equation immediately gives us $\Gamma_2(4)$, though we should be careful not to read too much into what Gauss wrote.

More interesting is Gauss' use of the reduction theory of positive definite quadratic forms as developed in Disquisitiones Arithmeticae (see [11, § 171]). This can be used to determine fundamental domains as follows. A positive definite quadratic form $ax^2 + 2bxy + cy^2$ may be written $a\,|\,x - \tau y\,|^2$ where $\tau \in \mathfrak{H}$. An easy computation shows that this form is equivalent via an element γ of $SL(2, \mathbf{Z})$ to another form $a'\,|\,x - \tau' y\,|^2$ if and only if $\tau' = \gamma^{-1}\tau$. Then, given $\tau \in \mathfrak{H}$, Gauss applies the reduction theory mentioned above to $|\,x - \tau y\,|^2$ and obtains a $SL(2, \mathbf{Z})-$ equivalent form $A\,|\,x - \tau' y\,|^2 = Ax^2 + 2Bxy + Cy^2$ which is reduced, i.e.

$$2\,|\,B\,| \leqslant A \leqslant C$$

(see [11, § 171] and [12, X.1, p. 225]). These inequalities easily imply that $|\operatorname{Re}\tau'| \leqslant 1/2$, $|\operatorname{Re} 1/\tau'| \leqslant 1/2$, so that τ' lies in the shaded region



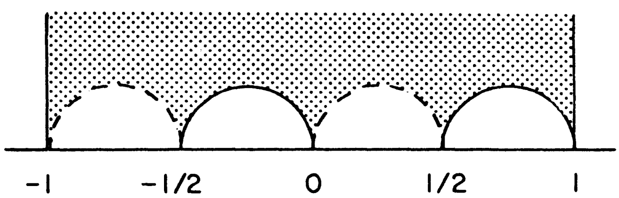
which is well known to be the fundamental domain of $SL(2, \mathbf{Z})$ acting on $\mathfrak{H}$ (see [29, Ch. VII, Thm. 1]).

This seems quite compelling, but Gauss never gave a direct connection between reduction theory and fundamental domains. Instead, he used reduction as follows: given $\tau \in \mathfrak{H}$, the reduction algorithm gives $\tau' = \gamma\tau$ as above and *at the same time* decomposes γ into a continued fraction similar to (2.14). Gauss then applies this to relate $p(\tau')$ and $p(\tau)$, etc., bringing us back to (2.15) (see [12, X.1, p. 225]). But in another place we find such continued fraction decompositions in close conjunction with geometric pictures similar to F_1 and the above (see [12, VIII, pp. 103-105]). Based on this kind of evidence, Gauss' editors decided that he did see the connection (see [12, X.2, pp. 105-106]). Much of this is still a matter of conjecture, but the fact remains that reduction theory is a powerful tool for finding fundamental domains (see [6, Ch. 12]) and that Gauss was aware of some of this power.

Having led the reader on a rather long digression, it is time for us to return to the arithmetic-geometric mean.

Step 3. The Simplest Value

Let $F^\wedge = \{\tau \in F : |\tau - 1/4| > 1/4, |\tau + 3/4| > 1/4\}$. We may picture $F^\wedge$ as follows.



$F^\wedge$ shaded

Let $a, b \in \mathbf{C}^*$ be as usual, and let $\tau \in \mathfrak{H}$ satisfy $k'(\tau) = b/a$. From Lemma 2.3 we know that $\mu = a/p(\tau)^2$ is a value of $M(a, b)$. The goal of Step 3 is to prove the following lemma.

LEMMA 2.9. *If* $\tau \in F^\wedge$, *then* μ *is the simplest value of* $M(a, b)$.

Proof. From Lemma 2.3 we know that

$$a_n = \mu\, p(2^n\tau)^2\,, \qquad b_n = \mu\, q(2^n\tau)^2\,, \qquad n = 0, 1, 2, \ldots \tag{2.16}$$

gives us good sequences converging to μ. We need to show that b_{n+1} is the right choice for $(a_n b_n)^{1/2}$ for all $n \geqslant 0$.

The following equivalences are very easy to prove:

$$|a_{n+1} - b_{n+1}| \leqslant |a_{n+1} + b_{n+1}| \Leftrightarrow \operatorname{Re}\left(\frac{b_{n+1}}{a_{n+1}}\right) \geqslant 0$$

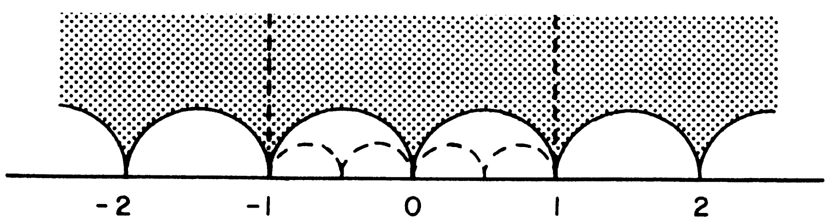
$$|a_{n+1} - b_{n+1}| = |a_{n+1} + b_{n+1}| \Leftrightarrow \operatorname{Re}\left(\frac{b_{n+1}}{a_{n+1}}\right) = 0\,.$$

Recalling the definition of the right choice, we see that we have to prove, for all $n \geqslant 0$, that $\operatorname{Re}\left(\frac{b_{n+1}}{a_{n+1}}\right) \geqslant 0$, and if $\operatorname{Re}\left(\frac{b_{n+1}}{a_{n+1}}\right) = 0$, then $\operatorname{Im}\left(\frac{b_{n+1}}{a_{n+1}}\right) > 0$. From (2.16) we see that

$$\frac{b_{n+1}}{a_{n+1}} = \frac{q(2^{n+1}\tau)^2}{p(2^{n+1}\tau)^2} = k'(2^{n+1}\tau)\,,$$

so that we are reduced to proving that if $\tau \in F^\wedge$, then for all $n \geqslant 0$, $\operatorname{Re}(k'(2^{n+1}\tau)) \geqslant 0$, and if $\operatorname{Re}(k'(2^{n+1}\tau)) = 0$, then $\operatorname{Im}(k'(2^{n+1}\tau)) > 0$.

Let $\tilde{F}_1$ denote the region obtained by translating F_1 by ± 2, ± 4, etc. The drawing below pictures both $\tilde{F}_1$ and F.

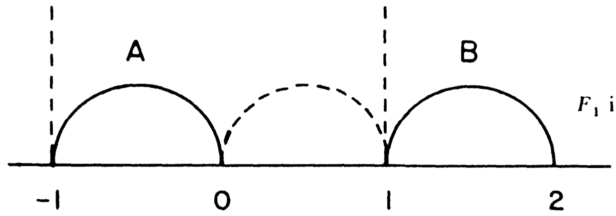


$\tilde{F}_1$ shaded
F indicated by dashed lines

Since $k'(\tau)$ has period 2 and its real part is nonnegative on F_1 by Lemma 2.8, it follows that the real part of $k'(\tau)$ is nonnegative on all of $\tilde{F}_1$. Furthermore, it is clear that on F_1, $\operatorname{Re}(k'(\tau)) = 0$ can occur only on ∂F_1. The product expansions (2.6) show that $k'(\tau)$ is real when $\operatorname{Re}\tau = \pm 1$, so that on F_1, $\operatorname{Re}(k'(\tau)) = 0$ can occur only on the boundary semicircles. From the periodicity of $k'(\tau)$ we conclude that $k'(\tau)$ has positive real part on the interior $\tilde{F}_1^0$ of $\tilde{F}_1$.

If $\tau \in F^\wedge$, then the above drawing makes it clear that $2^{n+1}\tau \in \tilde{F}_1$ for $n \geqslant 0$ and that $2^{n+1}\tau \in \tilde{F}_1^0$ for $n \geqslant 1$. We thus see that $\operatorname{Re}(k'(2^{n+1}\tau)) > 0$ for $n \geqslant 0$ unless $n = 0$ and $2\tau \in \partial\tilde{F}_1$. Thus the lemma will be proved once we show that $\operatorname{Im}(k'(2\tau)) > 0$ when $\tau \in F^\wedge$ and $2\tau \in \partial\tilde{F}_1$.

These last two conditions imply that 2τ lies on one of the semicircles A and B pictured below.



F_1 indicated by dashed lines

By periodicity, k' takes the same values on A and B. Thus it suffices to show that $\operatorname{Im}(k'(2\tau)) > 0$ for $2\tau \in A$. Since $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ maps the line $\operatorname{Re}\sigma = 1$ to A, we can write $2\tau = -1/\sigma$, where $\operatorname{Re}\sigma = 1$. Then, using (2.7), we obtain

$$k'(2\tau) = k'(-1/\sigma) = \frac{q(-1/\sigma)^2}{p(-1/\sigma)^2} = \frac{r(\sigma)^2}{p(\sigma)^2}.$$

Since $\operatorname{Re}\sigma = 1$, the product expansions (2.6) easily show that

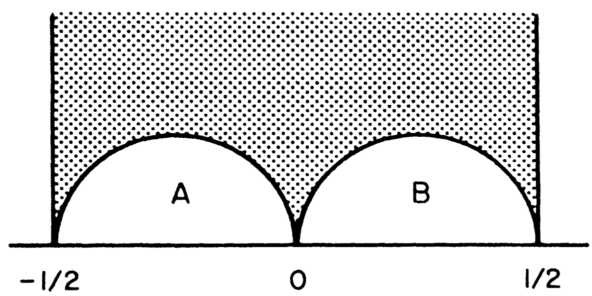
$$\operatorname{Im}(r(\sigma)^2/p(\sigma)^2) > 0,$$

which completes the proof of Lemma 2.9. QED

Step 4. Conclusion of the Proof.

We can now prove Theorem 2.2. Recall that at the end of Step 1 we were left with three tasks: to find all solutions τ of $k'(\tau) = b/a$, to relate the values of $a/p(\tau)^2$ thus obtained, and to show that all values of $M(a, b)$ arise in this way.

We are given $a, b \in \mathbf{C}^*$ with $a \neq \pm b$ and $|a| \geqslant |b|$. We will first find $\tau_0 \in F_2 \cap F^\wedge$ such that $k'(\tau_0) = b/a$. Since $|b/a| \leqslant 1$, Lemma 2.8 gives us $\tau_0 \in F_2$ with $k'(\tau_0) = b/a$. Could τ_0 fail to lie in $F^\wedge$? From the definition of $F^\wedge$, this only happens when τ_0 lies in the semicircle B pictured below.



F_2 shaded

However, $\gamma = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \in \Gamma_2(4)$ takes B to the semicircle A. Since k' is invariant under $\Gamma_2(4)$, we have $k'(\gamma\tau_0) = k'(\tau_0) = b/a$. Thus, replacing τ_0 by $\gamma\tau_0$, we may assume that $\tau_0 \in F_2 \cap F^\wedge$.

It is now easy to solve the first two of our tasks. Since k' induces a bijection $\mathfrak{H}/\Gamma_2(4) \cong \mathbf{C} - \{0, \pm 1\}$, it follows that all solutions of $k'(\tau) = b/a$ are given by $\tau = \gamma\tau_0, \gamma \in \Gamma_2(4)$. This gives us the following set of values of $M(a, b)$:

$$\{a/p(\gamma\tau_0)^2 : \gamma \in \Gamma_2(4)\}\,.$$

Recalling the statement of Theorem 2.2, it makes sense to look at the reciprocals of these values:

$$R = \{p(\gamma\tau_0)^2/a : \gamma \in \Gamma_2(4)\}$$

By (2.12), $p(\gamma\tau_0)^2 = (c\tau_0+d)\,p(\tau_0)^2$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(4) \subseteq \Gamma(2)_0$. Setting $\mu = a/p(\tau_0)^2$, we have

$$\begin{aligned} R &= \left\{(c\tau_0+d)\,p(\tau_0)^2/a : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(4)\right\} \\ &= \left\{(c\tau_0+d)/\mu : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(4)\right\}. \end{aligned}$$

An easy exercise in number theory shows that the bottom rows (c, d) of elements of $\Gamma_2(4)$ are precisely those pairs (c, d) satisfying $GCD(c, d) = 1$, $c \equiv 0 \bmod 4$ and $d \equiv 1 \bmod 4$. We can therefore write

$$R = \{(c\tau_0+d)/\mu : GCD(c, d) = 1, \quad c \equiv 0 \bmod 4, \quad d \equiv 1 \bmod 4\}\,.$$

Then setting $\lambda = i\mu/\tau_0$ gives us

$$(2.17)\quad R = \left\{\frac{d}{\mu} + \frac{ic}{\lambda} : GCD(c, d) = 1, \quad d \equiv 1 \bmod 4, \quad c \equiv 0 \bmod 4\right\}.$$

Finally, we will show that μ and λ are the simplest values of $M(a, b)$ and $M(a+b, a-b)$ respectively. This is easy to see for μ: since $\tau_0 \in F^\wedge$, Lemma 2.9 implies that $\mu = a/p(\tau_0)^2$ is the simplest value of $M(a, b)$. Turning to λ, recall from Lemma 2.3 that $a = \mu p(\tau_0)^2$ and $b = \mu q(\tau_0)^2$. Thus by (2.8) and (2.7),

$$a + b = \mu\big(p(\tau_0)^2 + q(\tau_0)^2\big) = 2\mu\, p(2\tau_0)^2 = 2\mu\left(\frac{i}{2\tau_0}\right) p\left(\frac{-1}{2\tau_0}\right)^2,$$

$$a - b = \mu(p(\tau_0)^2 - q(\tau_0)^2) = 2\mu\, r(2\tau_0)^2 = 2\mu \left(\frac{i}{2\tau_0}\right) q\left(\frac{-1}{2\tau_0}\right)^2,$$

which implies that

$$a + b = \lambda\, p(-1/2\tau_0)^2, \quad a - b = \lambda\, q(-1/2\tau_0)^2.$$

Hence λ is a value of $M(a+b, a-b)$. To see that it is the simplest value, we must show that $-1/2\tau_0 \in F^\wedge$ (by Lemma 2.9). Since $\tau_0 \in F_2$, we have $2\tau_0 \in F_1$. But F_1 is stable under $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that $-1/2\tau_0 \in F_1$. The inclusion $F_1 \subseteq F^\wedge$ is obvious, and $-1/2\tau_0 \in F^\wedge$ follows. This completes our first two tasks.

Our third and final task is to show that (2.17) gives the reciprocals of *all* values of $M(a, b)$. This will finish the proof of Theorem 2.2. So let μ' be a value of $M(a, b)$, and let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be the good sequences such that $\mu' = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Then there is some m such that b_{n+1} is the right choice for $(a_n b_n)^{1/2}$ for all $n \geqslant m$, and thus μ' is the simplest value of $M(a_m, b_m)$. Since $k' : F' \to \mathbf{C} - \{0, \pm 1\}$ is surjective by Lemma 2.7, we can find $\tau \in F'$ such that $k'(\tau) = b_m/a_m$. Arguing as above, we may assume that $\tau \in F^\wedge$. Then Lemma 2.9 shows that $\mu' = a_m/p(\tau)^2$ and also that for $n \geqslant m$,

$$a_n = \mu'\, p(2^{n-m}\tau)^2, \quad b_n = \mu'\, q(2^{n-m}\tau)^2. \tag{2.18}$$

Let us study a_{m-1} and b_{m-1}. Their sum and product are $2\,a_m$ and b_m^2 respectively. From the quadratic formula we see that

$$\{a_{m-1}, b_{m-1}\} = \{a_m \pm (a_m^2 - b_m^2)^{1/2}\}.$$

Using (2.9), we obtain

$$a_m^2 - b_m^2 = \mu'^2(p(\tau)^4 - q(\tau)^4) = \mu'^2 r(\tau)^4,$$

so that, again using (2.9), we have

$$a_m \pm (a_m^2 - b_m^2)^{1/2} = \mu'(p(\tau)^2 \pm r(\tau)^2) = \begin{cases} \mu' p(\tau/2)^2 \\ \mu' q(\tau/2)^2. \end{cases}$$

Thus, either

$$a_{m-1} = \mu'\, p(\tau/2)^2,\ b_{m-1} = \mu'\, q(\tau/2)^2 \text{ or } a_{m-1} = \mu'\, q(\tau/2)^2,\ b_m = \mu'\, p(\tau/2)^2.$$

In the former case, set $\tau_1 = \tau/2$. Then from (2.18) we easily see that for $n \geqslant m-1$,

$$a_n = \mu' \, p(2^{n-m+1}\tau_1)^2, \quad b_n = \mu' \, q(2^{n-m+1}\tau_1)^2. \tag{2.19}$$

If the latter case holds, let $\tau_1 = \tau/2 + 1$. From (2.7) we see that $a_{m-1} = \mu' \, p(\tau_1)^2$, $b_{m-1} = \mu' \, q(\tau_1)^2$, and it also follows easily that $p(2^{n-m+1}\tau_1) = p(2^{n-m}\tau)$ and $q(2^{n-m+1}\tau_1) = q(2^{n-m}\tau)$ for all $n \geqslant m$. Thus (2.19) holds for this choice of τ_1 and $n \geqslant m-1$.

By induction, this argument shows that there is $\tau_m \in \mathfrak{H}$ such that for all $n \geqslant 0$,

$$a_n = \mu' \, p(2^n\tau_m)^2, \quad b_n = \mu' \, q(2^n\tau_m)^2.$$

In particular, $\mu' = a/p(\tau_m)^2$ and $k'(\tau_m) = b/a$. Thus $(\mu')^{-1} = p(\tau_m)^2/a$ is in the set R of (2.17). This shows that R consists of the reciprocals of all values of $M(a, b)$, and the proof of Theorem 2.2 is now complete. QED

We should point out that the proof just given, though arrived at independently, is by no means original. The first proofs of Theorem 2.2 appeared in 1928 in [15] and [35]. Geppert's proof [15] is similar to ours in the way it uses the theory of theta functions and modular functions. The other proof [35], due to von David, is much shorter; it is a model of elegance and conciseness.

Let us discuss some consequences of the proof of Theorem 2.2. First, the formula $\lambda = i\mu/\tau_0$ obtained above is quite interesting. We say that τ_0 "uniformizes" the simplest value μ of $M(a, b)$, where

$$a = \mu \, p(\tau_0)^2, \quad b = \mu \, q(\tau_0)^2.$$

Writing the above formula as $\tau_0 = i\dfrac{\mu}{\lambda}$, we see how to *explicitly compute* τ_0 in terms of the simplest values of $M(a, b)$ and $M(a+b, a-b)$. This is especially useful when $a > b > 0$. Here, if we set $c = \sqrt{a^2 - b^2}$, then, using the notation of § 1, the simplest values are $M(a, b)$ and $M(a, c)$, so that

$$\tau_0 = i\frac{M(a, b)}{M(a, c)}. \tag{2.20}$$

A nice example is when $a = \sqrt{2}$ and $b = 1$. Then $c = 1$, which implies $\tau_0 = i$! Thus $M(\sqrt{2}, 1) = \sqrt{2}/p(i)^2 = 1/q(i)^2$. From § 1 we know $M(\sqrt{2}, 1) = \pi/\varpi$, which gives us the formulas

$$\begin{aligned} \varpi/\pi &= 2^{-1/2}p(i)^2 = 2^{-1/2}(1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\ldots)^2\,, \\ \varpi/\pi &= q(i)^2 = (1-2e^{-\pi}+2e^{-4\pi}-2e^{-9\pi}+\ldots)^2\,. \end{aligned} \tag{2.21}$$

We will discuss the importance of this in § 3.

Turning to another topic, note that $M(a, b)$ is clearly homogeneous of degree 1, i.e., if μ is a value of $M(a, b)$, then $c\mu$ is a value of $M(ca, cb)$ for $c \in \mathbf{C}^*$. Thus, it suffices to study $M(1, b)$ for $b \in \mathbf{C} - \{0, \pm 1\}$. Its values are given by $\mu = 1/p(\tau)^2$ where $k'(\tau) = b$. Since $k' : \mathfrak{H} \to \mathbf{C} - \{0, \pm 1\}$ is a local biholomorphism, it follows that $M(1, b)$ is a multiple valued holomorphic function. To make it single valued, we pull back to the universal cover via k', giving us $M(1, k'(\tau))$. We thus obtain

$$M(1, k'(\tau)) = 1/p(\tau)^2\,.$$

This shows that the agM may be regarded as a meromorphic modular form of weight -1.

Another interesting multiple valued holomorphic function is the elliptic integral $\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2}d\phi$. This is a function of $k \in \mathbf{C} - \{0, \pm 1\}$. If we pull back to the universal cover via $k : \mathfrak{H} \to \mathbf{C} - \{0, \pm 1\}$ (recall from Step 2 that $k(\tau) = r(\tau)^2/p(\tau)^2$), then it is well known that

$$\frac{2}{\pi}\int_0^{\pi/2} (1-k(\tau)^2\sin^2\phi)^{-1/2}d\phi = p(\tau)^2$$

(see [36, p. 500]). Combining the above two equations, we obtain

$$\frac{1}{M(1, k'(\tau))} = p(\tau)^2 = \frac{2}{\pi}\int_0^{\pi/2} (1-k(\tau)^2\sin^2\phi)^{-1/2}d\phi\,,$$

which may be viewed as a rather amazing generalization of (1.9).

Finally, let us make some remarks about the set $\mathscr{M}$ of values of $M(a, b)$, where a and b are fixed. If μ denotes the simplest value of $M(a, b)$, then it can be shown that $|\mu| \geqslant |\mu'|$ for $\mu' \in \mathscr{M}$, and $|\mu|$ is a strict maximum if $\text{ang}(a, b) \neq \pi$. This may be proved directly from the definitions (see [35]). Another proof proceeds as follows. We know that any $\mu' \in \mathscr{M}$ can be written

$$\mu' = \mu/(c\tau_0+d)\,, \tag{2.22}$$

where $\tau_0 \in F_2$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(4)$. Thus it suffices to prove that $|c\tau_0 + d| \geqslant 1$ whenever $\tau_0 \in F_2$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(4)$. This is left as an exercise for the reader.

We can also study the accumulation points of $\mathscr{M}$. Since $|c\tau_0 + d|$ is a positive definite quadratic form in c and d, it follows from (2.22) that $0 \in \mathbf{C}$ is the only accumulation point of $\mathscr{M}$. This is very satisfying once we recall from Proposition 2.1 that $0 \in \mathbf{C}$ is the common limit of all non-good sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ coming from (2.1).

The proof of Theorem 2.2 makes one thing very clear: we have now seen "an entirely new field of analysis." However, before we can say that Gauss' prediction of May 30, 1799 has been fulfilled, we need to show that the proof given above reflects what Gauss actually did. Since we know from Step 2 about his work with the theta functions $p(\tau)$, $q(\tau)$ and $r(\tau)$ and the modular function $k'(\tau)$, it remains to see how he applied all of this to the arithmetic-geometric mean.

The connections we seek are found in several places in Gauss' notes. For example, he states very clearly that if

$$a = \mu\, p(\tau)^2\,, \quad b = \mu\, q(\tau)^2\,, \tag{2.23}$$

then the sequences $a_n = \mu\, p(2^n\tau)^2$, $b_n = \mu\, q(2^n\tau)^2$ satisfy the agM algorithm (2.1) with μ as their common limit (see [12, III, p. 385 and pp. 467-468]). This is precisely our Lemma 2.3. In another passage, Gauss defines the "einfachste Mittel" (simplest mean) to be the limit of those sequences where $\mathrm{Re}(b_{n+1}/a_n) > 0$ for all $n \geqslant 0$ (see [12, III, p. 477]). This is easily seen to be equivalent to our definition of simplest value when $\mathrm{ang}(a, b) \neq \pi$. On the same page, Gauss then asserts that for $\tau \in F_2$, μ is the simplest value of $M(a, b)$ for a, b as in (2.23). This is a weak form of Lemma 2.9. Finally, consider the following quote from [12, VIII, p. 101]: "In order to solve the equation $\dfrac{q(t)}{p(t)} = A$, one sets $A^2 = n/m$ and takes the agM of m and n; let this be μ. One further takes the agM of m and $\sqrt{m^2 - n^2}$, or, what is the same, of $m + n$ and $m - n$; let this be λ. One then has $t = \mu/\lambda$. This gives only one value of t; all others are contained in the formula

$$t' = \frac{\alpha t - 2\beta i}{\delta - 2\gamma t i},$$

where α, β, γ, δ signify all integers which satisfy the equation $\alpha\delta - 4\beta\gamma = 1$." Recall that $\operatorname{Re} t > 0$, so that our τ is just ti. Note also that the last assertion is not quite correct.

Unfortunately, in spite of these compelling fragments, Gauss never actually stated Theorem 2.2. The closest he ever came is the following quote from [12, X.1, p. 219]: "The agM changes, when one chooses the negative value for one of n', n'', n''' etc.: however all resulting values are of the following form:

$$\frac{1}{(\mu)} = \frac{1}{\mu} + \frac{4ik}{\lambda}.\text{"} \tag{2.24}$$

Here, Gauss is clearly dealing with $M(m, n)$ where $m > n > 0$. The fraction $1/\mu$ in (2.24) is correct: in fact, it can be shown that if the negative value of $n^{(r)}$ is chosen, and all other choices are the right choice, then the corresponding value μ' of $M(m, n)$ satisfies

$$\frac{1}{\mu'} = \frac{1}{\mu} + \frac{2^{r+1}i}{\lambda}$$

(see [13, p. 140]). So (2.24) is only a very special case of Theorem 2.2.

There is one final piece of evidence to consider: the 109th entry in Gauss' mathematical diary. It reads as follows:

> Between two given numbers there are always infinitely many means both arithmetic-geometric and harmonic-geometric, the observation of whose mutual connection has been a source of happiness for us.

(See [12, X.1, p. 550]. The harmonic-geometric mean of a and b is $M(a^{-1}, b^{-1})^{-1}$.) What is amazing is the date of this entry: June 3, 1800, a little more than a year after May 30, 1799. We know from § 1 that Gauss' first proofs of Theorem 1.1 date from December 1799. So less than six months later Gauss was aware of the multiple valued nature of $M(a, b)$ and of the relations among these values! One tantalizing question remains: does the phrase "mutual connection" refer only to (2.24), or did Gauss have something more like Theorem 2.2 in mind? Just how much did he know about modular functions as of June 3, 1800? In order to answer these questions, we need to examine the history of the whole situation more closely.

3. Historical remarks

The main difficulty in writing about the history of mathematics is that so much has to be left out. The mathematics we are studying has a richness which can never be conveyed in one article. For instance, our discussion of Gauss' proofs of Theorem 1.1 in no way does justice to the complexity of his mathematical thought; several important ideas were simplified or omitted altogether. This is not entirely satisfactory, yet to rectify such gaps is beyond the scope of this paper. As a compromise, we will explore the three following topics in more detail:

A. The history of the lemniscate,

B. Gauss' work on inverting lemniscatic integrals, and

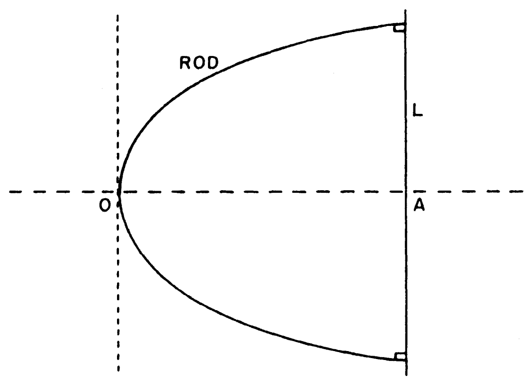
C. The chronology of Gauss' work on the agM and theta functions.

A. The lemniscate was discovered by Jacob Bernoulli in 1694. He gives the equation in the form

$$xx + yy = a\sqrt{xx - yy}$$

(in § 1 we assumed that $a = 1$), and he explains that the curve has "the form of a figure 8 on its side, as of a band folded into a knot, or of a lemniscus, or of a knot of a French ribbon" (see [2, p. 609]). "Lemniscus" is a Latin word (taken from the Greek) meaning a pendant ribbon fastened to a victor's garland.

More interesting is that the integral $\int_0^1 (1-z^4)^{-1/2}dz$, which gives one-quarter of the arc length of the lemniscate, had been discovered three years earlier in 1691! This was when Bernoulli worked out the equation of the so-called *elastic curve*. The situation is as follows: a thin elastic rod is bent until the two ends are perpendicular to a given line L.



After introducing cartesian coordinates as indicated and letting a denote 0A, Bernoulli was able to show that the upper half of the curve is given by the equation

$$y = \int_0^x \frac{z^2 dz}{\sqrt{a^4 - z^4}}, \tag{3.1}$$

where $0 \leqslant x \leqslant a$ (see [2, pp. 567-600]).

It is convenient to assume that $a = 1$. But as soon as this is done, we no longer know how long the rod is. In fact, (3.1) implies that the arc length from the origin to a point (x, y) on the rescaled elastic curve is $\int_0^x (1-z^4)^{-1/2} dz$. Thus the length of the whole rod is $2\int_0^1 (1-z^4)^{-1/2} dz$, which is precisely Gauss' ϖ!

How did Bernoulli get from here to the lemniscate? He was well aware of the transcendental nature of the elastic curve, and so he used a standard seventeenth century trick to make things more manageable: he sought "an algebraic curve... whose rectification should agree with the rectification of the elastic curve" (this quote is from Euler [9, XXI, p. 276]).

Jacob actually had a very concrete reason to be interested in arc length: in 1694, just after his long paper on the elastic curve was published, he solved a problem of Leibniz concerning the "isochrona paracentrica" (see [2, pp. 601-607]). This called for a curve along which a falling weight recedes from or approaches a given point equally in equal times. Since Bernoulli's solution involved the arc length of the elastic curve, it was natural for him to seek an algebraic curve with the same arc length. Very shortly thereafter, he found the equation of the lemniscate (see [2, pp. 608-612]). So we really can say that the arc length of the lemniscate was known well before the curve itself.

But this is not the full story. In 1694 Jacob's younger brother Johann independently discovered the lemniscate! Jacob's paper on the isochrona paracentrica starts with the differential equation

$$(xdx + ydy)\sqrt{y} = (xdy - ydx)\sqrt{a},$$

which had been derived earlier by Johann, who, as Jacob rather bluntly points out, hadn't been able to solve it. Johann saw this comment for the first time when it appeared in June 1694 in Acta Eruditorum. He took up the challenge and quickly produced a paper on the isochrona paracentrica which gave the equation of the lemniscate and its relation to the elastic curve. This appeared in Acta Eruditorum in October 1694 (see [3, pp. 119-

122]), but unfortunately for Johann, Jacob's article on the lemniscate appeared in the September issue of the same journal. There followed a bitter priority dispute. Up to now relations between the brothers had been variable, sometimes good, sometimes bad, with always a strong undercurrent of competition between them. After this incident, amicable relations were never restored. (For details of this controversy, as well as a fuller discussion of Jacob's mathematical work, see [18].)

We need to mention one more thing before going on. Near the end of Jacob's paper on the lemniscate, he points out that the y-value $\int_0^x z^2(a^4-z^4)^{-1/2}dz$ of the elastic curve can be expressed as the difference of an arc of the ellipse with semiaxes $a\sqrt{2}$ and a, and an arc of the lemniscate (see [2, pp. 611-612]). This observation is an easy consequence of the equation

$$\int_0^x \frac{a^2dz}{(a^4-z^4)^{1/2}} + \int_0^x \frac{z^2dz}{(a^4-z^4)^{1/2}} = \int_0^x \left(\frac{a^2+z^2}{a^2-z^2}\right)^{1/2} dz\,. \tag{3.2}$$

What is especially intriguing is that the ratio $\sqrt{2}:1$, so important in Gauss' observation of May 30, 1799, was present at the very birth of the lemniscate.

Throughout the eighteenth century the elastic curve and the lemniscate appeared in many papers. A lot of work was done on the integrals $\int_0^1 (1-z^4)^{-1/2}dz$ and $\int_0^1 z^2(1-z^4)^{-1/2}dz$. For example, Stirling, in a work written in 1730, gives the approximations

$$\int_0^1 \frac{dz}{\sqrt{1-z^4}} = 1.31102877714605987$$

$$\int_0^1 \frac{z^2dz}{\sqrt{1-z^4}} = .59907011736779611$$

(see [31, pp. 57-58]). Note that the second number doubled is 1.19814023473559222, which agrees with $M(\sqrt{2}, 1)$ to sixteen decimal places. Stirling also comments that these two numbers add up to one half the circumference of an ellipse with $\sqrt{2}$ and 1 as axes, a special case of Bernoulli's observation (3.2).

Another notable work on the elastic curve was Euler's paper "De miris proprietatibus curvae elasticae sub equatione $y = \int \frac{xxdx}{\sqrt{1-x^4}}$ contentae" which appeared posthumously in 1786. In this paper Euler gives approximations to the above integrals (not as good as Stirling's) and, more importantly, proves the amazing result that

$$\int_0^1 \frac{dz}{\sqrt{1-z^2}} \cdot \int_0^1 \frac{z^2 dz}{\sqrt{1-z^4}} = \frac{\pi}{4} \tag{3.3}$$

(see [9, XXI, pp. 91-118]). Combining this with Theorem 1.1 we see that

$$M(\sqrt{2}, 1) = 2\int_0^1 \frac{z^2 dz}{\sqrt{1-z^4}},$$

so that the coincidence noted above has a sound basis in fact.

We have quoted these two papers on the elastic curve because, as we will see shortly, Gauss is known to have read them. Note that each paper has something to contribute to the equality $M(\sqrt{2}, 1) = \pi/\varpi$: from Stirling, we get the ratio $\sqrt{2}$: 1, and from Euler we get the idea of using an equation like (3.3).

Unlike the elastic curve, the story of the lemniscate in the eighteenth century is well known, primarily because of the key role it played in the development of the theory of elliptic integrals. Since this material is thoroughly covered elsewhere (see, for example, [1, Ch. 1-3], [8, pp. 470-496], [19, § 1-§ 4] and [21, § 19.4]), we will mention only a few highlights. One early worker was C. G. Fagnano who, following some ideas of Johann Bernoulli, studied the ways in which arcs of ellipses and hyperbolas can be related. One result, known as Fagnano's Theorem, states that the sum of two appropriately chosen arcs of an ellipse can be computed algebraically in terms of the coordinates of the points involved. He also worked on the lemniscate, starting with the problem of halving that portion of the arc length of the lemniscate which lies in one quadrant. Subsequently he found methods for dividing this arc length into n equal pieces, where $n = 2^m$, $3 \cdot 2^m$ or $5 \cdot 2^m$. These researches of Fagnano's were published in the period 1714-1720 in an obscure Venetian journal and were not widely known. In 1750 he had his work republished, and he sent a copy to the Berlin Academy. It was given to Euler for review on December 23, 1751. Less than five weeks later, on January 27, 1752, Euler read a paper giving new derivations for Fagnano's results on elliptic and hyperbolic arcs as well as significantly new results on lemniscatic arcs. By 1753 he had a general addition theorem for lemniscatic integrals, and by 1758 he had the addition theorem for elliptic integrals (see [9, XX, pp. VII-VIII]). This material was finally published in 1761, and for the first time there was a genuine theory of elliptic integrals. For the next twenty years Euler and Lagrange made significant contributions, paving the way for Legendre to cast the field in its classical form which we glimpsed at the end of § 1. Legendre published his definitive treatise on elliptic integrals in two volumes in 1825 and 1826. The irony is that in 1828 he had to publish a third volume describing the groundbreaking papers of Abel and Jacobi which rendered obsolete much of his own work (see [23]).

An important problem not mentioned so far is that of computing tables of elliptic integrals. Such tables were needed primarily because of the many applications of elliptic integrals to mechanics. Legendre devoted the entire second volume of his treatise to this problem. Earlier Euler had computed these integrals using power series similar to (1.8) (see also [9, XX, pp. 21-55]), but these series often converged very slowly. The real breakthrough came in Lagrange's 1785 paper "Sur une nouvelle méthode de calcul intégral" (see [22, pp. 253-312]). Among other things, Lagrange is concerned with integrals of the form

$$\int \frac{M\,dy}{\sqrt{(1+p^2y^2)(1+q^2y^2)}}, \tag{3.4}$$

where M is a rational function of y^2 and $p \geqslant q > 0$. He defines sequences $p, p', p'', \ldots, q, q', q'', \ldots$ as follows:

$$\begin{aligned} p' &= p + (p^2-q^2)^{1/2}, & q' &= p - (p^2-q^2)^{1/2}, \\ p'' &= p' + (p'^2-q'^2)^{1/2}, & q'' &= p' - (p'^2-q'^2)^{1/2}, \\ &\ \ \vdots & &\ \ \vdots \end{aligned} \tag{3.5}$$

and then, using the substitution

$$y' = \frac{y((1+p^2y^2)(1+q^2y^2))^{1/2}}{1+q^2y^2} \tag{3.6}$$

he shows that

$$((1+p^2y^2)(1+q^2y^2))^{-1/2}dy = ((1+p'^2y'^2)(1+q'^2y'^2))^{-1/2}dy'. \tag{3.7}$$

Two methods of approximation are now given. The first starts by observing that the sequence $p, p', p'', \ldots$ approaches $+\infty$ while $q, q', q'', \ldots$ approaches 0. Thus by iterating the substitution (3.6) in the integral of (3.4), one can eventually assume that $q = 0$, which gives an easily computable integral. The second method consists of doing the first backwards: from (3.5) one easily obtains

$$p = (p'+q')/2\,, \qquad q = (p'q')^{1/2}\,.$$

Lagrange then observes that continuing this process leads to sequences $p', p, {}'p, {}''p, \ldots, q', q, {}'q, {}''q, \ldots$ which converge to a common limit (see [22, p. 271]). Hence iterating (3.6) allows one to eventually assume $p = q$, again giving an easily computable integral.

So here we are in 1785, staring at the definition of the arithmetic-geometric mean, six years before Gauss' earliest work on the subject. By setting $py = \tan\phi$, one obtains

$$((1+p^2y^2)(1+q^2y^2))^{-1/2}dy = (p^2\cos^2\phi + q^2\sin^2\phi)^{-1/2}d\phi\,,$$

so that (3.6) and (3.7) are precisely (1.5) and (1.6) from the proof of Theorem 1.1. Thus Lagrange not only could have defined the agM, he could have also proved Theorem 1.1 effortlessly. Unfortunately, none of this happened; Lagrange never realized the power of what he had discovered.

One question emerges from all of this: did Gauss ever see Lagrange's article? The library of the Collegium Carolinum in Brunswick had some of Lagrange's works (see [4, p. 9]) and the library at Gottingen had an extensive collection (see [12, X.2, p. 22]). On the other hand, Gauss, in the research announcement of his 1818 article containing the proof of Theorem 1.1, claims that his work is independent of that of Lagrange and Legendre (see [12, III, p. 360]). A fuller discussion of these matters is in [12, X.2, pp. 12-22]. Assuming that Gauss did discover the agM independently, we have the amusing situation of Gauss, who anticipated so much in Abel, Jacobi and others, himself anticipated by Lagrange.

The elastic curve and the lemniscate were equally well known in the eighteenth century. As we will soon see, Gauss at first associated the integral $\int (1-z^4)^{-1/2}dz$ with the elastic curve, only later to drop it in favor of the lemniscate. Subsequent mathematicians have followed his example. Today, the elastic curve has been largely forgotten, and the lemniscate has suffered the worse fate of being relegated to the polar coordinates section of calculus books. There it sits next to the formula for arc length in polar coordinates, which can never be applied to the lemniscate since such texts know nothing of elliptic integrals.

B. Our goal in describing Gauss' work on the lemniscate is to learn more of the background to his observation of May 30, 1799. We will see that the lemniscatic functions played a key role in Gauss' development of the arithmetic-geometric mean.

Gauss began innocently enough in September 1796, using methods of Euler to find the formal power series expansion of the inverse function of first $\int (1-x^3)^{-1/2}dx$, and then more generally $\int (1-x^n)^{-1/2}dx$ (see [12, X.1, p. 502]). Things became more serious on January 8, 1797. The 51st entry in his mathematical diary, bearing this date, states that "I have begun to investigate the elastic curve depending on $\int (1-x^4)^{-1/2}dx$." Notes written at the same time show that Gauss was reading the works of Euler and Stirling on the elastic curve, as discussed earlier. Significantly, Gauss later struck out the word "elastic" and replaced it with "lemniscatic" (see [12, X.1, pp. 147 and 510]).

Gauss was strongly motivated by the analogy to the circular functions. For example, notice the similarity between $\varpi/2 = \int_0^1 (1-z^4)^{-1/2}dz$ and $\pi/2 = \int_0^1 (1-z^2)^{-1/2}dz$. (This similarity is reinforced by the fact that many eighteenth century texts used ϖ to denote π — see [12, X.2, p. 33].) Gauss then defined the lemniscatic functions as follows:

$$\text{sinlemn}\left(\int_0^x (1-z^4)^{-1/2}dz\right) = x$$

$$\text{coslemn}\left(\varpi/2 - \int_0^x (1-z^4)^{-1/2}dz\right) = x$$

(see [12, III, p. 404]). Gauss often used the abbreviations sl ϕ and cl ϕ for sinlemn ϕ and coslemn ϕ respectively, a practice we will adopt. From Euler's addition theorem one easily obtains

$$\mathrm{sl}^2\phi + \mathrm{cl}^2\phi + \mathrm{sl}^2\phi\,\mathrm{cl}^2\phi = 1 \tag{3.8}$$

$$\mathrm{sl}(\phi+\phi') = \frac{\mathrm{sl}\,\phi\,\mathrm{cl}\,\phi' + \mathrm{sl}\,\phi'\,\mathrm{cl}\,\phi}{1 - \mathrm{sl}\,\phi\,\mathrm{sl}\,\phi'\,\mathrm{cl}\,\phi\,\mathrm{cl}\,\phi'} \tag{3.9}$$

(see [12, X.1, p. 147]).

Other formulas can now be derived in analogy with the trigonometric functions (see [25, pp. 155-156] for a nice treatment), but Gauss went much, much farther. A series of four diary entries made in March 1797 reveal the amazing discoveries that he made in the first three months of 1797. We will need to describe these results in some detail.

Gauss started with Fagnano's problem of dividing the lemniscate into n equal parts. Since this involved an equation of degree n^2, Gauss realized that most of the roots were complex (see [12, X.1, p. 515]). This led him to define $\operatorname{sl}\phi$ and $\operatorname{cl}\phi$ for complex numbers ϕ. The first step is to show that

$$\operatorname{sl}(iy) = i \operatorname{sl} y\,, \qquad \operatorname{cl}(iy) = 1/\operatorname{cl}(y)\,,$$

(the first follows from the change of variable $z = iz'$ in $\int (1-z^4)^{-1/2}dz$, and the second follows from (3.8)). Then (3.9) implies that

$$\operatorname{sl}(x+iy) = \frac{\operatorname{sl} x + i \operatorname{sl} y \operatorname{cl} x \operatorname{cl} y}{\operatorname{cl} y - i \operatorname{sl} x \operatorname{sl} y \operatorname{cl} x}$$

(see [12, X.1, p. 154]).

It follows easily that $\operatorname{sl}\phi$ is doubly periodic, with periods 2ϖ and $2i\varpi$. The zeros and poles of $\operatorname{sl}\phi$ are also easy to determine; they are given by $\phi = (m+in)\varpi$ and $\phi = \big((2m-1)+i(2n-1)\big)(\varpi/2)$, $m, n \in \mathbf{Z}$, respectively. Then Gauss shows that $\operatorname{sl}\phi$ can be written as

$$\operatorname{sl}\phi = \frac{M(\phi)}{N(\phi)}$$

where $M(\phi)$ and $N(\phi)$ are entire functions which are doubly indexed infinite products whose factors correspond to the zeros and poles respectively (see [12, X.1, pp. 153-155]). In expanding these products, Gauss writes down the first examples of Eisenstein series (see [12, X.1, pp. 515-516]). He also obtains many identities involving $M(\phi)$ and $N(\phi)$, such as

$$N(2\phi) = M(\phi)^4 + N(\phi)^4 \tag{3.10}$$

(see [12, X.1, p. 157]). Finally, Gauss notices that the numbers $N(\varpi)$ and $e^{\pi/2}$ agree to four decimal places (see [12, X.1, p. 158]). He comments that a proof of their equality would be "a most important advancement of analysis" (see 12, X.1, p. 517]).

Besides being powerful mathematics, what we have here is almost a rehearsal for what Gauss did with the arithmetic-geometric mean: the observation that two numbers are equal, the importance to analysis of proving this, and the passage from real to complex numbers in order to get at the real depth of the subject. Notice also that identities such as (3.10) are an important warm-up to the theta function identities needed in § 2.

Two other discoveries made at this time require comment. First, only a year after constructing the regular 17-gon by ruler and compass, Gauss found a ruler and compass construction for dividing the lemniscate into five equal pieces (see [12, X.1, p. 517]). This is the basis for the remarks concerning $\int (1-x^4)^{-1/2}dx$ made in Disquisitiones Arithmeticae (see [11, § 335]). Second, Gauss discovered the complex multiplication of elliptic functions when he gave formulas for $\mathrm{sl}(1+i)\phi$, $N(1+i)\phi$, etc. (see [12, III, pp. 407 and 411]). These discoveries are linked: complex multiplication on the elliptic curve associated to the lemniscate enabled Abel to determine all n for which the lemniscate can be divided into n pieces by ruler and compass. (The answer is the same as for the circle! See [28] for an excellent modern account of Abel's theorem.)

After this burst of progress, Gauss left the lemniscatic functions to work on other things. He returned to the subject over a year later, in July 1798, and soon discovered that there was a better way to write $\mathrm{sl}\,\phi$ as a quotient of entire functions. The key was to introduce the new variable $s = \sin\left(\frac{\pi}{\varpi}\phi\right)$. Since $\mathrm{sl}\,\phi$ has period 2ϖ, it can certainly be written as a function of s. By expressing the zeros and poles of $\mathrm{sl}\,\phi$ in terms of s, Gauss was able to prove that

$$\mathrm{sl}\,\phi = \frac{P(\phi)}{Q(\phi)},$$

where

$$P(\phi) = \frac{\varpi}{\pi}s\left(1 + \frac{4s^2}{(e^{\pi}-e^{-\pi})^2}\right)\left(1 + \frac{4s^2}{(e^{2\pi}-e^{-2\pi})^2}\right)\cdots$$

$$Q(\phi) = \left(1 - \frac{4s^2}{(e^{\pi/2}+e^{-\pi/2})^2}\right)\left(1 - \frac{4s^2}{(e^{3\pi/2}+e^{-3\pi/2})^2}\right)\cdots$$

(see [12, III, pp. 415-416]). Relating these to the earlier functions $M(\phi)$ and $N(\phi)$, Gauss obtains (letting $\phi = \psi\varpi$)

$$M(\psi\varpi) = e^{\pi\psi^2/2}\,P(\psi\varpi),$$

$$N(\psi\varpi) = e^{\pi\psi^2/2}\, Q(\psi\varpi),$$

(see [12, III, p. 416]). Notice that $N(\varpi) = e^{\pi/2}$ is an immediate consequence of the second formula.

Many other things were going on at this time. The appearance of π/ϖ sparked Gauss' interest in this ratio. He found several ways of expressing ϖ/π, for example

$$\text{(3.11)} \qquad \frac{\varpi}{\pi} = \frac{\sqrt{2}}{2}\left(1 + \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{3}{8}\right)^2 \frac{1}{4} + \left(\frac{5}{16}\right)^2 \frac{1}{8} + \dots\right),$$

and he computed ϖ/π to fifteen decimal places (see [12, X.1, p. 169]). He also returned to some of his earlier notes and, where the approximation $2\int_0^1 z^2(1-z^4)^{-1/2}dz \approx 1.198$ appears, he added that this is π/ϖ (see [12, X.1, pp. 146 and 150]). Thus in July 1798 Gauss was intimately familiar with the right-hand side of the equation $M(\sqrt{2}, 1) = \pi/\varpi$. Another problem he studied was the Fourier expansion of sl ϕ. Here, he first found the numerical value of the coefficients, i.e.

$$\operatorname{sl} \psi\varpi = .95500599 \sin \psi\pi - .04304950 \sin 3\,\psi\pi + \dots,$$

and then he found a formula for the coefficients, obtaining

$$\operatorname{sl} \psi\varpi = \frac{4\pi}{\varpi(e^{\pi/2} + e^{-\pi/2})} \sin \psi\pi - \frac{4\pi}{\varpi(e^{3\pi/2} + e^{-3\pi/2})} \sin 3\,\psi\pi + \dots$$

see [12, X.1, p. 168 and III, p. 417]).

The next breakthrough came in October 1798 when Gauss computed the Fourier expansions of $P(\phi)$ and $Q(\phi)$. As above, he first computed the coefficients numerically and then tried to find a general formula for them. Since he suspected that numbers like $e^{-\pi}$, $e^{-\pi/2}$, etc., would be involved, he computed several of these numbers (see [12, III, pp. 426-432]). The final formulas he found were

$$P(\psi\varpi) = 2^{3/4}(\pi/\varpi)^{1/2}\left(e^{-\pi/4} \sin \psi\pi - e^{-9\pi/4} \sin 3\,\psi\pi + e^{-25\pi/4} \sin 5\psi\pi - \dots\right)$$

$$\text{(3.12)} \qquad Q(\psi\varpi) = 2^{-1/4}(\pi/\varpi)^{1/2}\left(1 + 2e^{-\pi} \cos 2\psi\pi + 2e^{-4\pi} \cos 4\psi\pi + 2e^{-9\pi} \cos 6\psi\pi + \dots\right)$$

(see [12, X.1, pp. 536-537]). A very brief sketch of how Gauss proved these formulas may be found in [12, X.2, pp. 38-39].

These formulas are remarkable for several reasons. First, recall the theta functions Θ_1 and Θ_3:

$$(3.13)\qquad \begin{aligned} \Theta_1(z, q) &= 2q^{1/4}\sin z - 2q^{9/4}\sin 3z + 2q^{25/4}\sin 5z - \ldots \\ \Theta_3(z, q) &= 1 + 2q\cos 2z + 2q^4\cos 4z + 2q^9\cos 6z + \ldots \end{aligned}$$

(see [36, p. 464]). Up to the constant factor $2^{-1/4}(\pi/\varpi)^{1/2}$, we see that $P(\psi\varpi)$ and $Q(\psi\varpi)$ are precisely $\Theta_1(\psi\pi, e^{-\pi})$ and $\Theta_3(\psi\pi, e^{-\pi})$ respectively. Even though this is just a special case, one can easily discern the general form of the theta functions from (3.12). (For more on the relation between theta functions and sl ϕ, see [36, pp. 524-525]).

Several interesting formulas can be derived from (3.12) by making specific choice for ψ. For example, if we set $\psi = 1$, we obtain

$$\sqrt{\varpi/\pi} = 2^{-1/4}(1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\ldots).$$

Also, if we set $\psi = 1/2$ and use the nontrivial fact that $P(\varpi/2) = Q(\varpi/2) = 2^{-1/4}$ (this is a consequence of the formula $Q(2\phi) = P(\phi)^4 + Q(\phi)^4$ – see (3.10)), we obtain

$$(3.14)\qquad \begin{aligned} \sqrt{\varpi/\pi} &= 2(e^{-\pi/4}+e^{-9\pi/4}+e^{-25\pi/4}+\ldots) \\ \sqrt{\varpi/\pi} &= 1 - 2e^{-\pi} + 2e^{-4\pi} - 2e^{-9\pi} + \ldots . \end{aligned}$$

Gauss wrote down these last two formulas in October 1798 (see [12, III, p. 418]). We, on the other hand, derived the first and third formulas as (2.21) in § 2, only after a very long development. Thus Gauss had some strong signposts to guide his development of modular functions.

These results, all dating from 1798, were recorded in Gauss' mathematical diary as the 91st and 92nd entries (in July) and the 95th entry (in October). The statement of the 92nd entry is especially relevant: "I have obtained most elegant results concerning the lemniscate, which surpasses all expectation—indeed, by methods which open an entirely new field to us" (see [12, X.1, p. 535]). There is a real sense of excitement here; instead of the earlier "advancement of analysis" of the 63rd entry, we have the much stronger phrase "entirely new field." Gauss knew that he had found something of importance. This feeling of excitement is confirmed by the

95th entry: "A new field of analysis is open before us, that is, the investigation of functions, etc." (see [12, X.1, p. 536]). It's as if Gauss were so enraptured he didn't even bother to finish the sentence.

More importantly, this "new field of analysis" is clearly the same "entirely new field of analysis" which we first saw in § 1 in the 98th entry. Rather than being an isolated phenomenon, it was the culmination of years of work. Imagine Gauss' excitement on May 30, 1799: this new field which he had seen grow up around the lemniscate and reveal such riches, all of a sudden expands yet again to encompass the arithmetic-geometric mean, a subject he had known since age 14. All of the powerful analytic tools he had developed for the lemniscatic functions were now ready to be applied to the agM.

C. In studying Gauss' work on the agM, it makes sense to start by asking where the observation $M(\sqrt{2}, 1) = \pi/\varpi$ came from. Using what we have learned so far, part of this question can now be answered: Gauss was very familiar with π/ϖ, and from reading Stirling he had probably seen the ratio $\sqrt{2} : 1$ associated with the lemniscate. (In fact, this ratio appears in most known methods for constructing the lemniscate—see [24, pp. 111-117].) We have also seen, in the equation $N(\varpi) = e^{\pi/2}$, that Gauss often used numerical calculations to help him discover theorems. But while these facts are enlightening, they still leave out one key ingredient, the idea of taking the agM of $\sqrt{2}$ and 1. Where did this come from? The answer is that every great mathematical discovery is kindled by some intuitive spark, and in our case, the spark came on May 30, 1799 when Gauss decided to compute $M(\sqrt{2}, 1)$.

We are still missing one piece of our picture of Gauss at this time: how much did he know about the agM? Unfortunately, this is a very difficult question to answer. Only a few scattered fragments dealing with the agM can be dated before May 30, 1799 (see [12, X.1, pp. 172-173 and 260]). As for the date 1791 of his discovery of the agM, it comes from a letter he wrote in 1816 (see [12, X.1, p. 247]), and Gauss is known to have been sometimes wrong in his recollections of dates. The only other knowledge we have about the agM in this period is an oral tradition which holds that Gauss knew the relation between theta functions and the agM in 1794 (see [12, III, 493]). We will soon see that this claim is not as outrageous as one might suspect.

It is not our intention to give a complete account of Gauss' work on the agM. This material is well covered in other places (see [10], [12, X.2, pp. 62-114], [13], [14] and [25]—the middle three references are especially complete), and furthermore it is impossible to give the full story of what happened. To explain this last statement, consider the following formulas:

$$\begin{aligned} B + (1/4)B^3 + (9/64)B^5 + \dots &= (2z^{1/2}+2z^{9/2}+\dots)^2 = r^2\,, \\ \frac{a}{M(a,b)} &= 1 + (1/4)B^2 + (9/64)B^4 + \dots\,, \end{aligned} \tag{3.15}$$

where $B = \left(1-(b/a)^2\right)^{1/2}$. These come from the first surviving notes on the agM that Gauss wrote after May 30, 1799 (see [12, X.1, pp. 177-178]). If we set $a = 1$ and $b = k' = \sqrt{1-k^2}$, then $B = k$, and we obtain

$$\begin{aligned} \frac{1}{M(1,k')} &= 1 + (1/4)k^2 + (9/64)k^4 + \dots \\ \frac{k}{M(1,k')} &= (2z^{1/2}+2z^{9/2}+\dots)^2 = r^2\,. \end{aligned} \tag{3.16}$$

The first formula is (1.8), and the second, with $z = e^{\pi i \tau/2}$, follows easily from what we learned in § 2 about theta functions and the agM. Yet the formulas (3.15) appear with neither proofs nor any hint of where they came from. The discussion at the end of § 1 sheds some light on the bottom formula of (3.15), but there is nothing to prepare us for the top one.

It is true that Gauss had a long-standing interest in theta functions, going back to when he first encountered Euler's wonderful formula

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{(3n^2+n)/2} = \prod_{n=1}^{\infty} (1-x^n)\,.$$

The right-hand side appears in a fragment dating from 1796 (see [12, X.1, p. 142]), and the 7th entry of his mathematical diary, also dated 1796, gives a continued fraction expansion for

$$1 - 2 + 8 - 64 + \dots\,.$$

Then the 58th entry, dated February 1797, generalizes this to give a continued fraction expansion for

$$1 - a + a^3 - a^6 + a^{10} - \dots$$

(see [12, X.1, pp. 490 and 513]). The connection between these series and lemniscatic functions came in October 1798 with formulas such as (3.14).

This seems to have piqued his interest in the subject, for at this time he also set himself the problem of expressing

$$(3.17) \qquad 1 + x + x^3 + x^6 + x^{10} + \dots$$

as an infinite product (see [12, X.1, p. 538]). Note also that the first formula of (3.14) gives r with $z = e^{-\pi/2}$.

Given these examples, we can conjecture where (3.15) came from. Gauss could easily have defined p, q and r in general and then derived identities (2.8)-(2.9) (recall the many identities obtained in 1798 for $P(\phi)$ and $Q(\phi)$—see (3.10) and [12, III, p. 410]). Then (3.15) would result from noticing that these identities formally satisfy the agM algorithm, which is the basic content of Lemma 2.3. This conjecture is consistent with the way Gauss initially treated z as a purely formal variable (the interpretation $z = e^{-\pi i \tau/2}$ was only to come later—see [12, X.1, pp. 262-263 and X.2, pp. 65-66]).

The lack of evidence makes it impossible to verify this or any other reasonable conjecture. But one thing is now clear: in Gauss' observation of May 30, 1799, we have not two but three distinct streams of his thought coming together. Soon after (or simultaneous with) observing that $M(\sqrt{2}, 1) = \pi/\varpi$, Gauss knew that there were inimate connections between lemniscatic functions, the agM, and theta functions. The richness of the mathematics we have seen is in large part due to the many-sided nature of this confluence.

There remain two items of unfinished business. From § 1, we want to determine more precisely when Gauss first proved Theorem 1.1. And recall from § 2 that on June 3, 1800, Gauss discovered the "mutual connection" among the infinitely many values of $M(a, b)$. We want to see if he really knew the bulk of § 2 by this date. To answer these questions, we will briefly examine the main notebook Gauss kept between November 1799 and July 1800 (the notebook is "Scheda Ac" and appears as pp. 184-206 in [12, X.1]).

The starting date of this notebook coincides with the 100th entry of Gauss' mathematical diary, which reads "We have uncovered many new things about arithmetic-geometric means" (see [12, X.1, p. 544]). After several pages dealing with geometry, one all of a sudden finds the formula (3.11) for ϖ/π. Since Gauss knew (3.15) at this time, we get an immediate proof of $M(\sqrt{2}, 1) = \pi/\varpi$. Gauss must have had this in mind, for otherwise why would he so carefully recopy a formula proved in July 1798? Yet one could also ask why such a step is necessary: isn't Theorem 1.1 an immediate consequence of (3.15)? Amazingly enough, it appears that Gauss wasn't yet aware of this connection (see [12, X.1, p. 262]). Part of the problem is that he had been distracted by the power series, closely related to (3.15), which gives the arc length of the ellipse (see [12, X.1, p. 177]). This distraction was actually a bonus, for an asymptotic formula of Euler's for the arc length of the ellipse led Gauss to write

$$M(x, 1) = \frac{(\pi/2)\,(x-\alpha x^{-1}-\beta x^{-3}-\ldots)}{\log(1/z)} \tag{3.18}$$

where $x = k^{-1}$, and z and k are as in (3.16) (see [12, X.1, pp. 186 and 268-270]). He was then able to show that the power series on top was $(k\,M(1, k'))^{-1}$, which implies that

$$z = \exp\left(-\frac{\pi}{2}\cdot\frac{M(1, k')}{M(1, k)}\right)$$

(see [12, X.1, pp. 187 and 190]). Letting $z = e^{\pi i\tau/2}$, we obtain formulas similar to (2.20). More importantly, we see that Gauss is now in a position to uniformize the agM; z is no longer a purely formal variable.

In the process of studying (3.18), Gauss also saw the relation between the agM and complete elliptic integrals of the first kind. The formula

$$\frac{1}{M(1, k')} = \frac{2}{\pi}\int_0^1 ((1-x^2)\,(1-k^2x^2))^{-1/2}dx$$

follows easily from [12, X.1, p. 187], and this is trivially equivalent to (1.7). Furthermore, we know that this page was written on December 14, 1799 since on this date Gauss wrote in his mathematical diary that the agM was the quotient of two transcendental functions (see (3.18)), one of which was itself an integral quantity (see the 101st entry, [12, X.1, 544]). Thus Theorem 1.1 was proved on December 14, 1799, nine days earlier than our previous estimate.

Having proved this theorem, Gauss immediately notes one of its corollaries, that the "constant term" of the expression $(1+\mu\cos^2\phi)^{-1/2}$ is $M(\sqrt{1+\mu}, 1)^{-1}$ (see [12, X.1, p. 188]). By "constant term" Gauss means the coefficient A in the Fourier expansion

$$(1+\mu\cos^2\phi)^{-1/2} = A + A'\cos\phi + A''\cos 2\phi + \ldots .$$

Since A is the integral $\frac{2}{\pi}\int_0^{\pi/2}(1+\mu\cos^2\phi)^{-1/2}d\phi$, the desired result follows from Theorem 1.1. This interpretation is important because these coefficients are useful in studying secular perturbations in astronomy (see [12, X.1, pp. 237-242]). It was in this connection that Gauss published his 1818 paper [12, III, pp. 331-355] from which we got our proof of Theorem 1.1.

What Gauss did next is unexpected: he used the agM to generalize the lemniscate functions to arbitrary elliptic functions, which for him meant inverse functions of elliptic integrals of the form

$$\int (1+\mu^2 \sin^2\phi)^{-1/2} d\phi = \int ((1-x^2)(1+\mu^2 x^2))^{-1/2} dx .$$

Note that $\mu = 1$ corresponds to the lemniscate. To start, he first set $\mu = \tan \nu$,

$$\varpi = \frac{\pi \cos \nu}{M(1, \cos\nu)}, \qquad \varpi' = \frac{\pi \cos \nu}{M(1, \sin\nu)}$$

and finally

$$z = \exp\left(-\frac{\pi}{2}\cdot\frac{\varpi'}{\varpi}\right) = \exp\left(-\frac{\pi}{2}\cdot\frac{M(1, \cos\nu)}{M(1, \sin\nu)}\right). \tag{3.19}$$

Then he defined the elliptic function $S(\phi)$ by $S(\phi) = \dfrac{T(\phi)}{W(\phi)}$ where

$$\begin{aligned} T(\psi\varpi) &= 2\mu^{-1/2}\sqrt{M(1, \cos\nu)}\,(z^{1/2}\sin\psi\pi - z^{9/2}\sin 3\psi\pi + \ldots) \\ W(\psi\varpi) &= \sqrt{M(1, \cos\nu)}\,(1 + 2z^2\cos 2\psi\pi + 2z^8\cos 4\psi\pi + \ldots) \end{aligned} \tag{3.20}$$

(see [12, X.1, pp. 194-195 and 198]). In the pages that follow, we find the periods 2ϖ and $2i\varpi'$, the addition formula, and the differential equation connecting $S(\phi)$ to the above elliptic integral. Thus Gauss had a complete theory of elliptic functions.

In general, there are two basic approaches to this subject. One involves direct inversion of the elliptic integral and requires a detailed knowledge of the associated Riemann surface (see [17, Ch. VII]). The other more common approach defines elliptic functions as certain series ($\wp$-functions) or quotients of series (theta functions). The difficulty is proving that such functions invert all elliptic integrals. Classically, this uniformization problem is solved by studying a function such as $k(\tau)^2$ (see [36, § 20.6 and § 21.73]) or $j(\tau)$ (as in most modern texts—see [30, § 4.2]). Gauss uses the agM to solve this problem: (3.19) gives the desired uniformizing parameter! (In this connection, the reader should reconsider the from [12, VIII, p. 101] given near the end of § 2.)

For us, the most interesting aspect of what Gauss did concerns the functions T and W. Notice that (3.20) is a direct generalization of (3.12); in fact, in terms of (3.13), we have

$$T(\psi\varpi) = \mu^{-1/2}\sqrt{M(1, \cos v)}\,\Theta_1(\psi\pi, z^2),$$

$$W(\psi\varpi) = \sqrt{M(1, \cos v)}\,\Theta_3(\psi\pi, z^2).$$

Gauss also introduces $T(\varpi/2-\phi)$ and $W(\varpi/2-\phi)$, which are related to the theta functions Θ_2 and Θ_4 by similar formulas (see [12, X.1, pp. 196 and 275]). He then studies the squares of these functions and he obtains identities such as

$$2\Theta_3(0, z^4)\,\Theta_3(2\phi, z^4) = \Theta_3(\phi, z^2)^2 + \Theta_4(\phi, z^2)^2$$

(this, of course, is the modern formulation—see [12, X.1, pp. 196 (Eq. 14) and 275]). When ϕ =' 0, this reduces to the first formula

$$p(\tau)^2 + q(\tau)^2 = 2p(2\tau)^2$$

of (2.8). The other formulas of (2.8) appear similarly. Gauss also obtained product expansions for the theta functions (see [12, X.1, pp. 201-205]). In particular, one finds all the formulas of (2.6). These manipulations yielded the further result that

$$1 + z + z^3 + z^6\ z^{10} + \ldots - \prod_{n=1}^{\infty}(1-z)^{-1}(1-z^2),$$

solving the problem he had posed a year earlier in (3.17).

From Gauss' mathematical diary, we see that the bulk of this work was done in May 1800 (see entries 105, 106 and 108 in [12, X.1, pp. 546-549]). The last two weeks were especially intense as Gauss realized the special role played by the agM. The 108th entry, dated June 3, 1800, announces completion of a general theory of elliptic functions ("sinus lemniscatici universalissime accepti"). On the same day he recorded his discovery of the "mutual connection" among the values of the agM!

This is rather surprising. We've seen that Gauss knew the basic identities (2.6), (2.8) and (2.9), but the formulas (2.7), which tell us how theta functions behave under linear fractional transformations, are nowhere to be seen, nor do we find any hint of the fundamental domains used in § 2. Reading this notebook makes it clear that Gauss now knew the basic observation of

Lemma 2.3 that theta functions satisfy the agM algorithm, but there is no way to get from here to Theorem 2.2 without knowing (2.7). It is not until 1805 that this material appears in Gauss' notes (see [12, X.2, pp. 101-103]). Thus some authors, notably Markushevitch [25], have concluded that on June 3, 1800, Gauss had nothing approaching a proof of Theorem 2.2.

Schlesinger, the last editor of Gauss' collected works, feels otherwise. He thinks that Gauss knew (2.7) at this time, though knowledge of the fundamental domains may have not come until 1805 (see [12, X.2, p. 106]). Schlesinger often overestimates what Gauss knew about modular functions, but in this case I agree with him. As evidence, consider pp. 287-307 in [12, X.1]. These reproduce twelve consecutive pages from a notebook written in 1808 (see [12, X.1, p. 322]), and they contain the formulas (2.7), a clear statement of the basic observation of Lemma 2.3, the infinite product manipulations described above, and the equations giving the division of the agM into 3, 5 and 7 parts (in analogy with the division of the lemniscate). The last item is especially interesting because it relates to the second half of the 108th entry: "Moreover, in these same days, we have discovered the principles according to which the agM series should be interpolated, so as to enable us to exhibit by algebraic equations the terms in a given progression pertaining to any rational index" (see [12, X.1, p. 548]). There is no other record of this in 1800, yet here it is in 1808 resurfacing with other material (the infinite products) dating from 1800. Thus it is reasonable to assume that the rest of this material, including (2.7), also dates from 1800. Of course, to really check this conjecture, one would have to study the original documents in detail.

Given all of (2.6)-(2.9), it is still not clear where Gauss got the basic insight that $M(a, b)$ is a multiple valued function. One possible source of inspiration is the differential equation (1.12) whose solution (1.13) suggests linear combinations similar to those of Theorem 2.2. We get even closer to this theorem when we consiser the periods of $S(\phi)$:

$$m\varpi + in\varpi' = \pi \cos \nu \left(\frac{m}{M(1, \cos\nu)} + \frac{in}{M(1, \sin\nu)} \right)$$

where m, n are even integers. Gauss' struggles during May 1800 to understand the imaginary nature of these periods (see [12, X.2, pp. 70-71]) may have influenced his work on the agM. (We should point out that the above comments are related: Theorem 2.2 can be proved by analyzing the monodromy group—$\Gamma_2(4)$ in this case—of the differential equation (1.12).) On the other hand, Geppert suggests that Gauss may have taken a completely different route, involving the asymptotic formula (3.18), of arriving at Theorem 2.2 (see [14, pp. 173-175]). We will of course never really know how Gauss arrived at this theorem.

For many years, Gauss hoped to write up these results for publication. He mentions this in Disquisitiones Arithmeticae (see [11, § 335]) and in the research announcement to his 1818 article (see [12, III, p. 358]). Two manuscripts written in 1800 (one on the agM, the other on lemniscatic functions) show that Gauss made a good start on this project (see [12, III, pp. 360-371 and 413-415]). He also periodically returned to earlier work and rewrote it in more complete form (the 1808 notebook is an example of this). Aside from the many other projects Gauss had to distract him, it is clear why he never finished this one: it was simply too big. Given his predilection for completeness, the resulting work would have been enormous. Gauss finally gave up trying in 1827 when the first works of Abel and Jacobi appeared. As he wrote in 1828, "I shall most likely not soon prepare my investigations on transcendental functions which I have had for many years—since 1798—because I have many other matters which must be cleared up. Herr Abel has now, I see, anticipated me and relieved me of the burden in regard to one third of these matters, particularly since he carried out all developments with great concision and elegance" (see [12, X.1, p. 248]).

The other two thirds "of these matters" encompass Gauss' work on the agM and modular functions. The latter were studied vigorously in the nineteenth century and are still an active area of research today. The agM, on the other hand, has been relegated to the history books. This is not entirely wrong, for the history of this subject is wonderful. But at the same time the agM is also wonderful as mathematics, and this mathematics deserves to be better known.

REFERENCES

[1] ALLING, N. L. *Real Elliptic Curves*. North-Holland Mathematics Studies, Vol. 54, North-Holland, Amsterdam, 1981.

[2] BERNOULLI, Jacob. *Opera*, Vol. I. Geneva, 1744.

[3] BERNOULLI, Johann. *Opera omnia*, Vol. I. Lausanne, 1742.

[4] BÜHLER, W. K. *Gauss: A Biographical Study*. Springer-Verlag, Berlin-Heidelberg-New York, 1981.

[5] CARLSON, B. C. Algorithms involving Arithmetic and Geometric Means. *Amer. Math. Monthly 78* (1971), 496-505.

[6] CASSELS, J. W. S. *Rational Quadratic Forms*. Academic Press, New York, 1978.

[7] COPSON, E. T. *An Introduction to the Theory of Functions of a Complex Variable*. Oxford U. Press, London, 1935.

[8] ENNEPER, A. *Elliptische Functionen: Theorie und Geschichte*. Halle, 1876.

[9] EULER, L. *Opera Omnia*, Series Prima, Vol. XX and XXI. Teubner, Leipzig and Berlin, 1912-1913.

[10] FUCHS, W. Das arithmetisch-geometrische Mittel in den Untersuchungen von Carl Friedrich Gauss. *Gauss-Gesellschaft Göttingen, Mittelungen No. 9* (1972), 14-38.

[11] GAUSS, C. F. *Disquisitiones Arithmeticae*. Translated by A. Clark, Yale U. Press, New Haven, 1965 (see also [12, I]).

[12] —— *Werke*. Göttingen-Leipzig, 1868-1927.

[13] GEPPERT, H. *Bestimmung der Anziehung eines elliptischen Ringes*. Ostwald's Klassiker, Vol. 225, Akademische Verlag, Leipzig, 1927.

[14] —— Wie Gauss zur elliptischen Modulfunktion kam. *Deutsche Mathematik 5* (1940), 158-175.

[15] —— Zur Theorie des arithmetisch-geometrischen Mittels. *Math. Annalen 99* (1928), 162-180.

[16] GRADSHTEYN, I. S. and I. M. RYZHIK. *Table of Integrals, Series and Products*. Academic Press, New York, 1965.

[17] HANCOCK, H. *Lectures on the Theory of Elliptic Functions*. Vol. I. Wiley, New York, 1910.

[18] HOFFMAN, J. E. Über Jakob Bernoullis Beiträge zur Infinitesimalmathematik. *L'Enseignement Math. 2* (1956), 61-171.

[19] HOUZEL, C. Fonctions Elliptiques et Intégrals Abéliennes. In *Abrégé d'histoire des mathématiques 1700-1900*, Vol. II. Ed. by J. Dieudonné, Hermann, Paris, 1978, 1-112.

[20] JACOBI, C. C. J. *Gesammelte Werke*. G. Reimer, Berlin, 1881.

[21] KLINE, M. *Mathematical Thought from Ancient to Modern Times*. Oxford U. Press, New York, 1972.

[22] LAGRANGE, J. L. *Œuvres*, Vol. II. Gauthier-Villars, Paris, 1868.

[23] LEGENDRE, A. M. *Traité des Fonctions Elliptiques*. Paris, 1825-1828.

[24] LOCKWOOD, E. H. *A Book of Curves*. Cambridge U. Press, Cambridge, 1971.

[25] MARKUSHEVITCH, A. I. Die Arbeiten von C. F. Gauss über Funktionentheorie. In *C. F. Gauss Gedenkband Anlässlich des 100. Todestages am 23. Februar 1955*. Ed. by H. Reichart, Teubner, Leipzig, 1957, 151-182.

[26] MIEL, G. Of Calculations Past and Present: The Archimedean Algorithm. *Amer. Math. Monthly 90* (1983), 17-35.

[27] MUMFORD, D. *Tata Lectures on Theta I*. Progress in Mathematics Vol. 28, Birkhäuser, Boston, 1983.

[28] ROSEN, M. Abel's Theorem on the Lemniscate. *Amer. Math. Monthly 88* (1981), 387-395.
[29] SERRE, J.-P. *Cours d'Arithmétique*. Presses U. de France, Paris, 1970.
[30] SHIMURA, G. *Introduction to the Arithmetic Theory of Automorphic Functions*. Princeton U. Press, Princeton, 1971.
[31] STIRLING, J. *Methodus Differentialis*. London, 1730.
[32] TANNERY, J. and J. MOLK. *Eléments de la Théorie des Fonctions Elliptiques*, Vol. 2. Gauthiers-Villars, Paris, 1893.
[33] TODD, J. The Lemniscate Constants. *Comm. of the ACM 18* (1975), 14-19.
[34] van der POL, B. Démonstration Elémentaire de la Relation $\Theta_3^4 = \Theta_0^4 + \Theta_2^4$ entre les Différentes Fonctions de Jacobi. *L'Enseignement Math. 1* (1955), 258-261.
[35] von DAVID, L. Arithmetisch-geometrisches Mittel und Modulfunktion. *J. für die Reine u. Ang. Math. 159* (1928), 154-170.
[36] WHITTAKER, E. T. and G. N. WATSON. *A Course of Modern Analysis*, 4th ed. Cambridge U. Press, Cambridge, 1963.

(Reçu le 21 novembre 1983)

David A. Cox

Department of Mathematics
Amherst College
Amherst, MA 01002 (USA)

SIAM REVIEW
Vol. 26, No. 3, July 1984

002

THE ARITHMETIC-GEOMETRIC MEAN AND FAST COMPUTATION OF ELEMENTARY FUNCTIONS*

J. M. BORWEIN† AND P. B. BORWEIN†

Abstract. We produce a self contained account of the relationship between the Gaussian arithmetic-geometric mean iteration and the fast computation of elementary functions. A particularly pleasant algorithm for π is one of the by-products.

Introduction. It is possible to calculate 2^n decimal places of π using only n iterations of a (fairly) simple three-term recursion. This remarkable fact seems to have first been explicitly noted by Salamin in 1976 [16]. Recently the Japanese workers Y. Tamura and Y. Kanada have used Salamin's algorithm to calculate π to 2^{23} decimal places in 6.8 hours. Subsequently 2^{24} places were obtained ([18] and private communication). Even more remarkable is the fact that all the elementary functions can be calculated with similar dispatch. This was proved (and implemented) by Brent in 1976 [5]. These extraordinarily rapid algorithms rely on a body of material from the theory of elliptic functions, all of which was known to Gauss. It is an interesting synthesis of classical mathematics with contemporary computational concerns that has provided us with these methods. Brent's analysis requires a number of results on elliptic functions that are no longer particularly familiar to most mathematicians. Newman in 1981 stripped this analysis to its bare essentials and derived related, though somewhat less computationally satisfactory, methods for computing π and log. This concise and attractive treatment may be found in [15].

Our intention is to provide a mathematically intermediate perspective and some bits of the history. We shall derive implementable (essentially) quadratic methods for computing π and all the elementary functions. The treatment is entirely self-contained and uses only a minimum of elliptic function theory.

1. 3.141592653589793238462643383279502884197. The calculation of π to great accuracy has had a mathematical import that goes far beyond the dictates of utility. It requires a mere 39 digits of π in order to compute the circumference of a circle of radius 2×10^{25} meters (an upper bound on the distance travelled by a particle moving at the speed of light for 20 billion years, and as such an upper bound on the radius of the universe) with an error of less than 10^{-12} meters (a lower bound for the radius of a hydrogen atom).

Such a calculation was in principle possible for Archimedes, who was the first person to develop methods capable of generating arbitrarily many digits of π. He considered circumscribed and inscribed regular n-gons in a circle of radius 1. Using $n = 96$ he obtained

$$3.1405 \cdots = \frac{6336}{2017.25} < \pi < \frac{14688}{4673.5} = 3.1428.$$

If $1/A_n$ denotes the area of an inscribed regular 2^n-gon and $1/B_n$ denotes the area of a circumscribed regular 2^n-gon about a circle of radius 1 then

$$A_{n+1} = \sqrt{A_n B_n}, \qquad B_{n+1} = \frac{A_{n+1} + B_n}{2}. \tag{1.1}$$

*Received by the editors February 8, 1983, and in revised form November 21, 1983. This research was partially sponsored by the Natural Sciences and Engineering Research Council of Canada.

†Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 4H8.

This two-term iteration, starting with $A_2 := 1/2$ and $B_2 := 1/4$, can obviously be used to calculate π. (See Edwards [9, p. 34].) A_{15}^{-1}, for example, is 3.14159266 which is correct through the first seven digits. In the early sixteen hundreds Ludolph von Ceulen actually computed π to 35 places by Archimedes' method [2].

Observe that $A_n := 2^{-n} \operatorname{cosec}(\theta/2^n)$ and $B_n := 2^{-n-1} \operatorname{cotan}(\theta/2^{n+1})$ satisfy the above recursion. So do $A_n := 2^{-n} \operatorname{cosech}(\theta/2^n)$ and $B_n := 2^{-n-1} \operatorname{cotanh}(\theta/2^{n+1})$. Since in both cases the common limit is $1/\theta$, the iteration can be used to calculate the standard inverse trigonometric and inverse hyperbolic functions. (This is often called Borchardt's algorithm [6], [19].)

If we observe that

$$A_{n+1} - B_{n+1} = \frac{1}{2(\sqrt{A_n}/\sqrt{B_n} + 1)}(A_n - B_n)$$

we see that the error is decreased by a factor of approximately four with each iteration. This is linear convergence. To compute n decimal digits of π, or for that matter arcsin, arcsinh or log, requires $O(n)$ iterations.

We can, of course, compute π from arctan or arcsin using the Taylor expansion of these functions. John Machin (1680–1752) observed that

$$\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$$

and used this to compute π to 100 places. William Shanks in 1873 used the same formula for his celebrated 707 digit calculation. A similar formula was employed by Leonhard Euler (1707–1783):

$$\pi = 20 \arctan\left(\frac{1}{7}\right) + 8 \arctan\left(\frac{3}{79}\right).$$

This, with the expansion

$$\arctan(x) = \frac{y}{x}\left(1 + \frac{2}{3}y + \frac{2.4}{3.5}y^2 + \cdots\right)$$

where $y = x^2/(1 + x^2)$, was used by Euler to compute π to 20 decimal places in an hour. (See Beckman [2] or Wrench [21] for a comprehensive discussion of these matters.) In 1844 Johann Dase (1824–1861) computed π correctly to 200 places using the formula

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right).$$

Dase, an "idiot savant" and a calculating prodigy, performed this "stupendous task" in "just under two months." (The quotes are from Beckman, pp. 105 and 107.)

A similar identity:

$$\pi = 24 \arctan\left(\frac{1}{8}\right) + 8 \arctan\left(\frac{1}{57}\right) + 4 \arctan\left(\frac{1}{239}\right)$$

was employed, in 1962, to compute 100,000 decimals of π. A more reliable "idiot savant", the IBM 7090, performed this calculation in a mere 8 hrs. 43 mins. [17].

There are, of course, many series, products and continued fractions for π. However, all the usual ones, even cleverly evaluated, require $O(\sqrt{n})$ operations ($+$, $\times$, $\div$, $\sqrt{\ }$) to arrive at n digits of π. Most of them, in fact, employ $O(n)$ operations for n digits, which is essentially linear convergence. Here we consider only full precision operations. For a time complexity analysis and a discussion of time efficient algorithms based on binary splitting see [4].

The algorithm employed in [17] requires about 1,000,000 operations to compute 1,000,000 digits of π. We shall present an algorithm that reduces this to about 200 operations. The algorithm, like Salamin's and Newman's requires some very elementary elliptic function theory. The circle of ideas surrounding the algorithm for π also provides algorithms for all the elementary functions.

2. Extraordinarily rapid algorithms for algebraic functions. We need the following two measures of speed of convergence of a sequence (a_n) with limit L. If there is a constant C_1 so that

$$|a_{n+1} - L| \leq C_1 |a_n - L|^2$$

for all n, then we say that (a_n) converges to L *quadratically*, or with *second order*. If there is a constant $C_2 > 1$ so that, for all n,

$$|a_n - L| \leq C_2^{-2^n}$$

then we say that (a_n) converges to L *exponentially*. These two notions are closely related; quadratic convergence implies exponential convergence and both types of convergence guarantee that a_n and L will "agree" through the first $O(2^n)$ digits (provided we adopt the convention that .9999. . .9 and 1.000. . .0 agree through the required number of digits).

Newton's method is perhaps the best known second order iterative method. Newton's method computes a zero of $f(x) - y$ by

$$x_{n+1} := x_n - \frac{f(x_n) - y}{f'(x_n)} \tag{2.1}$$

and hence, can be used to compute f^{-1} quadratically from f, at least locally. For our purposes, finding suitable starting values poses little difficulty. Division can be performed by inverting $(1/x) - y$. The following iteration computes $1/y$:

$$x_{n+1} := 2x_n - x_n^2 y. \tag{2.2}$$

Square root extraction ($\sqrt{y}$) is performed by

$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{y}{x_n}\right). \tag{2.3}$$

This ancient iteration can be traced back at least as far as the Babylonians. From (2.2) and (2.3) we can deduce that division and square root extraction are of the same order of complexity as multiplication (see [5]). Let $M(n)$ be the "amount of work" required to multiply two n digit numbers together and let $D(n)$ and $S(n)$ be, respectively, the "amount of work" required to invert an n digit number and compute its square root, to n digit accuracy. Then

$$D(n) = O(M(n))$$

and

$$S(n) = O(M(n)).$$

We are not bothering to specify precisely what we mean by work. We could for example count the number of single digit multiplications. The basic point is as follows. It requires

$O(\log n)$ iterations of Newton's method (2.2) to compute $1/y$. However, at the ith iteration, one need only work with accuracy $O(2^i)$. In this sense, Newton's method is self-correcting. Thus,

$$D(n) = O\left(\sum_{i=1}^{\log n} M(2^i)\right) = O(M(n))$$

provided $M(2^i) \geqq 2M(2^{i-1})$. The constants concealed beneath the order symbol are not even particularly large. Finally, using a fast multiplication, see [12], it is possible to multiply two n digits numbers in $O(n \log(n) \log\log(n))$ single digit operations.

What we have indicated is that, for the purposes of asymptotics, it is reasonable to consider multiplication, division and root extraction as equally complicated and to consider each of these as only marginally more complicated than addition. Thus, when we refer to operations we shall be allowing addition, multiplication, division and root extraction.

Algebraic functions, that is roots of polynomials whose coefficients are rational functions, can be approximated (calculated) exponentially using Newton's method. By this we mean that the iterations converge exponentially and that each iterate is itself suitably calculable. (See [13].)

The difficult trick is to find a method to exponentially approximate just one elementary transcendental function. It will then transpire that the other elementary functions can also be exponentially calculated from it by composition, inversion and so on.

For this Newton's method cannot suffice since, if f is algebraic in (2.1) then the limit is also algebraic.

The only familiar iterative procedure that converges quadratically to a transcendental function is the arithmetic-geometric mean iteration of Gauss and Legendre for computing complete elliptic integrals. This is where we now turn. We must emphasize that it is difficult to exaggerate Gauss' mastery of this material and most of the next section is to be found in one form or another in [10].

3. The real AGM iteration. Let two positive numbers a and b with $a > b$ be given. Let $a_0 := a$, $b_0 := b$ and define

$$a_{n+1} := \frac{1}{2}(a_n + b_n), \qquad b_{n+1} := \sqrt{a_n b_n} \tag{3.1}$$

for n in $\mathbb{N}$.

One observes, as a consequence of the arithmetic-geometric mean inequality, that $a_n \geqq a_{n+1} \geqq b_{n+1} \geqq b_n$ for all n. It follows easily that (a_n) and (b_n) converge to a common limit L which we sometimes denote by $AG(a, b)$. Let us now set

$$c_n := \sqrt{a_n^2 - b_n^2} \quad \text{for } n \in \mathbb{N}. \tag{3.2}$$

It is apparent that

$$c_{n+1} = \frac{1}{2}(a_n - b_n) = \frac{c_n^2}{4a_{n+1}} \leqq \frac{c_n^2}{4L}, \tag{3.3}$$

which shows that (c_n) converges quadratically to zero. We also observe that

$$a_n = a_{n+1} + c_{n+1} \quad \text{and} \quad b_n = a_{n+1} - c_{n+1} \tag{3.4}$$

which allows us to define a_n, b_n and c_n for negative n. These negative terms can also be generated by the *conjugate scale* in which one starts with $a_0' := a_0$ and $b_0' := c_0$ and defines

(a'_n) and (b'_n) by (3.1). A simple induction shows that for any integer n

$$a'_n = 2^{-n}a_{-n}, \quad b'_n = 2^{-n}c_{-n}, \quad c'_n = 2^{-n}b_{-n}. \tag{3.5}$$

Thus, backward iteration can be avoided simply by altering the starting values. For future use we define the quadratic *conjugate* $k' := \sqrt{1 - k^2}$ for any k between 0 and 1.

The limit of (a_n) can be expressed in terms of a *complete elliptic integral of the first kind,*

$$I(a, b) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}. \tag{3.6}$$

In fact

$$I(a, b) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}} \tag{3.7}$$

as the substitution $t := a\tan\theta$ shows. Now the substitution of $u := \frac{1}{2}(t - (ab/t))$ and some careful but straightforward work [15] show that

$$I(a, b) = I\left(\left(\frac{a + b}{2}\right), \sqrt{ab}\right). \tag{3.8}$$

It follows that $I(a_n, b_n)$ is independent of n and that, on interchanging limit and integral,

$$I(a_0, b_0) = \lim_{n\to\infty} I(a_n, b_n) = I(L, L).$$

Since the last integral is a simple arctan (or directly from (3.6)) we see that

$$I(a_0, b_0) = \frac{\pi}{2} AG(a_0, b_0). \tag{3.9}$$

Gauss, of course, had to derive rather than merely verify this remarkable formula. We note in passing that $AG(\cdot, \cdot)$ is positively homogeneous.

We are now ready to establish the underlying limit formula.

PROPOSITION 1.

$$\lim_{k\to 0^+}\left[\log\left(\frac{4}{k}\right) - I(1, k)\right] = 0. \tag{3.10}$$

Proof. Let

$$A(k) := \int_0^{\pi/2} \frac{k'\sin\theta\, d\theta}{\sqrt{k^2 + (k')^2\cos^2\theta}}$$

and

$$B(k) := \int_0^{\pi/2} \sqrt{\frac{1 - k'\sin\theta}{1 + k'\sin\theta}}\, d\theta.$$

Since $1 - (k'\sin\theta)^2 = \cos^2\theta + (k\sin\theta)^2 = (k'\cos\theta)^2 + k^2$, we can check that

$$I(1, k) = A(k) + B(k).$$

Moreover, the substitution $u := k'\cos\theta$ allows one to evaluate

$$A(k) := \int_0^{k'} \frac{du}{\sqrt{u^2 + k^2}} = \log\left(\frac{1 + k'}{k}\right). \tag{3.11}$$

Finally, a uniformity argument justifies

$$\lim_{k\to 0^+} B(k) = B(0) = \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{1+\sin\theta} = \log 2, \tag{3.12}$$

and (3.11) and (3.12) combine to show (3.10). □

It is possible to give various asymptotics in (3.10), by estimating the convergence rate in (3.12).

PROPOSITION 2. *For* $k \in (0, 1]$

$$\left|\log\left(\frac{4}{k}\right) - I(1,k)\right| \leqq 4k^2 I(1,k) \leqq 4k^2(8 + |\log k|). \tag{3.13}$$

Proof. Let

$$\Delta(k) := \log\left(\frac{4}{k}\right) - I(1,k).$$

As in Proposition 1, for $k \in (0, 1]$,

(3.14)

$$|\Delta(k)| \leqq \left|\log\left(\frac{2}{k}\right) - \log\left(\frac{1+k'}{k}\right)\right| + \left|\int_0^{\pi/2}\left[\sqrt{\frac{1-k'\sin\theta}{1+k'\sin\theta}} - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\right]d\theta\right|.$$

We observe that, since $1 - k' = 1 - \sqrt{1-k^2} < k^2$,

$$\left|\log\left(\frac{2}{k}\right) - \log\left(\frac{1+k'}{k}\right)\right| = \left|\log\left(\frac{1+k'}{2}\right)\right| \leqq 1 - k' \leqq k^2. \tag{3.15}$$

Also, by the mean value theorem, for each θ there is a $\gamma \in [0, k]$, so that

$$\begin{aligned} 0 &\leqq \left[\sqrt{\frac{1-k'\sin\theta}{1+k'\sin\theta}} - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\right] \\ &\leqq \left[\sqrt{\frac{1-(1-k^2)\sin\theta}{1+(1-k^2)\sin\theta}} - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\right] \\ &= \left[\frac{\sqrt{1+(1-\gamma^2)\sin\theta}}{\sqrt{1-(1-\gamma^2)\sin\theta}} \cdot \frac{2\gamma\sin\theta}{(1+(1-\gamma^2)\sin\theta)^2}\right]k \\ &\leqq \frac{2\gamma k}{\sqrt{1-(1-\gamma^2)\sin\theta}} \leqq \frac{2k^2}{\sqrt{1-(1-k^2)\sin\theta}}. \end{aligned}$$

This yields

$$\left|\int_0^{\pi/2}\left[\sqrt{\frac{1-k'\sin\theta}{1+k'\sin\theta}} - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\right]d\theta\right| \leqq 2k^2\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k'\sin\theta}} \leqq 2\sqrt{2}\,k^2 I(1,k)$$

which combines with (3.14) and (3.15) to show that

$$|\Delta(k)| \leqq (1 + 2\sqrt{2})k^2 I(1,k) \leqq 4k^2 I(1,k).$$

We finish by observing that

$$kI(1,k) \leqq \frac{\pi}{2}$$

allows us to deduce that

$$I(1, k) \leqq 2\pi k + \log\left(\frac{4}{k}\right). \qquad \square$$

Similar considerations allow one to deduce that

$$|\Delta(k) - \Delta(h)| \leqq 2\pi |k - h| \tag{3.16}$$

for $0 < k, h < 1/\sqrt{2}$.

The next proposition gives all the information necessary for computing the elementary functions from the AGM.

PROPOSITION 3. *The* AGM *satisfies the following identity* (*for all initial values*):

$$\lim_{n\to\infty} 2^{-n}\frac{a'_n}{a_n}\log\left(\frac{4a_n}{c_n}\right) = \frac{\pi}{2}. \tag{3.17}$$

Proof. One verifies that

$$\begin{aligned}\frac{\pi}{2} &= \lim_{n\to\infty} a'_n I(a'_0, b'_0) &&\text{(by (3.9))}\\ &= \lim_{n\to\infty} a'_n I(a'_{-n}, b'_{-n}) &&\text{(by (3.8))}\\ &= \lim_{n\to\infty} a'_n I(2^n a_n, 2^n c_n) &&\text{(by 3.5)).}\end{aligned}$$

Now the homogeneity properties of $I(\cdot, \cdot)$ show that

$$I(2^n a_n, 2^n c_n) = \frac{2^{-n}}{a_n} I\left(1, \frac{c_n}{a_n}\right).$$

Thus

$$\frac{\pi}{2} = \lim_{n\to\infty} 2^{-n}\frac{a'_n}{a_n} I\left(1, \frac{c_n}{a_n}\right),$$

and the result follows from Proposition 1. $\square$

From now on we fix $a_0 := a'_0 := 1$ and consider the iteration as a function of $b_0 := k$ and $c_0 := k'$. Let P_n and Q_n be defined by

$$P_n(k) := \left(\frac{4a_n}{c_n}\right)^{2^{1-n}}, \qquad Q_n(k) := \frac{a_n}{a'_n}, \tag{3.18}$$

and let $P(k) := \lim_{n\to\infty} P_n(k)$, $Q(k) := \lim_{n\to\infty} Q_n(k)$. Similarly let $a := a(k) := \lim_{n\to\infty} a_n$ and $a' := a'(k) := \lim_{n\to\infty} a'_n$.

THEOREM 1. *For* $0 < k < 1$ *one has*

$$\begin{aligned}&\text{(a)}\quad P(k) = \exp(\pi Q(k)),\\ &\text{(b)}\quad 0 \leqq P_n(k) - P(k) \leqq \frac{16}{1-k^2}\left(\frac{a_n - a}{a}\right),\\ &\text{(c)}\quad |Q_n(k) - Q(k)| \leqq \frac{a'|a - a_n| + a|a' - a'_n|}{(a')^2}.\end{aligned} \tag{3.19}$$

Proof. (a) is an immediate rephrasing of Proposition 3, while (c) is straightforward.

To see (b) we observe that

$$P_{n+1} = P_n \cdot \left(\frac{a_{n+1}}{a_n}\right)^{2^{1-n}} \tag{3.20}$$

because $4a_{n+1}c_{n+1} = c_n^2$, as in (3.3). Since $a_{n+1} \leq a_n$ we see that

$$\begin{aligned} O \leq P_n - P_{n+1} &\leq \left[1 - \left(\frac{a_{n+1}}{a_n}\right)^{2^{1-n}}\right] P_n \leq \left(1 - \frac{a_{n+1}}{a_n}\right) P_0, \\ P_n - P_{n+1} &\leq \left(\frac{a_n - a_{n+1}}{a}\right) P_0 \end{aligned} \tag{3.21}$$

since a_n decreases to a. The result now follows from (3.21) on summation. □

Thus, the theorem shows that both P and Q can be computed exponentially since (a_n) can be so calculated. In the following sections we will use this theorem to give implementable exponential algorithms for π and then for all the elementary functions.

We conclude this section by rephrasing (3.19a). By using (3.20) repeatedly we derive that

$$P = \frac{16}{1-k^2} \prod_{n=0}^{\infty} \left(\frac{a_{n+1}}{a_n}\right)^{2^{1-n}}. \tag{3.22}$$

Let us note that

$$\frac{a_{n+1}}{a_n} = \frac{a_n + b_n}{2a_n} = \frac{1}{2}\left(1 + \frac{b_n}{a_n}\right),$$

and $x_n := b_n/a_n$ satisfies the one-term recursion used by Legendre [14]

$$x_{n+1} := \frac{2\sqrt{x_n}}{x_n + 1} \qquad x_0 := k. \tag{3.23}$$

Thus, also

$$P_{n+1}(k) = \frac{16}{1-k^2} \prod_{j=0}^{n+1} \left(\frac{1+x_j}{2}\right)^{2^{1-j}} = \left(\frac{1+x_n}{1-x_n}\right)^{2^{-n}}. \tag{3.24}$$

When $k := 2^{-1/2}$, $k = k'$ and one can explicitly deduce that $P(2^{-1/2}) = e^{\pi}$. When $k = 2^{-1/2}$ (3.22) is also given in [16].

4. Some interrelationships. A centerpiece of this exposition is the formula (3.17) of Proposition 3.

$$\lim_{n\to\infty} \frac{1}{2^n} \log\left(\frac{4a_n}{c_n}\right) = \frac{\pi}{2} \lim_{n\to\infty} \frac{a_n}{a_n'}, \tag{4.1}$$

coupled with the observation that both sides converge exponentially. To approximate $\log x$ exponentially, for example, we first find a starting value for which

$$\left(\frac{4a_n}{c_n}\right)^{1/2^n} \to x.$$

This we can do to any required accuracy quadratically by Newton's method. Then we compute the right limit, also quadratically, by the AGM iteration. We can compute exp analogously and since, as we will show, (4.1) holds for complex initial values we can also get the trigonometric functions.

There are details, of course, some of which we will discuss later. An obvious detail is that we require π to desired accuracy. The next section will provide an exponentially converging algorithm for π also based only on (4.1). The principle for it is very simple. If we differentiate both sides of (4.1) we lose the logarithm but keep the π!

Formula (3.10), of Proposition 1, is of some interest. It appears in King [11, pp. 13, 38] often without the "4" in the log term. For our purposes the "4" is crucial since without it (4.1) will only converge linearly (like $(\log 4)/2^n$). King's 1924 monograph contains a wealth of material on the various iterative methods related to computing elliptic integrals. He comments [11, p. 14]:

"The limit [(4.1) without the "4"] does not appear to be generally known, although an equivalent formula is given by Legendre (*Fonctions èliptiques*, t. I, pp. 94–101)."

King adds that while Gauss did not explicitly state (4.1) he derived a closely related series expansion and that none of this "appears to have been noticed by Jacobi or by subsequent writers on elliptic functions." This series [10, p. 377] gives (4.1) almost directly.

Proposition 1 may be found in Bowman [3]. Of course, almost all the basic work is to be found in the works of Abel, Gauss and Legendre [1], [10] and [14]. (See also [7].) As was noted by both Brent and Salamin, Proposition 2 can be used to estimate log given π. We know from (3.13) that, for $0 < k \leqq 10^{-3}$,

$$\left| \log\left(\frac{4}{k}\right) - I(1, k) \right| < 10k^2 |\log k|.$$

By subtraction, for $0 < x < 1$, and $n \geqq 3$,

$$|\log(x) - [I(1, 10^{-n}) - I(1, 10^{-n}x)]| < n\, 10^{-2(n-1)} \tag{4.2}$$

and we can compute log exponentially from the AGM approximations of the elliptic integrals in the above formula. This is in the spirit of Newman's presentation [15]. Formula (4.2) works rather well numerically but has the minor computational drawback that it requires computing the AGM for small initial values. This leads to some linear steps (roughly $\log(n)$) before quadratic convergence takes over.

We can use (3.16) or (4.2) to show directly that π is exponentially computable. With $k := 10^{-n}$ and $h := 10^{-2n} + 10^{-n}$ (3.16) yields with (3.9) that, for $n \geqq 1$,

$$\left| \log(10^{-n} + 1) - \frac{\pi}{2}\left[\frac{1}{AG(1, 10^{-n})} - \frac{1}{AG(1, 10^{-n} + 10^{-2n})}\right] \right| \leqq 10^{1-2n}.$$

Since $|\log(x + 1)/x - 1| \leqq x/2$ for $0 < x < 1$, we derive that

$$\left| \frac{2}{\pi} - \left[\frac{10^n}{AG(1, 10^{-n})} - \frac{10^n}{AG(1, 10^{-n} + 10^{-2n})}\right] \right| \leqq 10^{1-n}. \tag{4.3}$$

Newman [15] gives (4.3) with a rougher order estimate and without proof. This analytically beautiful formula has the serious computational drawback that obtaining n digit accuracy for π demands that certain of the operations be done to twice that precision.

Both Brent's and Salamin's approaches require *Legendre's relation*: for $0 < k < 1$

$$I(1, k)J(1, k') + I(1, k')J(1, k) - I(1, k)I(1, k') = \frac{\pi}{2} \tag{4.4}$$

where $J(a, b)$ is the *complete elliptic integral of the second kind* defined by

$$J(a, b) := \int_0^{\pi/2} \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}\, d\theta.$$

The elliptic integrals of the first and second kind are related by

$$J(a_0, b_0) = \left(a_0^2 - \frac{1}{2}\sum_{n=0}^{\infty} 2^n c_n^2\right) I(a_0, b_0) \tag{4.5}$$

where, as before, $c_n^2 = a_n^2 - b_n^2$ and a_n and b_n are computed from the AGM iteration.

Legendre's proof of (4.4) can be found in [3] and [8]. His elegant elementary argument is to differentiate (4.4) and show the derivative to be constant. He then evaluates the constant, essentially by Proposition 1. Strangely enough, Legendre had some difficulty in evaluating the constant since he had problems in showing that $k^2 \log(k)$ tends to zero with k [8, p. 150].

Relation (4.5) uses properties of the ascending Landen transformation and is derived by King in [11].

From (4.4) and (4.5), noting that if k equals $2^{-1/2}$ then so does k', it is immediate that

$$\pi = \frac{[2AG(1, 2^{-1/2})]^2}{1 - \sum_{n=1}^{\infty} 2^{n+1} c_n^2}. \tag{4.6}$$

This concise and surprising exponentially converging formula for π is used by both Salamin and Brent. As Salamin points out, by 1819 Gauss was in possession of the AGM iteration for computing elliptic integrals of the first kind and also formula (4.5) for computing elliptic integrals of second kind. Legendre had derived his relation (4.4) by 1811, and as Watson puts it [20, p. 14] "in the hands of Legendre, the transformation [(3.23)] became a most powerful method for computing elliptic integrals." (See also [10], [14] and the footnotes of [11].) King [11, p. 39] derives (4.6) which he attributes, in an equivalent form, to Gauss. It is perhaps surprising that (4.6) was not suggested as a practical means of calculating π to great accuracy until recently.

It is worth emphasizing the extraordinary similarity between (1.1) which leads to linearly convergent algorithms for all the elementary functions, and (3.1) which leads to exponentially convergent algorithms.

Brent's algorithms for the elementary functions require a discussion of incomplete elliptic integrals and the Landen transform, matters we will not pursue except to mention that some of the contributions of Landen and Fagnano are entertainingly laid out in an article by G.N. Watson entitled "The Marquis [Fagnano] and the Land Agent [Landen]" [20]. We note that Proposition 1 is also central to Brent's development though he derives it somewhat tangentially. He also derives Theorem 1(a) in different variables via the Landen transform.

5. An algorithm for π. We now present the details of our exponentially converging algorithm for calculating the digits of π. Twenty iterations will provide over two million digits. Each iteration requires about ten operations. The algorithm is very stable with all the operations being performed on numbers between ½ and 7. The eighth iteration, for example, gives π correctly to 694 digits.

THEOREM 2. *Consider the three-term iteration with initial values*

$$\alpha_0 := \sqrt{2}, \quad \beta_0 := 0, \quad \pi_0 := 2 + \sqrt{2}$$

given by

(i) $\alpha_{n+1} := \dfrac{1}{2}(\alpha_n^{1/2} + \alpha_n^{-1/2})$,

(ii) $\beta_{n+1} := \alpha_n^{1/2}\left(\dfrac{\beta_n + 1}{\beta_n + \alpha_n}\right)$,

(iii) $\pi_{n+1} := \pi_n \beta_{n+1}\left(\dfrac{1 + \alpha_{n+1}}{1 + \beta_{n+1}}\right)$.

Then π_n *converges exponentially to* π *and*

$$|\pi_n - \pi| \leqq \frac{1}{10^{2^n}}.$$

Proof. Consider the formula

$$\frac{1}{2^n}\log\left(4\frac{a_n}{c_n}\right) - \frac{\pi}{2}\frac{a_n}{a'_n} \tag{5.1}$$

which, as we will see later, converges exponentially at a uniform rate to zero in some (complex) neighbourhood of $1/\sqrt{2}$. (We are considering each of $a_n, b_n, c_n, a'_n, b'_n, c'_n$ as being functions of a complex initial value k, i.e. $b_0 = k$, $b'_0 = \sqrt{1-k^2}$, $a_0 = a'_0 = 1$.)

Differentiating (5.1) with respect to k yields

$$\frac{1}{2^n}\left(\frac{\dot a_n}{a_n} - \frac{\dot c_n}{c_n}\right) - \frac{\pi}{2}\frac{a_n}{a'_n}\left(\frac{\dot a_n}{a_n} - \frac{\dot a'_n}{a'_n}\right) \tag{5.2}$$

which also converges uniformly exponentially to zero in some neighbourhood of $1/\sqrt{2}$. (This general principle for exponential convergence of differentiated sequences of analytic functions is a trivial consequence of the Cauchy integral formula.) We can compute $\dot a_n$, $\dot b_n$ and $\dot c_n$ from the recursions

$$\begin{aligned} \dot a_{n+1} &:= \frac{\dot a_n + \dot b_n}{2}, \\ \dot b_{n+1} &:= \frac{1}{2}\left(\dot a_n\sqrt{\frac{b_n}{a_n}} + \dot b_n\sqrt{\frac{a_n}{b_n}}\right), \\ \dot c_{n+1} &:= \frac{1}{2}(\dot a_n - \dot b_n), \end{aligned} \tag{5.3}$$

where $\dot a_0 := 0$, $\dot b_0 := 1$, $a_0 := 1$ and $b_0 := k$.

We note that a_n and b_n map $\{z \mid Re(z) > 0\}$ into itself and that $\dot a_n$ and $\dot b_n$ (for sufficiently large n) do likewise.

It is convenient to set

$$\alpha_n := \frac{a_n}{b_n} \quad \text{and} \quad \beta_n := \frac{\dot a_n}{\dot b_n} \tag{5.4}$$

with

$$\alpha_0 := \frac{1}{k} \quad \text{and} \quad \beta_0 := 0.$$

We can derive the following formulae in a completely elementary fashion from the basic relationships for a_n, b_n and c_n and (5.3):

$$\dot a_{n+1} - \dot b_{n+1} = \frac{1}{2}(\sqrt{a_n} - \sqrt{b_n})\left(\frac{\dot a_n}{\sqrt{a_n}} - \frac{\dot b_n}{\sqrt{b_n}}\right), \tag{5.5}$$

(5.6) $$1 - \frac{a_{n+1}}{\dot{a}_{n+1}} \frac{\dot{c}_{n+1}}{c_{n+1}} = \frac{2(\alpha_n - \beta_n)}{(\alpha_n - 1)(\beta_n + 1)},$$

(5.7) $$\alpha_{n+1} = \frac{1}{2}(\alpha_n^{1/2} + \alpha_n^{-1/2}),$$

(5.8) $$\beta_{n+1} = \alpha_n^{1/2}\left(\frac{\beta_n + 1}{\beta_n + \alpha_n}\right),$$

(5.9) $$\alpha_{n+1} - 1 = \frac{1}{2\alpha_n^{1/2}}(\alpha_n^{1/2} - 1)^2,$$

(5.10) $$\alpha_{n+1} - \beta_{n+1} = \frac{\alpha_n^{1/2}}{2} \frac{(1 - \alpha_n)(\beta_n - \alpha_n)}{\alpha_n(\beta_n + \alpha_n)},$$

(5.11) $$\frac{\alpha_{n+1} - \beta_{n+1}}{\alpha_{n+1} - 1} = \frac{(1 + \alpha_n^{1/2})^2}{(\beta_n + \alpha_n)} \cdot \frac{(\alpha_n - \beta_n)}{(\alpha_n - 1)}.$$

From (5.7) and (5.9) we deduce that $\alpha_n \to 1$ uniformly with second order in compact subsets of the open right half-plane. Likewise, we see from (5.8) and (5.10) that $\beta_n \to 1$ uniformly and exponentially. Finally, we set

(5.12) $$\gamma_n := \frac{1}{2^n}\left(\frac{\alpha_n - \beta_n}{\alpha_n - 1}\right).$$

We see from (5.11) that

(5.13) $$\gamma_{n+1} = \frac{(1 + \alpha_n^{1/2})}{2(\beta_n + \alpha_n)} \gamma_n$$

and also from (5.6) that

(5.14) $$\frac{\gamma_n}{1 + \beta_n} = \frac{1}{2^{n+1}}\left(1 - \frac{a_{n+1}}{\dot{a}_{n+1}} \frac{\dot{c}_{n+1}}{c_{n+1}}\right).$$

Without any knowledge of the convergence of (5.1) one can, from the preceding relationships, easily and directly deduce the exponential convergence of (5.2), in $\{z \,|\, |z - 1/2| \leq c < 1/2\}$. We need the information from (5.1) only to see that (5.2) converges to zero.

The algorithm for π comes from multiplying (5.2) by $a_n/\dot{a}_n$ and starting the iteration at $k := 2^{-1/2}$. For this value of k $a_n' = a_n$, $(\dot{a}_n') = -\dot{a}_n$ and

$$\frac{1}{2^{n+1}}\left(1 - \frac{a_{n+1}}{\dot{a}_{n+1}} \frac{\dot{c}_{n+1}}{c_{n+1}}\right) \to \pi$$

which by (5.14) shows that

$$\pi_n := \frac{\gamma_n}{1 + \beta_n} \to \pi.$$

Some manipulation of (5.7), (5.8) and (5.13) now produces (iii). The starting values for α_n, β_n and γ_n are computed from (5.4). Other values of k will also lead to similar, but slightly more complicated, iterations for π.

To analyse the error one considers

$$\frac{\gamma_{n+1}}{1 + \beta_{n+1}} - \frac{\gamma_n}{1 + \beta_n} = \left[\frac{(1 + \alpha_n^{1/2})^2}{2(\beta_n + \alpha_n)(1 + \beta_{n+1})} - \frac{1}{(1 + \beta_n)}\right] \gamma_n$$

and notes that, from (5.9) and (5.10), for $n \geqq 4$,

$$|\alpha_n - 1| \leqq \frac{1}{10^{2^n+2}} \quad \text{and} \quad |\beta_n - 1| \leqq \frac{1}{10^{2^n+2}}.$$

(One computes that the above holds for $n = 4$.) Hence,

$$\left|\frac{\gamma_{n+1}}{1+\beta_{n+1}} - \frac{\gamma_n}{1+\beta_n}\right| \leqq \left|\frac{1}{10^{2^n+1}}\right| |\gamma_n|$$

and

$$\left|\frac{\gamma_n}{1+\beta_n} - \pi\right| \leqq \frac{1}{10^{2^n}}. \qquad \square$$

In fact one can show that the error is of order $2^n e^{-\pi 2^{n+1}}$.

If we choose integers in $[\delta, \delta^{-1}]$, $0 < \delta < 1/2$ and perform n operations $(+, -, \times, \div, \sqrt{})$ then the result is always less than or equal to δ^{2^n}. Thus, if $\gamma > \delta$, it is not possible, using the above operations and integral starting values in $[\delta, \delta^{-1}]$, for every n to compute π with an accuracy of $O(\gamma^{-2^n})$ in n steps. In particular, convergence very much faster than that provided by Theorem 2 is not possible.

The analysis in this section allows one to derive the Gauss-Salamin formula (4.6) without using Legendre's formula or second integrals. This can be done by combining our results with problems 15 and 18 in [11]. Indeed, the results of this section make quantitative sense of problems 16 and 17 in [11]. King also observes that Legendre's formula is actually equivalent to the Gauss–Salamin formula and that each may be derived from the other using only properties of the AGM which we have developed and equation (4.5).

This algorithm, like the algorithms of §4, is not self correcting in the way that Newton's method is. Thus, while a certain amount of time may be saved by observing that some of the calculations need not be performed to full precision it seems intrinsic (though not proven) that $O(\log n)$ full precision operations must be executed to calculate π to n digits. In fact, showing that π is intrinsically more complicated from a time complexity point of view than multiplication would prove that π is transcendental [5].

6. The complex AGM iteration. The AGM iteration

$$a_{n+1} := \frac{1}{2}(a_n + b_n), \qquad b_{n+1} := \sqrt{a_n b_n}$$

is well defined as a complex iteration starting with $a_0 := 1$, $b_0 := z$. Provided that z does not lie on the negative real axis, the iteration will converge (to what then must be an analytic limit). One can see this geometrically. For initial z in the right half-plane the limit is given by (3.9). It is also easy to see geometrically that a_n and b_n are always nonzero.

The iteration for $x_n := b_n/a_n$ given in the form (3.23) as $x_{n+1} := 2\sqrt{x_n}/x_{n+1}$ satisfies

$$(x_{n+1} - 1) = \frac{(1 - \sqrt{x_n})^2}{1 + x_n}. \tag{6.1}$$

This also converges in the cut plane $\mathbb{C} - (-\infty, 0]$. In fact, the convergence is uniformly exponential on compact subsets (see Fig. 1). With each iteration the angle θ_n between x_n and 1 is at least halved and the real parts converge uniformly to 1.

It is now apparent from (6.1) and (3.24) that

$$P_n(k) := \left(\frac{4a_n}{c_n}\right)^{2^{1-n}} = \left(\frac{1+x_n}{1-x_n}\right)^{2^{-n}} \tag{6.2}$$

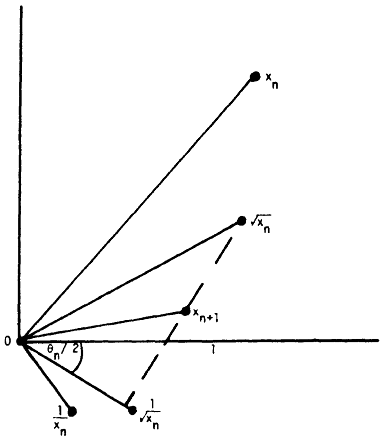


FIG. 1.

and also,

$$Q_n(k) := \frac{a_n}{a_n'}$$

converge exponentially to analytic limits on compact subsets of the complex plane that avoid

$$D := \{z \in \mathbb{C} \mid z \notin (-\infty, 0] \cup [1, \infty)\}.$$

Again we denote the limits by P and Q. By standard analytic reasoning it must be that (3.19a) still holds for k in D.

Thus one can compute the complex exponential—and so also cos and sin—exponentially using (3.19). More precisely, one uses Newton's method to approximately solve $Q(k) = z$ for k and then computes $P_n(k)$. The outcome is e^z. One can still perform the root extractions using Newton's method. Some care must be taken to extract the correct root and to determine an appropriate starting value for the Newton inversion. For example $k := 0.02876158$ yields $Q(k) = 1$ and $P_4(k) = e$ to 8 significant places. If one now uses k as an initial estimate for the Newton inversions one can compute $e^{1+i\theta}$ for $|\theta| \leq \pi/8$. Since, as we have observed, e is also exponentially computable we have produced a sufficient range of values to painlessly compute $\cos\theta + i\sin\theta$ with no recourse to any auxiliary computations (other than π and e, which can be computed once and stored). By contrast Brent's trigonometric algorithm needs to compute a different logarithm each time.

The most stable way to compute P_n is to use the fact that one may update c_n by

$$c_{n+1} = \frac{c_n^2}{4a_{n+1}}. \tag{6.3}$$

One then computes a_n, b_n and c_n to desired accuracy and returns

$$\left(\frac{4a_n}{c_n}\right)^{1/2^n} \quad \text{or} \quad \left(\frac{2(a_n + b_n)}{c_n}\right)^{1/2^n}.$$

This provides a feasible computation of P_n, and so of exp or log.

In an entirely analogous fashion, formula (4.2) for log is valid in the cut complex plane. The given error estimate fails but the convergence is still exponential. Thus (4.2) may also be used to compute all the elementary functions.

7. Concluding remarks and numerical data. We have presented a development of the AGM and its uses for rapidly computing elementary functions which is, we hope, almost entirely self-contained and which produces workable algorithms. The algorithm for π is particularly robust and attractive. We hope that we have given something of the flavour of this beautiful collection of ideas, with its surprising mixture of the classical and the modern. An open question remains. Can one entirely divorce the central discussion from elliptic integral concerns? That is, can one derive exponential iterations for the elementary functions without recourse to some nonelementary transcendental functions? It would be particularly nice to produce a direct iteration for e of the sort we have for π which does not rely either on Newton inversions or on binary splitting.

The algorithm for π has been run in an arbitrary precision integer arithmetic. (The algorithm can be easily scaled to be integral.) The errors were as follows:

Iterate	Digits correct	Iterate	Digits correct
1	3	6	170
2	8	7	345
3	19	8	694
4	41	9	1392
5	83	10	2788

Formula (4.2) was then used to compute 2 log (2) and log (4), using π estimated as above and the same integer package. Up to 500 digits were computed this way. It is worth noting that the error estimate in (4.2) is of the right order.

The iteration implicit in (3.22) was used to compute e^{π} in a double precision Fortran. Beginning with $k := 2^{-1/2}$ produced the following data:

Iterate	$P_n - e^{\pi}$	$a_n/b_n - 1$
1	1.6×10^{-1}	1.5×10^{-2}
2	2.8×10^{-9}	2.8×10^{-5}
3	1.7×10^{-20}	9.7×10^{-11}
4	$< 10^{-40}$	1.2×10^{-21}

Identical results were obtained from (6.3). In this case $y_n := 4a_n/c_n$ was computed by the two term recursion which uses x_n, given by (3.23), and

$$y_0^2 := \frac{16}{1-k^2}, \qquad y_{n+1} = \left(\frac{1+x_n}{2}\right)^2 y_n^2. \tag{7.1}$$

One observes from (7.1) that the calculation of y_n is very stable.

We conclude by observing that the high precision root extraction required in the AGM [18], was actually calculated by inverting $y = 1/x^2$. This leads to the iteration

$$x_{n+1} = \frac{3x_n - x_n^3 y}{2} \tag{7.2}$$

for computing $y^{-1/2}$. One now multiplies by y to recapture $\sqrt{y}$. This was preferred because it avoided division.

REFERENCES

[1] N. H. Abel, *Oeuvres complètes*, Grondahl and Son, Christiana, 1881.

[2] P. Beckman, *A History of Pi*, Golem Press, Ed. 4, 1977.

[3] F. Bowman, *Introduction to Elliptic Functions*, English Univ. Press, London, 1953.

[4] R. P. Brent, *Multiple-precision zero-finding and the complexity of elementary function evaluation*, in Analytic Computational Complexity, J. F. Traub, ed., Academic Press, New York, 1975, pp. 151–176.

[5] ———, *Fast multiple-precision evaluation of elementary functions*, J. Assoc. Comput. Mach., 23 (1976), pp. 242–251.

[6] B. C. Carlson, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly, 78(1971), pp. 496–505.

[7] A. Cayley, *An Elementary Treatise on Elliptic Functions*, Bell and Sons, 1895, republished by Dover, New York, 1961.

[8] A. Eagle, *The Elliptic Functions as They Should Be*, Galloway and Porter, Cambridge, 1958.

[9] C. H. Edwards, Jr. *The Historical Development of the Calculus*, Springer-Verlag, New York, 1979.

[10] K. F. Gauss, *Werke*, Bd 3, Gottingen, 1866, 361–403.

[11] L. V. King, *On the direct numerical calculation of elliptic functions and integrals*, Cambridge Univ. Press, Cambridge, 1924.

[12] D. Knuth, *The Art of Computer Programming. Vol.* 2: *Seminumerical Algorithms*, Addison-Wesley, Reading, MA, 1969.

[13] H. T. Kung and J. F. Traub, *All algebraic functions can be computed fast*, J. Assoc. Comput. Mach., 25(1978), 245–260.

[14] A. M. Legendre, *Exercices de calcul integral*, Vol. 1, Paris, 1811.

[15] D. J. Neuman, *Rational approximation versus fast computer methods*, in Lectures on Approximation and Value Distribution, Presses de l'Université de Montréal, Montreal, 1982, pp. 149–174.

[16] E. Salamin, *Computation of π using arithmetic-geometric mean*, Math. Comput. 135(1976), pp. 565–570.

[17] D. Shanks and J. W. Wrench, Jr. *Calculation of π to* 100,000 *decimals*, Math. Comput., 16 (1962), pp. 76–79.

[18] Y. Tamura and Y. Kanada, *Calculation of π to* 4,196,293 *decimals based on Gauss-Legendre algorithm*, preprint.

[19] J. Todd, *Basic Numerical Mathematics*, Vol. 1, Academic Press, New York, 1979.

[20] G. N. Watson, *The marquis and the land agent*, Math. Gazette, 17 (1933), pp. 5–17.

[21] J. W. Wrench, Jr. *The evolution of extended decimal approximations to π*. The Mathematics Teacher, 53(1960), pp. 644–650.

MATHEMATICS OF COMPUTATION
VOLUME 44, NUMBER 169
JANUARY 1985, PAGES 207-210

A Simplified Version of the Fast Algorithms of Brent and Salamin

By D. J. Newman*

Abstract. We produce more elementary algorithms than those of Brent and Salamin for, respectively, evaluating e^x and π. Although the Gauss arithmetic-geometric process still plays a central role, the elliptic function theory is now unnecessary.

In their remarkable papers, Brent [1] and Salamin [3], respectively, used the theory of elliptic functions to obtain "fast" computations of the function e^x and of the number π. In both cases rather heavy use of elliptic function theory, such as the transformation law of Landen, had to be utilized. Our purpose, in this note, is to give a highly simplified version of their constructions. In our approach, for example, the incomplete elliptic integral is never used.

We begin as they did with the Gauss arithmetic-geometric process, $T(a, b) = ((a + b)/2, \sqrt{ab})$ which maps couples with $a \geqslant b > 0$ into same. From the inequalities

$$\frac{(a+b)/2 - \sqrt{ab}}{(a+b)/2 + \sqrt{ab}} = \left(\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}\right)^2 \leqslant \left(\frac{a-b}{a+b}\right)^2,$$

and

$$\frac{(a+b)/2}{\sqrt{ab}} \leqslant \frac{a}{\sqrt{ab}} = \sqrt{\frac{a}{b}},$$

we see that $T^i(a, b)$ goes to its limiting couple (m, m) ($m = m(a, b)$ the so-called arithmetic-geometric mean) with "quadratic" speed. Indeed, $m(a, b)$ is determined to n places for an i of around $\log\log a/b + \log n$. The $\log\log$ from the $\sqrt{a/b}$ inequality expressing the time till the ratio first goes below 2, and the log from the $((a - b)/(a + b))^2$ inequality expressing the time for the error squaring to do its job.

Next, we recall Gauss' beautiful formula:

$$m(a, b) = \pi \Big/ \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}},$$

Received October 7, 1981; revised November 17, 1981, and March 22, 1984.
1980 *Mathematics Subject Classification*. Primary 65D15, 33A25, 41A25.
*Supported in part by NSF Grant MGS 7802171.

which follows from the fact that this (complete) elliptic integral is invariant under T. This fact, that namely

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{\left(t^2+((a+b)/2)^2\right)(t^2+ab)}},$$

is a simple consequence of the change of variables $t = (x - ab/x)/2$. Namely, we obtain

$$dt = \frac{x^2+ab}{2x^2}dx, \qquad t^2 + \left(\frac{a+b}{2}\right)^2 = \frac{(x^2+a^2)(x^2+b^2)}{4x^2},$$
$$t^2 + ab = \frac{(x^2+ab)^2}{4x^2},$$

$0 < x < \infty$, so that indeed we have

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \int_{0}^{\infty} \frac{2dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}$$
$$= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{\left(t^2+((a+b)/2)^2\right)(t^2+ab)}}.$$

Accordingly, a repeated use of this invariance gives

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \cdots = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+m^2)(x^2+m^2)}}$$
$$= \int_{-\infty}^{\infty} \frac{dx}{x^2+m^2} = \frac{\pi}{m},$$

and this is exactly Gauss' formula.

Actually, it is handier for us to work with what we might call the harmonic-geometric mean which can be defined by $h(a, b) = ab/m(a, b)$ or, alternatively, as the limit under repeated applications of S, rather than T, where

$$S(a, b) = \left(\sqrt{ab}, 2ab/(a+b)\right).$$

In these terms Gauss' formula reads

$$h(a, b) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(1+x^2/a^2)(1+x^2/b^2)}}.$$

The only place that we actually use this formula is to establish the asymptotic formula:

$$h(N, 1) = \frac{2}{\pi}\log 4N + O(1/N^2).$$

(This simple-looking formula certainly deserves an elementary proof independent of elliptic integrals, but we are unable to find one.)

So begin with

$$h(N, 1) = \frac{2}{\pi} \int_{0}^{\infty} \frac{dx}{\sqrt{(1+x^2)(1+x^2/N^2)}}$$

and observe that the map $x \to N/x$ leaves the integrand invariant. Thereby, we conclude

$$\int_0^{\sqrt{N}} \frac{dx}{\sqrt{(1+x^2)(1+x^2/N^2)}} = \int_{\sqrt{N}}^{\infty} \frac{dx}{\sqrt{(1+x^2)(1+x^2/N^2)}}$$

which gives us

$$\begin{aligned} h(N,1) &= \frac{4}{\pi}\int_0^{\sqrt{N}} \frac{dx}{\sqrt{(1+x^2)(1+x^2/N^2)}} \\ &= \frac{4}{\pi}\int_0^{\sqrt{N}} \frac{1}{\sqrt{(1+x^2)}}\left(1 - x^2/2N^2 + O(x^4/N^4)\right) dx \\ &= \frac{4}{\pi}\int_0^{\sqrt{N}} \left(\frac{1}{\sqrt{1+x^2}} - \frac{x}{2N^2}\right) dx + O(1/N^2) \\ &= \frac{4}{\pi}\left(\log(\sqrt{N} + \sqrt{N+1}) - 1/4N\right) + O(1/N^2) \end{aligned}$$

and so, since

$$\sqrt{N} + \sqrt{N+1} = 2\sqrt{N}\left(1 + 1/\left(2N + 2\sqrt{N(N+1)}\right)\right),$$

we obtain

$$\log(\sqrt{N} + \sqrt{N+1}) = \log 2\sqrt{N} + 1/4N + O\left(\frac{1}{N^2}\right),$$

which together with the previous gives

$$h(N,1) = \frac{4}{\pi}\log 2\sqrt{N} + O\left(\frac{1}{N^2}\right) = \frac{2}{\pi}\log 4N + O\left(\frac{1}{N^2}\right),$$

as required. (This result can also be found in [2].)

Summarizing, then, we have produced a fast method for obtaining n places of $2\log 4N/\pi$ (if N is of the size c^n). But, and here is the trick, this combination of π and the logarithm can be used to yield both of them separately, and we can thereby rederive both Salamin's and Brent's results.

To obtain π we examine the difference, $h(N+1,1) - h(N,1)$, and observe that $N(h(N+1,1) - h(N,1)) = 2/\pi + O(1/N)$ which gives n place accuracy for π if we choose, e.g., $N = 2^n$.

For the logarithm, on the other hand, we look to the quotient, $h(N+1,1)/h(N,1)$. This time we obtain

$$N\left(\frac{h(N+1,1)}{h(N,1)} - 1\right) = N\frac{\log(1+1/N) + O(1/N^2)}{\log 4N + O(1/N)} = \frac{1}{\log 4N + O(1/N)}.$$

From this we will be able to evaluate $\log x$ throughout the interval $(3,9)$, and so, of course, throughout any interval. And thereby, we will be able to obtain e^x, the inverse function, by the usual use of the (fast) Newton iteration scheme.

To obtain $\log x$, then, in the interval $(3,9)$, we first calculate $N = \frac{1}{4}x^n$, a process that takes only $\log n$ multiplications. But then the above formula becomes, upon substitution of this value of N,

$$\frac{1}{4}nx^n\left(\frac{h(\frac{1}{4}x^n + 1,1)}{h(\frac{1}{4}x^n,1)} - 1\right) = \frac{1}{\log x} + O\left(\frac{n}{x^n}\right) = \frac{1}{\log x} + O\left(\frac{n}{3^n}\right)$$

which does give the desired n place evaluation of $\log x$.

This trick of "differencing" $h(N+1,1)$ and $h(N,1)$, of course, carries a price. Thus we must compute these two quantities to $2n$ places and so the running time is around twice as long as the corresponding ones of Brent and Salamin.

Mathematics Department
Temple University
Broad and Montgomery Ave.
Philadelphia, Pennsylvania 19122

1. R. Brent, "Fast multiple evaluation of elementary functions," *J. Assoc. Comput. Mach.*, v. 23, 1976, pp. 242–251.
2. R. Brent, *Multiple-Precision Zero-Finding Methods and the Complexity of Elementary Function Evaluation* (J. F. Traub, ed.), Academic Press, New York, 1975, pp. 151–176.
3. E. Salamin, "Computation of π using the arithmetic-geometric mean," *Math. Comp.*, v. 30, 1976, pp. 565–570.

The Evidence

Is π Normal?
by Stan Wagon

The nature of the number π has intrigued mathematicians since the beginning of mathematical history. The most important properties of π are its irrationality and transcendence, which were established in 1761 and 1882, respectively. In the twentieth century the focus has been on a different sort of question, namely whether π, despite being irrational and transcendental, is normal.

The idea of normality, first introduced by E. Borel in 1909, is an attempt to formalize the notion of a real number being random. The definition is as follows: A real number x is *normal in base b* if in its representation in base b all digits occur, in an asymptotic sense, equally often. In addition, for each m, the b^m different m-strings must occur equally often. In other words, $\lim_{n\to\infty} N(s,n)/n = b^{-m}$ for each m-string s, where $N(s,n)$ is the number of occurrences of s in the first n base-b digits of x. A number that is normal in all bases is called *normal.* The apparent randomness of π's digits had been observed prior to the precise definition of normality. De Morgan, for example, pointed out that one would expect the digits to occur equally often, but yet the number of 7's in the first 608 digits is 44, much lower than expected. However, it turned out that his count was based on inaccurate data.

There are lots of normal numbers—Borel proved (see Niven or §9.12 of Hardy and Wright) that the set of non-normal numbers has measure zero—but it is difficult to provide concrete examples. While an undergraduate at Cambridge University, D. Champernowne proved that 0.12345678910111213 ... is normal in base 10, but an explicit example of a normal number is still lacking.

The question of π's normality only scratches the surface of the deeper question whether the digits of π are "random." That normality is not sufficient follows from the observation that a truly random sequence of digits ought to be normal when only digits in positions corresponding to perfect squares are examined. But if all such positions in a normal number are set to 0, the number is still normal. On the other hand, more rigorous definitions of "random" exclude π because π's decimal expansion is a recursive sequence. For an enlightening discussion of the general problem of defining randomness, see the section on "What is a Random Sequence" in volume 2 of Knuth's trilogy.

Thus deeper questions are lurking, but so little is known about π's decimal expansion that it is reasonable to focus on whether π is normal to base ten. To put our ignorance in perspective, note that it is not even known that all digits appear infinitely often: perhaps

$$\pi = 3.1415926.....0100100010000100000 1\ldots .$$

In order to gather evidence for π's normality one would like to examine as many digits as possible. Those who have pursued the remote digits of π have often been pejoratively referred to as "digit hunters," but certain recent developments have added some glamor to the centuries-old hunt. In 1975 Brent and Salamin, independently, discovered an algorithm that dramatically lowered the time needed to compute large numbers of digits of π. Moreover, the algorithm has important connections with efficient algorithms for computing various transcendental functions (sin, arctan, exp, log, elliptic integrals) to great accuracy.

The reader is probably aware that various arctangent formulas have been central in the computation of π. Early investigators used series such as the one for arctan $1/\sqrt{3}$, but convergence was much speeded by the use of formulas such as the following, by means of which Machin computed 100 decimals of π in 1706: $\pi = 16\arctan 1/5 - 4\arctan 1/239$. Indeed, this same formula was used in the first computer calculation, the ENIAC's computation of 2037 digits in 1949. Computers have become much faster, and various opti-

Column Editor: Stan Wagon, Department of Mathematics, Smith College, Northampton, Massachusetts 01063, USA.

Year	Time	Number of digits	Computer time per digit
1949	~70 hours	2,037	~2 minutes
1958	100 minutes	10,000	0.6 second
1961	8.43 hours	100,000	1/3 second
1973	23.3 hours	1,000,000	1/12 second
1983	<30 hours	16,000,000	<1/155 second

Sophisticated algorithms, combined with ever-faster machines, have led to increasingly more efficient computations of π's digits during the computer age.

Table 1

mizing tricks help speed up computations, but even the million-digit computation in 1973 used the same sort of formula, this one due to Gauss:

$$\pi = 48\arctan 1/18 + 32\arctan 1/57 \pm 20\arctan 1/239.$$

A limiting aspect of the arctangent formulas is the number of full-precision operations that must be carried out. By an *operation* we mean one of $+$, $\times$, $\div$, $\sqrt{\ }$. Since large-precision square roots and divisions can be performed in essentially as much time as that required by a full-precision multiplication (Newton's method can be used for both; see Borwein and Borwein), this measure of complexity (number of full-precision operations) is as good as the more usual "time complexity" for comparing algorithms. Now, an examination of the rate of convergence of the arctangent series shows that the arctangent method uses $O(n)$ (i.e., at most cn, c a constant) full-precision operations to compute n decimals of π; for example, the Shanks and Wrench computation of 100,000 decimals used just under 105,000 full-precision operations. Thus there are two basic time costs involved in the pushing of a calculation from n to $10n$ digits:

(1) the number of operations increases by a factor of 10, and
(2) the time for each full-precision operation is about 10 times greater.

The Brent-Salamin algorithm requires only $O(\log n)$ full-precision operations for n digits of π. Since $\log n$ barely increases when n is replaced by $10n$, the first of the two costs just mentioned is almost entirely eliminated!

The Brent-Salamin formula uses ideas that go back to Gauss and Legendre, but prior to the 1970s no one had thought to apply these ideas to evaluate π. The formula exploits the speed of convergence of the defining sequences for the arithmetic-geometric mean of two numbers. Given positive reals $a_0 > b_0$, their *arithmetic-geometric mean*, $AG(a_0,b_0)$, is defined to be the common limit of $\{a_n\}$, $\{b_n\}$, where $a_n = (a_{n-1} + b_{n-1})/2$ and $b_n = \sqrt{a_{n-1}\, b_{n-1}}$. Gauss had investigated such limits and had proved that $AG(a,b)$ equals $\pi/2I$, where

$$I = \int_0^{\pi/2} \frac{dt}{\sqrt{a^2\cos^2 t + b^2\sin^2 t}},$$

a complete elliptic integral of the first kind. To get a formula for π let a_0, b_0 be 1, $1/\sqrt{2}$, respectively, and define d_n to be $a_n^2 - b_n^2$; then

$$\pi = 4(AG(1,1/\sqrt{2}))^2 \Big/ \left(1 - \sum_{j=1}^{\infty} 2^{j+1} d_j\right).$$

If π_n is defined to be $4a_{n+1}^2/(1 - \sum_{j=1}^{n} 2^{j+1}d_j)$, then π_n converges quadratically to π. This means, roughly, that the number of correct digits *doubles* from one π_n to the next.

Quadratic convergence is most familiar from Newton's method of approximating solutions to algebraic equations. Thus, from a computational perspective, π behaves like an algebraic number.

A more precise error analysis shows that π_{16} is accurate to 178,000 digits, π_{19} to over a million, π_{22} to over ten million, and π_{26} to almost 200,000,000 digits. Since the computation of π_n requires $7n$ full-precision operations, the improvement over the classical algorithm is impressive: 100,000 digits require only 112 full-precision operations!

The fact that $c \log n$ full-precision operations yield n digits of π means that the time complexity (essentially, number of bit operations) of the computation is $O(n \log^2 n \log\log n)$; this uses the fact that n-digit multiplication is of complexity $O(n \log n \log\log n)$. Thus these fast algorithms for π are just about the fastest possible: it takes n steps just to write down n digits, and there is not much room between n and $n \log^2 n \log\log n$.

For refinements of the Brent-Salamin formulas, applications to the computation of transcendental functions, and some proofs, see the paper by Borwein and Borwein, who are preparing a book on the arithmetic-geometric mean. See the paper by Cox for more on the

Type of hand	Expected number	Actual number
No two digits the same	604,800	604,976
One pair	1,008,000	1,007,151
Two pair	216,000	216,520
Three of a kind	144,000	144,375
Full house	18,000	17,891
Four of a kind	9,000	8,887
Five of a kind	200	200

Distribution of the first two million poker hands in the digits of π.

Table 2

arithmetic-geometric mean and for a historical account of Gauss's work.

There is a surprising connection between these modern algorithms and the method of Archimedes. Archimedes used inscribed and circumscribed polygons to approximate π. Now, if A_n is the reciprocal of the circumference of a 2^n-gon inscribed in a unit circle and B_n likewise for a circumscribed 2^n-gon, then A_n and B_n satisfy: $B_{n+1} = \frac{1}{2}(A_n + B_n)$ and $A_{n+1} = \sqrt{A_n B_{n+1}}$. Thus the double sequence of Archimedes obeys a recursion almost identical to the defining recursion for AG(1/2, 1/4). Archimedes' sequences converge much more slowly, however: each iteration decreases the error by, approximately, a factor of four.

The Brent-Salamin algorithm has been implemented in Japan by Kanada, Tamura, Yoshino, and Ushiro who, in 1983, used it to compute 16 million decimal places. They checked the first 10,013,395 of these using Gauss's arctangent relation. The computation of 16 million digits took less than 30 hours of CPU time, although some time (10–20%) was saved by the reuse of intermediate values from earlier computations. See Table 1 for a comparison with previous computations. The check took only 24 hours, but the two times are not comparable since the arctangent computation was performed on a much faster computer, a Japanese Hitachi supercomputer with a speed of 630 MFLOPS (million floating-point operations per second).

A forthcoming paper by Kanada contains a statistical analysis of the first ten million digits, which show no unusual deviation from expected behavior. The frequencies for each of the ten digits are: 999,440; 999,333; 1,000,306; 999,964; 1,001,093; 1,000,466; 999,337; 1,000,207; 999,814; and 1,000,040. Moreover, the speed with which the relative frequencies are approaching 1/10 agrees with theory. Consider the digit 7 for example. Its relative frequencies in the first 10^i digits ($i = 1, \ldots, 7$) are 0, .08, .095, .097, .10025, .0998, .1000207, which seem to be approaching 1/10 at the speed predicted by probability theory for random digits, namely at a speed approximately proportional to $1/\sqrt{n}$. The poker test is relevant to the question of normality in base ten, and Table 2 contains the frequencies of poker hands from the first ten million digits; there is no significant deviation from the expected values.

Writers over the years have been fond of mentioning that 20 decimals of π suffice for any application imaginable. Moreover, the millions of digits now known shed absolutely no light on how to prove π's normality. But these criticisms miss the point. Huyghens, in using an extrapolative techinque to extend Archimedes' calculations, was the first to use an important technique that, in this century, has come to be known as the Romberg method for approximating definite integrals. And the arithmetic-geometric mean algorithms and their refinements are closely connected to the fastest known techniques for evaluating multiprecision transcendental functions. Thus digit-hunting has an importance that goes beyond the mere extension of the known decimal places of π.

References

J. M. Borwein and P. B. Borwein, The arithmetic-geometric mean and fast computation of elementary functions, *SIAM Review 26*, (1984) 351–366.

R. P. Brent, Multiple-precision zero-findings methods and the complexity of elementary function evaluation, in *Analytic Computational Complexity*, J. F. Traub, ed., New York: Academic Press, 1976, pp. 151–176.

D. Cox, The arithmetic-geometric mean of Gauss, *Ens. Math. 30*, (1984) 275–330.

G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, London: Oxford, 1975.

Y. Kanada, Y. Tamura, S. Yoshino, and Y. Ushiro, Calculation of π to 10,013,395 decimal places based on the Gauss-Legendre algorithm and Gauss arctangent relations, *Mathematics of Computation* (forthcoming).

D. E. Knuth, *The Art of Computer Programming*, vol. 2, Reading, Mass.: Addison-Wesley, 1969.

I. Niven, *Irrational Numbers*, Carus Mathematical Monographs, No. 11, The Mathematical Association of America. Distributed by Wiley, New York, 1967.

E. Salamin, Computation of π using arithmetic-geometric mean, *Mathematics of Computation 30*, (1976) 565–570.

Circle Digits
A Self-Referential Story

Michael Keith

For a time I
stood pondering on circle
sizes. The large computer
mainframe quietly processed all of its
assembly code. Inside my entire hope lay for
figuring out an elusive expansion. Value: pi.
Decimals expected soon. I nervously entered a format
procedure. The mainframe processed the request. Error.
I, again entering it, carefully retyped. This iteration gave
zero error printouts in all - success. Intently I waited.
Soon, roused by thoughts within me, appeared narrative mnemonics
relating digits to verbiage! The idea appeared to exist but only
in abbreviated fashion - little phrases typically. Pressing on I
then resolved, deciding firmly about a sum of decimals to use - likely
around four hundred, presuming the computer code soon halted!
Pondering these ideas, words appealed to me. But a problem of zeros did
exist. Pondering more, solution subsequently appeared. Zero suggests a
punctuation element. Very novel! My thoughts were culminated. No
periods, I concluded. All residual marks of punctuation = zeros. First
digit expansion answer then came before me. On examining some problems
unhappily arose. That imbecilic bug! The printout I possessed showed four
nine as foremost decimals. Manifestly troubling. Totally every number
looked wrong. Repairing the bug took much effort. A pi mnemonic with
letters truly seemed good. Counting of all the letters probably should
suffice. Reaching for a record would be helpful. Consequently, I
continued, expecting a good final answer from computer. First number
slowly displayed on the the flat screen - 3. Good. Trailing digits
apparently were right also. Now my memory scheme must probably be
implementable. The technique was chosen, elegant in scheme: by self
reference a tale mnemonically helpful was ensured. An able title
suddenly existed - "Circle Digits". Taking pen I began. Words
emanated uneasily. I desired more synonyms. Speedily I found my
(alongside me) Thesaurus. Rogets is probably an essential in
doing this, instantly I decided. I wrote and erased more.
The Rogets clearly assisted immensely. My story
proceeded (how lovely!) faultlessly. The end, above
all, would soon joyfully overtake. So, this memory
helper story is incontestably complete. Soon I
will locate publisher. There a narrative
will I trust immediately appear,
producing fame. THE END.

The previous self-referential story is a mnemonic for the first 402 decimals of the number pi. As it indicates, merely count the number of letters in each word of the story (beginning with the first word, "For", up to and including the final words, "The End") to obtain the successive decimals to pi. Any punctuation mark other than a period represents a zero digit (a period stands for no digit). Words of longer than 9 letters represent two adjacent digits (for example, a twelve-letter word represents the two digits 1 – 2). A digit written literally stands for the same digit in the expansion. This feature is only used once in the story (the first sentence of the seventh paragraph); overuse of this feature would be considered "cheating".

As far as I can determine, this story establishes a new record length for a literary pi mnemonic, although clearly the length of such a mnemonic is limited only by the patience of the constructor. This story has the added twist of self-reference, as it describes within itself its title, its method of construction, its length, and its subject. Incidentally, it has been checked by a computer program for correctness to the decimals of pi.

For those who want to compose even longer mnemonics using the same or similar rules, the following points may be of interest:

1. At decimal 601, the first triple-zero occurs. Clearly we can handle this with the present scheme, but a little ingenuity is required. No quadruple-zeros occur within at least the first 10,000 decimals, so we don't have to concern ourselves with that possibility.
2. At decimal 772 we encounter the amazing sequence 9999998. This seven-digit group has the largest digit sum of any seven-digit group in the first million decimals! Because of the resulting requirement for seven adjacent long words, it also poses quite a challenge in encoding.

We have seen pi-mnemonic sentences, poems, and now, a short story. Perhaps some day a complete novel?

D46 Abbington Drive
Hightstown, N.J. 08520

MATHEMATICS OF COMPUTATION
VOLUME 50, NUMBER 181
JANUARY 1988, PAGES 283-296

The Computation of π to 29,360,000 Decimal Digits Using Borweins' Quartically Convergent Algorithm

By David H. Bailey

Abstract. In a recent work [6], Borwein and Borwein derived a class of algorithms based on the theory of elliptic integrals that yield very rapidly convergent approximations to elementary constants. The author has implemented Borweins' quartically convergent algorithm for $1/\pi$, using a prime modulus transform multi-precision technique, to compute over 29,360,000 digits of the decimal expansion of π. The result was checked by using a different algorithm, also due to the Borweins, that converges quadratically to π. These computations were performed as a system test of the Cray-2 operated by the Numerical Aerodynamical Simulation (NAS) Program at NASA Ames Research Center. The calculations were made possible by the very large memory of the Cray-2.

Until recently, the largest computation of the decimal expansion of π was due to Kanada and Tamura [12] of the University of Tokyo. In 1983 they computed approximately 16 million digits on a Hitachi S-810 computer. Late in 1985 Gosper [9] reported computing 17 million digits using a Symbolics workstation. Since the computation described in this paper was performed, Kanada has reported extending the computation of π to over 134 million digits (January 1987).

This paper describes the algorithms and techniques used in the author's computation, both for converging to π and for performing the required multi-precision arithmetic. The results of statistical analyses of the computed decimal expansion are also included.

1. Introduction. The computation of the numerical value of the constant π has been pursued for centuries for a variety of reasons, both practical and theoretical. Certainly, a value of π correct to 10 decimal places is sufficient for most "practical" applications. Occasionally, there is a need for double-precision or even multi-precision computations involving π and other elementary constants and functions in order to compensate for unusually severe numerical difficulties in an extended computation. However, the author is not aware of even a single case of a "practical" scientific computation that requires the value of π to more than about 100 decimal places.

Beyond immediate practicality, the decimal expansion of π has been of interest to mathematicians, who have still not been able to resolve the question of whether the digits in the expansion of π are "random". In particular, it is widely suspected that the decimal expansions of $\pi, e, \sqrt{2}, \sqrt{2\pi}$, and a host of related mathematical constants all have the property that the limiting frequency of any digit is one tenth, and that the limiting frequency of any n-long string of digits is 10^{-n}. Such a guaranteed property could, for instance, be the basis of a reliable pseudo-random number generator. Unfortunately, this assertion has not been proven in even one instance. Thus, there is a continuing interest in performing statistical analyses on

Received January 27, 1986; revised April 27, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11-04, 65-04.

the decimal expansions of these numbers to see if there is any irregularity that would suggest this assertion is false.

In recent years, the computation of the expansion of π has assumed the role as a standard test of computer integrity. If even one error occurs in the computation, then the result will almost certainly be completely in error after an initial correct section. On the other hand, if the result of the computation of π to even 100,000 decimal places is correct, then the computer has performed billions of operations without error. For this reason, programs that compute the decimal expansion of π are frequently used by both manufacturers and purchasers of new computer equipment to certify system reliability.

2. History. The first serious attempt to calculate an accurate value for the constant π was made by Archimedes, who approximated π by computing the areas of equilateral polygons with increasing numbers of sides. More recently, infinite series have been used. In 1671 Gregory discovered the arctangent series

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

This discovery led to a number of rapidly convergent algorithms. In 1706 Machin used Gregory's series coupled with the identity

$$\pi = 16\tan^{-1}(1/5) - 4\tan^{-1}(1/239)$$

to compute 100 digits of π.

In the nearly 300 years since that time, most computations of the value of π, even those performed by computer, have employed some variation of this technique. For instance, a series based on the identity

$$\pi = 24\tan^{-1}(1/8) + 8\tan^{-1}(1/57) + 4\tan^{-1}(1/239)$$

was used in a computation of π to 100,000 decimal digits using an IBM 7090 in 1961 [15]. Readers interested in the history of the computation π are referred to Beckmann's entertaining book on the subject [2].

3. New Algorithms for π. Only very recently have algorithms been discovered that are fundamentally faster than the above techniques. In 1976 Brent [7] and Salamin [14] independently discovered an approximation algorithm based on elliptic integrals that yields quadratic convergence to π. With all of the previous techniques, the number of correct digits increases only linearly with the number of iterations performed. With this new algorithm, each additional iteration of the algorithm approximately *doubles* the number of correct digits. Kanada and Tamura employed this algorithm in 1983 to compute π to over 16 million decimal digits.

More recently, J. M. Borwein and P. B. Borwein [4] discovered another quadratically convergent algorithm for π, together with similar algorithms for fast computation of all the elementary functions. Their quadratically convergent algorithm for π can be stated as follows: Let $a_0 = \sqrt{2}$, $b_0 = 0$, $p_0 = 2 + \sqrt{2}$. Iterate

$$a_{k+1} = \frac{(\sqrt{a_k} + 1/\sqrt{a_k})}{2}, \quad b_{k+1} = \frac{\sqrt{a_k}(1 + b_k)}{a_k + b_k}, \quad p_{k+1} = \frac{p_k b_{k+1}(1 + a_{k+1})}{1 + b_{k+1}}.$$

Then p_k converges quadratically to π: Successive iterations of this algorithm yield 3, 8, 19, 40, 83, 170, 345, 694, 1392, and 2788 correct digits of the expansion of π.

However, it should be noted that this algorithm is not self-correcting for numerical errors, so that all iterations must be performed to full precision. In other words, in a computation of π to 2788 decimal digits using the above algorithm, each of the ten iterations must be performed with more than 2788 digits of precision.

Most recently, the Borweins [6] have discovered a general technique for obtaining even higher-order convergent algorithms for certain elementary constants. Their quartically convergent algorithm for $1/\pi$ can be stated as follows: Let $a_0 = 6-4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$\begin{aligned} y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}}, \\ a_{k+1} &= a_k(1+y_{k+1})^4 - 2^{2k+3}y_{k+1}(1+y_{k+1}+y_{k+1}^2). \end{aligned}$$

Then a_k converges quartically to $1/\pi$: Each successive iteration approximately *quadruples* the number of correct digits in the result. As in the previous case, each iteration must be performed to at least the level of precision desired for the final result.

4. Multi-Precision Arithmetic Techniques. A key element of a very high precision computation of this sort is a set of high-performance routines for performing multi-precision arithmetic. A naive approach to multi-precision computation would require a prohibitive amount of processing time and would, as a result, sharply increase the probability that an undetected hardware error would occur, rendering the result invalid. In addition to employing advanced algorithms for such key operations as multi-precision multiplication, it is imperative that these algorithms be implemented in a style that is conducive for high-speed computation on the computer being used.

The computer used for these computations is the Cray-2 at the NASA Ames Research Center. This computation was performed to test the integrity of the Cray-2 hardware, as well as the Fortran compiler and the operating system. The Cray-2 is particularly well suited for this computation because of its very large main memory, which holds $2^{28} = 268,435,456$ words (one word is 64 bits of data). With this huge capacity, all data for these computations can be contained entirely within main memory, insuring ease of programming and fast execution.

No attempt was made to employ more than one of the four central processing units in the Cray-2. Thus, at the same time these calculations were being performed, the computer was executing other jobs on the other processors. However, full advantage was taken of the vector operations and vector registers of the system. Considerable care was taken in programming to insure that the multi-precision algorithms were implemented in a style that would admit vector processing. Most key loops were automatically vectorized by the Cray-2 Fortran compiler. For those few that were not automatically vectorized, compiler directives were inserted to force vectorization. As a result of this effort, virtually all arithmetic operations were performed in vector mode, which on the Cray-2 is approximately 20 times faster than scalar mode. Because of the high level of vectorization that was achieved using the Fortran compiler, it was not necessary to use assembly language, nonstandard constructs, or library subroutines.

A multi-precision number is represented in these computations as an $(n+2)$-long array of floating-point whole numbers. The first cell contains the sign of the number, either 1, -1, or 0 (reserved for an exact zero). The second cell of the array contains the exponent (powers of the radix), and the remaining n cells contain the mantissa. The radix selected for the multi-precision numbers is 10^7. Thus the number 1.23456789 is represented by the array $1, 0, 1, 2345678, 9000000, 0, 0, \ldots, 0$.

A floating-point representation was chosen instead of an integer representation because the hardware of numerical supercomputers such as the Cray-2 is designed for floating-point computation. Indeed, the Cray-2 does not even have full-word integer multiply or divide hardware instructions. Such operations are performed by first converting the operands to floating-point form, using the floating-point unit, and converting the results back to fixed-point (integer) form. A decimal radix was chosen instead of a binary value because multiplications and divisions by powers of two are not performed any faster than normal on the Cary-2 (in vector mode). Since a decimal radix is clearly preferable to a binary radix for program troubleshooting and for input and output, a decimal radix was chosen. The value 10^7 was chosen because it is the largest power of ten that will fit in half of the mantissa of a single word. In this way two of these numbers may be multiplied to obtain the exact product using ordinary single-precision arithmetic.

Multi-precision addition and subtraction are not computationally expensive compared to multiplication, division, and square root extraction. Thus, simple algorithms suffice to perform addition and subtraction. The only part of these operations that is not immediately conducive to vector processing is releasing the carries for the final result. This is because the normal "schoolboy" approach of beginning at the last cell and working forward is a recursive operation. On a vector supercomputer this is better done by starting at the beginning and releasing the carry only one cell back for each cell processed. Unfortunately, it cannot be guaranteed that one application of this process will release all carries (consider the case of two or more consecutive 9999999's, followed by a number exceeding 10^7). Thus it is necessary to repeat this operation until all carries have been released (usually one or two additional times). In the rare cases where three applications of this vectorized process are not successful in releasing all carries, the author's program resorts to the scalar "schoolboy" method.

Provided a fast multi-precision multiplication procedure is available, multi-precision division and square root extraction may be performed economically using Newton's iteration, as follows. Let x_0 and y_0 be initial approximations to the reciprocal of a and to the reciprocal of the square root of a, respectively. Then

$$x_{k+1} = x_k(2 - ax_k), \qquad y_{k+1} = \frac{y_k(3 - ay_k^2)}{2}$$

both converge quadratically to the desired values. One additional full-precision multiplication yields the quotient and the square root, respectively. What is especially attractive about these algorithms is that the first iteration may be performed using ordinary single-precision arithmetic, and subsequent iterations may be performed using a level of precision that approximately doubles each time. Thus the total cost of computation is only about twice the cost of the final iteration, plus the one additional multiplication. As a result, a multi-precision division costs only about five times as much as a multi-precision multiplication, and a multi-precision square root costs only about seven times as much as a multi-precision multiplication.

5. Multi-Precision Multiplication. It can be seen from the above that the key component of a high-performance multi-precision arithmetic system is the multiply operation. For modest levels of precision (fewer than about 1000 digits), some variation of the usual "schoolboy" method is sufficient, although care must be taken in the implementation to insure that the operations are vectorizable. Above this level of precision, however, other more sophisticated techniques have a significant advantage. The history of the development of high-performance multiply algorithms will not be reviewed here. The interested reader is referred to Knuth [13]. It will suffice to note that all of the current state-of-the-art techniques derive from the following fact of Fourier analysis: Let $F(x)$ denote the discrete Fourier transform of the sequence $x = (x_0, x_1, x_2, \ldots, x_{N-1})$, and let $F^{-1}(x)$ denote the inverse discrete Fourier transform of x:

$$F_k(x) = \sum_{j=0}^{N-1} x_j \omega^{jk}, \qquad F_k^{-1}(x) = \frac{1}{N} \sum_{j=0}^{N-1} x_j \omega^{-jk},$$

where $\omega = e^{-2\pi i/N}$ is a primitive Nth root of unity. Let $C(x, y)$ denote the convolution of the sequences x and y:

$$C_k(x, y) = \sum_{j=0}^{N-1} x_j y_{k-j},$$

where the subscript $k-j$ is to be interpreted as $k-j+N$ if $k-j$ is negative. Then the "convolution theorem", whose proof is a straightforward exercise, states that

$$F[C(x, y)] = F(x)F(y),$$

or expressed another way,

$$C(x, y) = F^{-1}[F(x)F(y)].$$

This result is applicable to multi-precision multiplication in the following way. Let x and y be n-long representations of two multi-precision numbers (without the sign or exponent words). Extend x and y to length $2n$ by appending n zeros at the end of each. Then the multi-precision product z of x and y, except for releasing the carries, can be written as follows:

$$\begin{aligned}
z_0 &= x_0 y_0 \\
z_1 &= x_0 y_1 + x_1 y_0 \\
z_2 &= x_0 y_2 + x_1 y_1 + x_2 y_0 \\
&\vdots \\
z_{n-1} &= x_0 y_{n-1} + x_1 y_{n-2} + \cdots + x_{n-1} y_0 \\
&\vdots \\
z_{2n-3} &= x_{n-1} y_{n-2} + x_{n-2} y_{n-1} \\
z_{2n-2} &= x_{n-1} y_{n-1} \\
z_{2n-1} &= 0.
\end{aligned}$$

It can now be seen that this "multiplication pyramid" is precisely the convolution of the two sequences x and y, where $N = 2n$. The convolution theorem states that the multiplication pyramid can be obtained by performing two forward discrete Fourier transforms, one vector complex multiplication, and one reverse transform, each of length $N = 2n$. Once the resulting complex numbers have been rounded to the nearest integer, the final multi-precision product may be obtained by merely releasing the carries as described in the section above on addition and subtraction.

The key computational savings here is that the discrete Fourier transform may of course be economically computed using some variation of the "fast Fourier transform" (FFT) algorithm. It is most convenient to employ the radix two fast Fourier transform since there is a wealth of literature on how to efficiently implement this algorithm (see [1], [8], and [16]). Thus, it will be assumed from this point that $N = 2^m$ for some integer m.

One useful "trick" can be employed to further reduce the computational requirement for complex transforms. Note that the input data vectors x and y and the result vector z are purely real. This fact can be exploited by using a simple procedure ([8, p. 169]) for performing real-to-complex and complex-to-real transforms that obtains the result with only about half the work otherwise required.

One important item has been omitted from the above discussion. If the radix 10^7 is used, then the product of two cells will be in the neighborhood of 10^{14}, and the sum of a large number of these products cannot be represented exactly in the 48-bit mantissa of a Cray-2 floating-point word. In this case the rounding operation at the completion of the transform will not be able to recover the exact whole number result. As a result, for the complex transform method to work correctly, it is necessary to alter the above scheme slightly. The simplest solution is to use the radix 10^6 and to divide all input data into two words with only three digits each. Although this scheme greatly increases the memory space required, it does permit the complex transform method to be used for multi-precision computation up to several million digits on the Cray-2.

6. Prime Modulus Transforms. Some variation of the above method has been used in almost all high-performance multi-precision computer programs, including the program used by Kanada and Tamura. However, it appears to break down for very high-precision computation (beyond about ten million digits on the Cray-2), due to the round-off error problem mentioned above. The input data can be further divided into two digits per word or even one digit per word, but only with a substantial increase in run time and main memory. Since a principal goal in this computation was to remain totally within the Cray-2 main memory, a somewhat different method was used.

It can readily be seen that the technique of the previous section, including the usage of a fast Fourier transform algorithm, can be applied in any number field in which there exists a primitive Nth root of unity ω. This requirement holds for the field of the integers modulo p, where p is a prime of the form $p = kN + 1$ ([11, p. 85]). One significant advantage of using a prime modulus field instead of the field of complex numbers is that there is no need to worry about round-off error in the results, since all computations are exact.

However, there are some difficulties in using a prime modulus field for the transform operations above. The first is to find a prime p of the form $kN+1$, where $N=2^m$. The second is to find a primitive Nth root of unity modulo p. As it turns out, it is not too hard using a computer to find both of these numbers by direct search. Thirdly, one must compute the multiplicative inverse of N modulo p. This can be done using a variation of the Euclidean algorithm from elementary number theory. Note that each of these calculations needs to be performed one time only.

A more troublesome difficulty in using a prime modulus transform is the fact that the final multiplication pyramid results are only recovered modulo p. If p is greater than about 10^{24} then this is not a problem, but the usage of such a large prime would require *quadruple*-precision arithmetic operations to be performed in the inner loop of the fast Fourier transform, which would very greatly increase the run time. A simpler and faster approach to the problem is to use two primes, p_1 and p_2, each slightly greater than 10^{12}, and to perform the transform algorithm above using each prime. Then the Chinese remainder theorem may be applied to the results modulo p_1 and p_2 to obtain the results modulo the product p_1p_2. Since p_1p_2 is greater than 10^{24}, these results will be the exact multiplication pyramid numbers. Unfortunately, double-precision arithmetic must still be performed in the fast Fourier transform and in the Chinese remainder theorem calculation. However, the whole-number format of the input data simplifies these operations, and it is possible to program them in a vectorizable fashion.

Borodin and Munro ([3, p. 90]) have suggested using three transforms with three primes p_1, p_2 and p_3, each of which is just smaller than half of the mantissa, and using the Chinese remainder theorem to recover the results modulo $p_1p_2p_3$. In this way, double-precision operations are completely avoided in the inner loop of the FFT. This scheme runs quite fast, but unfortunately the largest transform that can be performed on the Cray-2 using this system is $N=2^{19}$, which corresponds to a maximum precision of about three million digits.

Readers interested in studying about prime modulus number fields, the Euclidean algorithm, or the Chinese remainder theorem are referred to any elementary text on number theory, such as [10] or [11]. Knuth [13] and Borodin [3] also provide excellent information on using these tools for computation.

7. Computational Results. The author has implemented all three of the above techniques for multi-precision multiplication on the Cray-2. By employing special high-performance techniques [1], the complex transform can be made to run the fastest, about four times faster than the two-prime transform method. However, the memory requirement of the two-prime scheme is significantly less than either the three-prime or the complex scheme, and since the two-prime scheme permits very high-precision computation, it was selected for the computations of π.

One of the author's computations used twelve iterations of Borweins' quartic algorithm for $1/\pi$, followed by a reciprocal operation, to yield 29,360,128 digits of π. In this computation, approximately 12 trillion arithmetic operations were performed. The run took 28 hours of processing time on one of the four Cray-2 central processing units and used 138 million words of main memory. It was started on January 7, 1986 and completed January 9, 1986. The program was not running this entire time—the system was taken down for service several times, and the run was frequently interrupted by other programs. Restarting the computation after a system down was a simple matter since the two key multi-precision number arrays were saved on disk after the completion of each iteration.

This computation was checked using 24 iterations of Borweins' quadratically convergent algorithm for π. This run took 40 hours processing time and 147 million words of main memory. A comparison of these output results with the first run found no discrepancies except for the last 24 digits, a normal truncation error. Thus it can be safely assumed that at least 29,360,000 digits of the final result are correct.

It was discovered after both computations were completed that one loop in the Chinese remainder theorem computation was inadvertently performed in scalar mode instead of vector mode. As a result, both of these calculations used about 25% more run time than would otherwise have been required. This error, however, did not affect the validity of the computed decimal expansions.

8. Statistical Analysis of π. Probably the most significant mathematical motivation for the computation of π, both historically and in modern times, has been to investigate the question of the randomness of its decimal expansion. Before Lambert proved in 1766 that π is irrational, there was great interest in checking whether or not its decimal expansion eventually repeats, thus disclosing that π is rational. Since that time there has been a continuing interest in the still unanswered question of whether the expansion is statistically random. It is of course strongly suspected that the decimal expansion of π, if computed to sufficiently high precision, will pass any reasonable statistical test for randomness. The most frequently mentioned conjecture along this line is that any sequence of n digits occurs with a limiting frequency of 10^{-n}.

With 29,360,000 digits, the frequencies of n-long strings may be studied for randomness for n as high as six. Beyond that level the expected number of any one string is too low for statistical tests to be meaningful. The results of tabulated frequencies for one and two digit strings are listed in Tables 1 and 2. In the first table the Z-score numbers are computed as the deviation from the mean divided by the standard deviation, and thus these statistics should be normally distributed with mean zero and variance one.

TABLE 1

Single digit statistics

Digit	Count	Deviation	Z-score
0	2935072	− 928	− 0.5709
1	2936516	516	0.3174
2	2936843	843	0.5186
3	2935205	− 795	− 0.4891
4	2938787	2787	1.7145
5	2936197	197	0.1212
6	2935504	− 496	− 0.3051
7	2934083	− 1917	− 1.1793
8	2935698	− 302	− 0.1858
9	2936095	95	0.0584

TABLE 2

Two digit frequency counts

00	293062	01	293970	02	293533	03	292893	04	294459
05	294189	06	292688	07	292707	08	294260	09	293311
10	294503	11	293409	12	293591	13	294285	14	294020
15	293158	16	293799	17	293020	18	293262	19	293469
20	293952	21	293226	22	293844	23	293382	24	293869
25	293721	26	293655	27	293969	28	293320	29	293905
30	293718	31	293542	32	293272	33	293422	34	293178
35	293490	36	293484	37	292694	38	294152	39	294253
40	294622	41	294793	42	293863	43	293041	44	293519
45	293998	46	294418	47	293616	48	293296	49	293621
50	292736	51	294272	52	293614	53	293215	54	293569
55	294194	56	293260	57	294152	58	293137	59	294048
60	293842	61	293105	62	294187	63	293809	64	293463
65	293544	66	293123	67	293307	68	293602	69	293522
70	292650	71	294304	72	293497	73	293761	74	293960
75	293199	76	293597	77	292745	78	293223	79	293147
80	292517	81	292986	82	293637	83	294475	84	294267
85	293600	86	293786	87	293971	88	293434	89	293025
90	293470	91	292908	92	293806	93	292922	94	294483
95	293104	96	293694	97	293902	98	294012	99	293794

The most appropriate statistical procedure for testing the hypothesis that the empirical frequencies of n-long strings of digits are random is the χ^2 test. The χ^2 statistic of the k observations $X_1, X_2, \ldots, X_k$ is defined as

$$\chi^2 = \sum_{i=1}^{k} \frac{(X_i - E_i)^2}{E_i}$$

where E_i is the expected value of the random variable X_i. In this case $k = 10^n$ and $E_i = 10^{-n}d$ for all i, where $d = 29,360,000$ denotes the number of digits. The mean of the χ^2 statistic in this case is $k-1$ and its standard deviation is $\sqrt{2(k-1)}$. Its distribution is nearly normal for large k. The results of the χ^2 analysis are shown in Table 3.

TABLE 3

Multiple digit χ^2 statistics

Length	χ^2 value	Z-score
1	4.869696	− 0.9735
2	84.52604	− 1.0286
3	983.9108	− 0.3376
4	10147.258	1.0484
5	100257.92	0.5790
6	1000827.7	0.5860

Another test that is frequently performed on long pseudo-random sequences is an analysis to check whether the number of n-long repeats for various n is within statistical bounds of randomness. An n-long repeat is said to occur if the n-long digit sequence beginning at two different positions is the same. The mean M and the variance V of the number of n-long repeats in d digits are (to an excellent approximation)

$$M = \frac{10^{-n}d^2}{2}, \qquad V = \frac{11 \cdot 10^{-n}d^2}{18}.$$

Tabulation of repeats in the expansion of π was performed by packing the string beginning at each position into a single Cray-2 word, sorting the resulting array, and counting equal contiguous entries in the sorted list. The results of this analysis are shown in Table 4.

TABLE 4

Long repeat statistics

10	42945	43100.	− 0.677
11	4385	4310.	1.033
12	447	431.	0.697
13	48	43.1	0.675
14	6	4.31	0.736
15	1	0.43	0.784

A third test frequently performed as a test for randomness is the runs test. This test compares the observed frequency of long runs of a single digit with the number of such occurrences that would be expected at random. The mean and variance of this statistic are the same as the formulas for repeats, except that d^2 is replaced by $2d$. Table 5 lists the observed frequencies of runs for the calculated expansion of π.

The frequencies of long runs are all within acceptable limits of randomness. The only phenomenon of any note in Table 5 is the occurrence of a 9-long run of sevens. However, there is a 29% chance that a 9-long run of some digit would occur in 29,360,000 digits, so this instance by itself is not remarkable.

TABLE 5

Single-digit run counts

	Length of Run				
Digit	5	6	7	8	9
0	308	29	3	0	0
1	281	21	1	0	0
2	272	23	0	0	0
3	266	26	5	0	0
4	296	40	6	1	0
5	292	30	4	0	0
6	316	33	3	0	0
7	315	37	6	2	1
8	295	36	3	0	0
9	306	40	7	0	0

9. Conclusion. The statistical analyses that have been performed on the expansion of π to 29,360,000 decimal places have not disclosed any irregularity. The observed frequencies of n-long strings of digits for n up to 6 are entirely unremarkable. The numbers of long repeating strings and single-digit runs are completely acceptable. Thus, based on these tests, the decimal expansion of π appears to be completely random.

Appendix

Selected Output Listing

```
Initial 1000 digits:

3.
14159265358979323846264338327950288419716939937510
58209749445923078164062862089986280348253421170679
82148086513282306647093844609550582231725359408128
48111745028410270193852110555964462294895493038196
44288109756659334461284756482337867831652712019091
45648566923460348610454326648213393607260249141273
72458700660631558817488152092096282925409171536436
78925903600113305305488204665213841469519415116094
33057270365759591953092186117381932611793105118548
07446237996274956735188575272489122793818301194912
98336733624406566430860213949463952247371907021798
60943702770539217176293176752384674818467669405132
00056812714526356082778577134275778960917363717872
14684409012249534301465495853710507922796892589235
42019956112129021960864034418159813629774771309960
51870721134999999837297804995105973173281609631859
50244594553469083026425223082533446850352619311881
71010003137838752886587533208381420617177669147303
59825349042875546873115956286388235378759375195778
18577805321712268066130019278766111959092164201989

Digits 4,999,001 to 5,000,000:

49480754784558100182731931632488412804488722296956
79855015464855780486736535227902836997918084867230
64962221004527085768335035212069684801817137616329
97561738425160340472537100056351640342162492027179
66824926458930960182645026923102266570541641475347
20341554913770421505764452807809035248393621093031
02288096238486877923145240841637271180953058890040
68843766781431498914299893621278545260143140439048
49938801556336059513116731891132765777881364690708
47036863411196323063886507480852125682842257852524
03086993703255692093960818587414181230484153204049
20234989002732447593020323794790776444752398445514
67304403210968985244961967143433964895893190552338
49818852746844924836314634250006421630628686858848
27453318669926734730642735036364002856022218966350
11429182634319974163253372368798553451111253055262
39104082639970934508146672521381105913047210052428
18988626533169469331951675296209306752291590715999
89846179288059262000848638138811280944056488021060
48865855191846702365421761783505181721320764619715
```

Digits 9,999,001 to 10,000,000:

55097818243516728227849910720400286757907904466335
12718202979525150617725334066894988956424703269230
15399820900390166275224338184424808589395293652582
53635658584175485536744818650289245188206447853280
79129675504865572929083083485483937583334671019089
12067114536955173140929461823466478725289529974204
02127635235923293305770179423865225963240694027480
60412880303092452481034941582735932443887273109397
41634889604695819245395151341043433998381874650972
33692635225791472454244401326312964396391209607800
16344851199125420819737407446045899742145731042313
64456486501937801063526603744056568823861389375443
97351681296831567911618884222251141477322612331396
18606080373110348692660933940438416300326143449280
50821131575737727739821551522286509997662432587213
93393445902091662272905493493827178205126669021149
47192311380933822311224099588372246332501222323378
96895269025366263941267010317327864987170257149617
76105155492579857592045532468944687427025046397905
65326553194060999469787333810631719481735348955897

Digits 14,999,001 to 15,000,000:

75161912582729034437123279749256311511925243956985
41466735069194815163837226073925151887751751659741
00622880726448602209456930414488539882981108512492
30626088375966783621649753412539683084922711342513
94953995693625441331401738133085848172315887473225
66862139251938540102249475575494947158395623512785
67033888824495551084462300472407612165952784386252
83059992302223284865934566262929748436827730812030
14434593689874259766415514412097984133998015934584
35393475650624323850160432731918805126406671871353
77555766214670931813151162879500509710551795152818
09093154481058044767364122166100032425098263166257
41730518220480715488224616563891344046934208103238
39903254029881746342496583186836947486194257533540
36331223838222392494056270856378033056213544686593
02986821714952808585949418676532291067339817684850
77576151785057277099880627370814385794117668763599
75814499149890314594098525960336377989988228138579
03954608500076180754880433958468619641092762653446
79645205263473393286074979323931503141172775669803

Digits 19,999,001 to 20,000,000:

```
24662421652199659486815804456870197576438951607697
86758526528445124126249995515004465281646092893016
37396198596248627116552469686381679679898926165214
19985145392716546108714664257998278750239431446690
24524827883001435830699295155565194378002452231513
03498450165135282534109758167508041457187906821950
98156889669401540575560430489547131781464796920586
99611799897126388736531564345333853581593559913668
62608486227029865668230856391322081859205243349223
41898466479821052634622968628766495150696262416056
24275201300452308788083860012754008114751496913646
62422297630443481605116791864334302662386921297850
27885235888942133721123400642720173755448172632485
38990548569368292370090889371435442648824207842546
28067400727949203553263884395310176843535902614634
76307233029969045465206192626213143248919480318684
24091340888618503237670440877047193079665717842568
49026897445701681738816789861189706430445720674936
81903857815020793466156644931359073005891342758785
95072447895232808191116291055801380049338634527644
```

Digits 24,999,001 to 25,000,000:

```
64626376657788401626872035835150250932381126804132
24527774629670113871130617683224437149346115597163
91099108362268853888484703799982396604187954247350
36635859521304516872709809678948655853409228442863
24948936001342207955968740967092110719683856558205
30816048151902240856062148774123551023529985810792
74189214723685203602121713995138514107079374902532
54350785997288413483911434952219864948321330490074
60146435121254311259573947301142531184570914224080
72612210306331872567179327168155609249989038137333
66960257521334843154895361888436208731274888674781
18373984739313750077149269011462219615798047067514
35050981335283641909759090614464729227662129370246
47057090874450108027231969863517024941726518038367
32762891741863822149208539226376382907305941739639
07549588865849168186491743776278287261919660505923
92475738836587226649359524383297861404378228288281
73596312642574370611956801297356036342637793562761
38037507909491563108238168922672241753290045253446
07864115924597806944245511285225546774836191884322
```

Digits 29,359,001 to 29,360,000:

```
3419284178891522964336847388197769853900574621 9846
```

NAS Systems Division
NASA Ames Research Center
Mail Stop 258-5
Moffett Field, California 94035

1. D. H. BAILEY, "A high-performance fast Fourier transform algorithm for the Cray-2," *J. Supercomputing*, v. 1, 1987, pp. 43–60.

2. P. BECKMANN, *A History of Pi*, Golem Press, Boulder, CO, 1971.

3. A. BORODIN & I. MUNRO, *The Computational Complexity of Algebraic and Numeric Problems*, American Elsevier, New York, 1975.

4. J. M. BORWEIN & P. B. BORWEIN, "The arithmetic-geometric mean and fast computation of elementary functions," *SIAM Rev.*, v. 26, 1984, pp. 351–366.

5. J. M. BORWEIN & P. B. BORWEIN, "More quadratically converging algorithms for π," *Math. Comp.*, v. 46, 1986, pp. 247–253.

6. J. M. BORWEIN & P. B. BORWEIN, *Pi and the AGM—A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, 1987.

7. R. P. BRENT, "Fast multiple-precision evaluation of elementary functions," *J. Assoc. Comput. Mach.*, v. 23, 1976, pp. 242–251.

8. E. O. BRIGHAM, *The Fast Fourier Transform*, Prentice-Hall, Englewood Cliffs, N. J., 1974.

9. W. GOSPER, private communication.

10. EMIL GROSSWALD, *Topics from the Theory of Numbers*, Macmillan, New York, 1966.

11. G. H. HARDY & E. M. WRIGHT, *An Introduction to the Theory of Numbers*, 5th ed., Oxford Univ. Press, London, 1984.

12. Y. KANADA & Y. TAMURA, *Calculation of π to 10,013,395 Decimal Places Based on the Gauss-Legendre Algorithm and Gauss Arctangent Relation*, Computer Centre, University of Tokyo, 1983.

13. D. KNUTH, *The Art of Computer Programming*, Vol. 2: *Seminumerical Algorithms*, Addison-Wesley, Reading, Mass., 1981.

14. E. SALAMIN, "Computation of π using arithmetic-geometric mean," *Math. Comp.*, v. 30, 1976, pp. 565–570.

15. D. SHANKS & J. W. WRENCH, JR., "Calculation of π to 100,000 decimals," *Math. Comp.*, v. 16, 1962, pp. 76–99.

16. P. SWARZTRAUBER, "FFT algorithms for vector computers," *Parallel Comput.*, v. 1, 1984, pp. 45–64.

Vectorization of Multiple-Precision Arithmetic Program and 201,326,000 Decimal Digits of π Calculation

Yasumasa Kanada

Computer Centre, University of Tokyo,
Bunkyo-ku Yayoi 2-11-16, Tokyo 113, Japan

More than 200 million decimal places of π were calculated using arithmetic-geometric mean formula independently discovered by Salamin and Brent in 1976. Correctness of the calculation were verified through Borwein's quartic convergent formula developed in 1983. The computation took CPU times of 5 hours 57 minutes for the main calculation and 7 hours 30 minutes for the verification calculation on the HITAC S-820 model 80 supercomputer with 256 Mb of main memory and 3 Gb of high speed semiconductor storage, Extended Storage, for shorten I/O time.

Computation was completed in 27th of January 1988. At that day two programs generated values up to 3×2^{26}, about 201 million. The two results agreed except for the last 21 digits. These results also agree with the 133,554,000 places calculation of π which was done by the author in January 1987. Compare to the record in 1987, 50% more decimal digits were calculated with about 1/6 of CPU time.

Computation was performed with real arithmetic based vectorized Fast Fourier Transform (FFT) multiplier and newly vectorized multiple-precision add, subtract and (single word) constant multiplication programs. Vectorizations for the later cases were realized through first order linear recurrence vector instruction on the S-820. Details of the computation and statistical tests on the first 200 million digits of π-3 are reported.

1. Introduction

Since the epoch-making calculation of π to 100,000 decimals [17], several computations have been performed as in Table 1. The development of new algorithms and programs suited to the calculation of π and new high speed computers with large memory and high speed large semiconductor disk, Extended Storage or Solid State Disk, threw more light on this fascinating number.

There are many arctangent relations for π[9]. However, all these computations until 1981 and verification for 10,000,000 decimal calculation of our previous record[21] used arctangent formulae such as:

$$\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}, \qquad \text{Machin}$$

$$= 24\arctan\frac{1}{8} + 8\arctan\frac{1}{57} + 4\arctan\frac{1}{239}, \qquad \text{Störmer}$$

$$= 48\arctan\frac{1}{18} + 32\arctan\frac{1}{57} - 20\arctan\frac{1}{239}, \qquad \text{Gauss}$$

$$= 32\arctan\frac{1}{10} - 4\arctan\frac{1}{239} - 16\arctan\frac{1}{515}. \qquad \text{Klingenstierna}$$

In 1976, an innovative quadratic convergent formula for the calculation of π was published independently by Salamin [14] and Brent [5]. Later in 1983, quadratic, cubic, quadruple and septet convergent product expansion for π, which are competitive with Salamin's and Brent's formula, were also discovered by two of Borwein[2]. These new formulae are based on the arithmetic-geometric mean, a process whose rapid convergence doubles, triples, quadruples and septates the number of significant digits at each step. The arithmetic-geometric mean is the basis of Gauss' method for the calculation of elliptic integrals. With the help of the elliptic integral relation of Legendre, π can be expressed in terms of the arithmetic-geometric mean and the resulting algorithm retains quadratic, cubic, quartic and septic convergence of the arithmetic-geometric mean process.

The author and Mr. Y. Tamura have calculated π up to more than 200 million decimal places by using the formula of Brent and Salamin and verified through Borwein's quartic convergent algorithm for π. Even for the quartic convergent algorithm, quadratic convergent algorithm of Brent and Salamin is faster through actual FORTRAN programs.

For reducing the computing time, theoretically fast multiple-precision multiplication algorithm was implemented through normal fast Fourier transform (FFA), inverse fast Fourier transform (FFS) and convolution operations[11] as before[19,9,21,8]. And in order to get more speed than before, vectorization schemes to the multiple-precision add, subtract and (single word) constant multiplication were introduced for the first time.

Calculation of 200 million decimal places of π was completed in January 27, 1988 and needed 5 hours 57 minutes of CPU time on a HITAC S-820 model 80 supercomputer at Hitachi Kanagawa Works under a VOS3/HAP/ES 31 bit addressing operating system. Main memory used was about 240 Mb and 2.7 Gb of Extended Storage was also required. If the machine had more memory, CPU time could be reduced and more Extended Storage, calculation decimals could be extended with minor changes in the FORTRAN programs.

The algorithms used in the 200 million place calculation are briefly explained in section 2. Programming in FORTRAN is discussed in section 3. Results of statistical tests and some interesting figures for the 200,000,000 decimals of π appear in section 4.

2. How to Calculate π: Algorithmic Aspects

In this section we briefly explain the algorithms used in the 200 million decimal place calculations.

2.1. The Gauss-Legendre Algorithm: Main Algorithm

The theoretical basis of the Gauss-Legendre algorithm for π is explained in the references [2,5,14]. Here, we summarize the quadratic algorithm for π. (Refer to the references for the details.)

We first define the arithmetic-geometric mean $agm(a_0,b_0)$. Let a_0, b_0 and c_0 be positive numbers satisfying $a_0^2 = b_0^2 + c_0^2$. Define a_n, the sequence of arithmetic means, and b_n, the sequence of geometric means, by

$$a_n = \frac{(a_{n-1} + b_{n-1})}{2}, \quad b_n = (a_{n-1} \times b_{n-1})^{1/2} .$$

Also, define a positive number sequence c_n:

$$c_n^2 = a_n^2 - b_n^2 . \tag{1}$$

Note that, two relations easily follow from these definitions.

$$c_n = \frac{(a_{n-1} - b_{n-1})}{2} = (a_{n-1} - a_n), \quad c_n^2 = 4 \times a_{n+1} \times c_{n+1} . \tag{2}$$

Then, $agm(a_0,b_0)$ is the common limit of the sequences a_n and b_n, namely

$$agm(a_0,b_0) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n .$$

Now, π can be expressed as follows:

$$\pi = \frac{4agm(1,k)agm(1,k')}{1 - \sum_{j=1}^{\infty} 2^j (c_j^2 + c'^2_j)}, \tag{4}$$

where $a_0 = a'_0 = 1$, $b_0 = k$, $b'_0 = k'$ and $k^2 + k'^2 = 1$. It is easier to compute squares of c_j and c'_j by Eq. (2) than to calculate c_j^2 and c'^2_j by Eq. (1).

The symmetric choice of $k = k' = 2^{-1/2}$ is recommendable for the actual calculation and it causes the two sequences of arithmetic-geometric means to coincide. Then, Eq. (3) becomes

$$\pi = \frac{4(agm(1,2^{-1/2}))^2}{1 - \sum_{j=1}^{\infty} 2^{j+1} c_j^2} . \tag{4}$$

After n square root operations in computing $agm = agm(1,2^{-1/2})$, π can be approximated by π_n:

$$\pi_n = \frac{4a_{n+1}^2}{1 - \sum_{j=1}^{n} 2^{j+1} c_j^2} = \frac{a_{n+1}^2}{0.25 - \sum_{j=1}^{n} 2^{j-1} c_j^2} . \qquad (5)$$

Then, the absolute value of $\pi - \pi_n$ is bounded as follows (Theorem 2b in reference [14]) :

$$|\pi - \pi_n| < (\pi^2 \times 2^{n+4} / agm^2) \exp(-\pi \times 2^{n+1}) . \qquad (6)$$

Thus, the formula has the quadratic convergence nature. We must note here that all operations and constants in Eq. (4) must be correct up to the required number of digits plus α (20 to 30 as the guard digits for 200 million decimal digits calculation).

Then the sequences of agm and agm related π are calculated by the following algorithm;

$A := 1; B := 2^{-1/2}; T := 1/4; X := 1;$

while $A - B > 2^{-n}$ **do begin**
$W := A*B; V := A; A := A+B; A := A/2;$
$V := V-A; V := V*V; V := V*X;$
$T := T-V; B := \sqrt{W}; X := 2*X$ **end;**

$A := A*B; B := 1/T; A := A*B;$
return A .

Here, A, B, T, V and W are full-precision variables and X is a double-precision variable. After twenty eight iterations of the main loop, π to 200 million decimal places is to be obtained.

2.2. Borwein's Quartic Convergent Algorithm: Verification Algorithm

Borwein's quartic convergent algorithm is explained as the following scheme:

$$a_0 = 6 - 4 \times \sqrt{2} , \quad y_0 = \sqrt{2} - 1$$

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 + y_k^4)^{1/4}} , \quad a_{k+1} = a_k \times (1 + y_{k+1})^4 - 2^{2k+3} \times y_{k+1} \times (1 + y_{k+1} + y_{k+1}^2),$$

$$\pi \approx \frac{1}{a_k} \quad \text{for large } k.$$

Here, precisions for a_k and y_k must be more than the desired digits. This algorithm is basically the same with that used for the main run of the 29 million decimal calculation done by Dr. D.H. Bailey[1]. Dr. Bailey used the following Borwein's quadratic convergent algorithm for the verification calculation:

$$a_0 = \sqrt{2} , \quad b_0 = 0 , \quad p_0 = 2 + \sqrt{2}$$

$$a_{k+1} = \frac{(\sqrt{a_k} + \frac{1}{\sqrt{a_k}})}{2} , \quad b_{k+1} = \frac{\sqrt{a_k} \times (1 + b_k)}{a_k + b_k} , \quad p_{k+1} = \frac{p_k \times b_{k+1} \times (1 + a_{k+1})}{1 + b_{k+1}} ,$$

$$\pi \approx p_k \quad \text{for large } k.$$

Here, precisions for a_k, b_k and p_k must be more than the desired digits. It is to be noted that the Gauss-Legendre algorithm is superior to the Borwein's algorithms explained here as in the Table 2.

2.3. Calculation of Reciprocals and Square Roots

As is easily seen from the algorithms explained above, arithmetic operations, reciprocals and square roots must be computed with high efficiency in order to reduce the computing time. Theoretical bases for fast calculation of reciprocals and square roots of multi-precision numbers appear in references [16, 12, 4, 5]. In this subsection we summarize the algorithms used in the actual calculation.

The reciprocal of C is obtained by the Newton iteration for the equation $f(x) = x^{-1} - C \equiv 0$:

$$x_{i+1} = x_i \times (2 - C \times x_i) . \qquad (7)$$

A single-, double- or quadruple-precision approximation of $1/C$ is a reasonable selection for x_0, i.e. initial starting value for the Newton iteration.

Square roots of C should be calculated through the multiplication of C and the result obtained by the Newton iteration for the equation $f(x) = x^{-2} - C \equiv 0$:

$$x_{i+1} = \frac{x_i \times (3 - C \times x_i^2)}{2} . \qquad (8)$$

In this case also, a single-, double- or quadruple-precision approximation of $1/\sqrt{C}$ is a reasonable selection for x_0, initial starting value.

Compare this to the well known Newton iteration for the equation $f(x) = x^2 - C \equiv 0$:

$$x_{i+1} = \frac{(x_i + \frac{C}{x_i})}{2} . \qquad (9)$$

The iteration of Eq. (8) is better in both computing complexity and in actual calculation than the iteration of Eq. (9). The order of convergence for the iterations of Eq. (7) and Eq. (8) is two. This convergence speed is favorable in the actual calculation. It is to be noted that in these iterations the whole operation need not to be done with full-precision at each step. That is, if k-precision calculation is done at step j, $2k$-precision calculation is sufficient for at step $j+1$.

2.4. Multiple-precision Multiplication

Schönhage-Strassen's algorithm [15], which uses the discrete Fourier transform with modulo $2^n + 1$, could be the key multiple-precision multiplication algorithm for speeding up the π calculation. However, this algorithm is so hard to implement and needs binary to decimal radix conversion for the final result. Dr. Bailey also used discrete Fourier transform but with three prime modulo computation followed by the reconstruction through Chinese Remainder Theorem for his 29 million decimal places calculation[1].

Now, we focused the special scheme which utilizes the fact of "*the Fourier transform of a convolution product is the ordinary product of the Fourier transforms.*"

Let consider the product C of two length n with radix X integers A and B. Note that the radix X need not to be a power of 2.

$$A = \sum_{i=0}^{2n-1} a_i X^i , B = \sum_{i=0}^{2n-1} b_i X^i ,$$

where $0 \le a_i < X, 0 \le b_i < X$ for $0 \le i < n-1$, $0 < a_{n-1} < X, 0 < b_{n-1} < X$ and $a_i = b_i = 0$, for $i < 0, n \le i$.

Then,

$$C \equiv A \cdot B = (\sum_{i=0}^{2n-1} a_i X^i) \cdot (\sum_{j=0}^{2n-1} b_j X^j) = \sum_{i=0}^{2n-1} X^i (\sum_{j=0}^{2n-1} a_j b_{i-j}) \equiv \sum_{i=0}^{2n-1} c_i X^i$$

Thus,

$$c_i = \sum_{j=0}^{2n-1} a_j b_{i-j}, \quad \text{for } i = 0, \cdots, 2n-2, \text{ and } c_{2n-1} = 0.$$

If $\omega = exp(2\pi i/2n)$ is a $2n$th root of unity, the one-dimensional Fourier transform of the sequences of complex numbers $(a_0, a_1, \cdots, a_{2n-1})$, $(b_0, b_1, \cdots, b_{2n-1})$ and $(c_0, c_1, \cdots, c_{2n-1})$ are defined as the sequences of $(\hat{a}_0, \hat{a}_1, \cdots, \hat{a}_{2n-1})$, $(\hat{b}_0, \hat{b}_1, \cdots, \hat{b}_{2n-1})$ and $(\hat{c}_0, \hat{c}_1, \cdots, \hat{c}_{2n-1})$, respectively. Here,

$$\hat{a}_l = \sum_{i=0}^{2n-1} a_i \omega^{il}, \hat{b}_l = \sum_{i=0}^{2n-1} b_i \omega^{il}, \hat{c}_l = \sum_{i=0}^{2n-1} c_i \omega^{il}, \quad \text{for } 0 \le l \le 2n-1 .$$

Then, $(\hat{a}_0\hat{b}_0, \hat{a}_1\hat{b}_1, \cdots, \hat{a}_{2n-1}\hat{b}_{2n-1})$ is equal to $(\hat{c}_0, \hat{c}_1, \cdots, \hat{c}_{2n-1})$ as the followings:

$$\hat{a}_k\hat{b}_k = (\sum_{i=0}^{2n-1} a_i \omega^{ki}) (\sum_{j=0}^{2n-1} b_j \omega^{kj}) = \sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1} a_i b_j \omega^{k(i+j)} = \sum_{l=0}^{2n-1} (\sum_{j=0}^{2n-1} a_j b_{l-j}) \omega^{kl} = \sum_{l=0}^{2n-1} c_l \omega^{kl}$$

In these discussions, radix X might be an any number. However, ω must be a complex number, namely *floating point real* arithmetic operations are needed in the process of Fourier transformation. If and only if all the *floating point real* arithmetic operations are performed exactly, this scheme should give the correct result as discussed in page 290-295 of reference[11]. Compared to the discrete Fourier transform based multiplier, *floating point real* arithmetic operations based Fourier transform seems ideal and dangerous, but attractive as for the actual multiple-precision multiplication method.

The reasons are:

1) The speed of double precision floating point operations is faster than integer operations in the available machine, especially for supercomputers. And in general, the number of bits obtainable in one double precision floating point instruction is longer than that obtainable in one single integer instruction.

2) Conversion from binary results to decimal results needs other techniques and coding. (Simple is best. Schönhage-Strassen's discrete FFT algorithm is the binary data multiplier.)

3) A qualified high-speed FFT routine was available as a library. (Qualified programming improves the reliability of the program. If we adopt Schönhage-Strassen's discrete FFT algorithm, we have to code for it.)

4) Chinese remainder based discrete FFT algorithm is not so fast as far as the Dr. Y. Ushiro's experiment in 1983[20] is concerned. (He showed the inferiority of Chinese remainder based discrete FFT to us prior to the Dr. Bailey's experiment[1]. We needed a faster multiple-precision multiplier.)

Followings are the algorithm used in our calculation. Here, let consider the multiplication of two $m \times 2^n$ bit ($= m \times (\log_{10}2) \times 2^n$ decimal digit) integers A and B through our schemes.

Step 1: Prepare two 2×2^n entry double precision floating point array.

Step 2: Convert both of $m \times 2^n$ bit integers into double precision floating point numbers. (The first half of 2×2^n entry contains information for $m \times 2^n$ bit, namely, m bit information per one double precision floating point array entry.)

Step 3: Initialize to double precision floating point zero for the second half of 2×2^n entry.

Step 4: Apply 2^{n+1} point normal Fourier transform, say FFA, operations to A and B giving A' and B', respectively.

Step 5: Do the convolution product operations between A' and B' giving new 2×2^n entry double precision array C'.

Step 6: Apply 2^{n+1} point inverse Fourier transform, say FFS, operations to C' giving C. (Now, C is the double precision floating point array of 2×2^n entry. If operations FFA, FFS and convolution product are performed in infinite precision, each entry of C should be the exact double precision floating point representation for integer with maximum value of $2^n \times (2^m-1)^2$. However, these representation are slightly deviated from exact integer in the actual operation. Because, infinite precision operations are impossible to perform.)

Step 7: Convert each entry of C (let it to be x) into integer representation. (Conversion should be done with IDNINT operation in FORTRAN. (IDNINT(x) = IDINT(x+0.5D0).) If absolute value of ($x-DFLOAT(IDNINT(x))$) is near to 0.5D0, the multiplication is considered to be incorrect. We don't know that criterion of 0.45D0 is sufficient or not. However criterion of 0.2D0, for example, was sufficient enough in the actual multiplication.)

Step 8: Normalize C under the suitable base. The base of 2^m or $10^{m(\log_{10}2)}$ is better for binary or decimal representation. Final result is the result of multiplication between A and B.

According to the theoretical and experimental analysis of error in FFT [6, 10], FFT attains rather stable error behavior. Theoretically roughly speaking, 2^l point FFT attains $l\ log(l)$ bits error at the maximum (worst case). This means that FFA and the convolution product followed by FFS would acquire $2\ l\ log(l)$ bits error at the maximum, even if the trigonometric functions at coefficients for the butterfly operation are calculated exactly.

Available mantissa bits are 56 for double-precision floating point data of the Japanese supercomputers. (CRAY and ETA machines adopt 48 bit mantissa representation for the floating point numbers.) Then, the following equation must be satisfied for the worst case;

bits available in the calculation $\geq$ (*guard bits* for *FFA* and *FFS operations*) +

(*necessary bits* for *preserving the results*)

$$56 \geq 2\,(n+1)\,log\,(n+1) + log\,(2^n \times (2^m-1)^2)\ .$$

Here n is an integer. Thus, even for $m = 1$,

$$n \leq 7\ .$$

The analysis suggests a maximum n of 7. However, the maximum n depends on the method of programming, errors in trigonometric function calculations, value of m, *etc.* (There is another error analysis for the *floating point real* FFT multiplier in reference[11] which specifies the radix X in our explanation to be the power of 2.)

The error analysis for the actual calculation is hard. Then, we have checked the availability of the FFT multiplier based on the above schemes through actual programs. We monitored the deviations from integer values after the stage of conversion to integer representation as explained in Step 7 of the above explanation. The monitoring secures a maximum n of 24, for the condition of one data point, a double word, holds 3 decimals at the maximum (m is about 10) on the first half of 2×2^{24} data points and zero on the second half of 2×2^{24} data points. Then we decided that the maximum length of multiplicand to be multiplied in-core is 3×2^{24} decimal places (about 50 million decimals) in the actual multiplication.

For multiplying 3×2^{26} decimal places numbers (multiple-precision data for 200 million decimal places), we used the classical $O(n^2)$ algorithm, e.g. school boy method, for the data of $2^{26}/2^{24} = 2^2 = 4$ units with base of $2^{24} \times 1{,}000$! It is possible to realize multiple-precision multiplier all through FFT, not through school boy method. In order to do so, we must write extra program for out-of-core version of FFT. We preferred to utilize the reliability of the numerical libraries.

2.5. Vectorization of Multiple-Precision Add, Subtract and (Single Word) Constant Multiplication

For the programs of multiple-precision add, subtract and (single word) constant multiplier, time consuming process is releasing the "borrow or carry."

If the machine has special instruction for vectorizing first order linear recurrence relation of the following, the process of releasing the borrow or carry could be vectorized. (Now, all of the Japanese supercomputers are equipped with such instructions.)

```
      DO 10 I = ...
10    A(I) = B(I) * A(I-1) + C(I)
```

Here, vector A has a nature of first order linear recurrence relation. The following program which were extracted form the source codes of the actual run will explain how to rewrite the program for multiple-precision adder. Same strategies can be applied to the multiple-precision routines for subtract and (single word) constant multiplication.

```
C integer*4 -- MA, MB, ICY(=carry), ICW(=work), IONE8(=base=10**6)
C real*8 -- Y(=work), Z(=carry), CY(=work), CW(=work)
C real*8 -- ONED8=DFLOAT(IONE8), ONEDM8=1.D0/ONED8, HLFDM8=.5D0*ONEDM8

      DO 10 J=NDA,1,-1                         DO 10 J=NDA,2,-1
        ICW=MA(J)+MB(J)+ICY                      Y(J)=MA(J)+MB(J)
        IF( ICW.GE.IONE8 ) THEN                  Z(J-1)=Y(J)*ONEDM8+Z(J)*ONEDM8
          MA(J)=ICW-IONE8                        Z(J)=DINT(Z(J)+HLFDM8)
          ICY=1                          10    CONTINUE
        ELSE                        ==>          Z(1)=DINT(Z(J)+HLFDM8)
          MA(J)=ICW                            DO 11 J=NDA,2,-1
          ICY=0                                  MA(J)=Y(J)+Z(J)-Z(J-1)*ONED8
        END IF                           11    CONTINUE
10    CONTINUE                                   CW=DFLOAT(MA(1)+MB(1))+Z(1)
                                                 CY=DINT(CW*ONEDM8+HLFDM8)
                                                 MA(1)=CW-CY*ONED8
```

3. How to Calculate π: Programming Aspects

The actual programs, written in FORTRAN, consisted of 3426 (main) and 3642 (verification) lines of source code with comment lines. The numbers of program units are 59 (main) and 64 (verification). In this section we briefly overview the actual programming.

About the half of the sub-programs are routines related to multiple-precision multiplication. Others are routines for addition, subtraction, reciprocal operations, square root operations, file I/O operations, *etc.*

The HITAC S-820 model 80 is a so called 2nd generation supercomputer in Japan. The machine is single processor model and has theoretical peak performance of 3Gflops. Maximum attachable main memory size is 512 Mb and Extended Storage size is 12 Gb. FFT routines, FFA and FFS, are carefully selected from the non-vectorized FORTRAN numerical library which was provided by the Hitachi and source codes were slightly modified by myself for the vectorization.

3.1. Layout of Storage

As explained in section 2, we used a *floating point real* FFT whose accuracy is secured under the conditions of

(1) 2^{24+1} point double-precision *floating point real* FFT,

(2) maximum number at each entry point for the first half entry is non negative number and must be bounded by 1,000,

(3) the second half entry contains zero.

These conditions allow in-core multiplication of numbers with $2^{24} \times 3$ (= 50,331,648) decimals. If we want more decimals to be calculated in-core (on main memory), we must reduce the above conditions, e.g.

(1) 2^{24+1} points -> 2^{29+1} points,

(2) maximum number at each entry point of 1,000 -> 100.

These new conditions would allow numbers of up to $2^{29} \times 2$ (= 1,073,741,824) decimals to be manipulated in-core. (We did not check the validity of these conditions through actual run. Our examples are only for explanation!) However, currently available maximum memory size prevent such higher precision calculations in-core. In order to run with such ideal conditions, at least 8 GB of main memory should be available. Now, the cases with 2^{17+1} points or 2^{18+1} points and maximum number at each entry point of 10,000 were applied for the previous our π calculations including 133 million decimals record. The increase of in-core operable FFT points is the major factor for the speed-up to the π calculation.

It was impossible to obtain 200 million decimal places through in-core operations because of available main memory size. Therefore we introduced the user controlled virtual memory scheme for saving high precision constants of $1/\sqrt{2}$, $\sqrt{2}$, π, and several working storage with compression factor of 6 decimals / 4 bytes. (Integer representation was used for saving storages on the extended storage. For FFT, a 3 decimals / 8 bytes scheme - double precision floating point representation - was needed as explained.) These schemes needed about 240 Mb of main memory for the working storage, input and output (I/O) buffer, object codes, *etc.* As for the extended storage size, 13.5 Mb / 1 million decimal places was needed.

3.2. Optimization for Speedup

We have employed several optimization schemes:

1) Multiplication by 1 -> normal copy operation.
2) Multiplication by 2 -> normal addition operation.
3) Deletion of unnecessary FFA and FFS operations.
4) Reuse of internal iteration results.

For 1) and 2), explanation is simple enough. In the following two subsections, we explain the details of 3) and 4).

3.2.1. How to Delete FFA and FFS Operations

As explained in section 2.4, the school boy method was employed for multiple-precision multiplication (for data length longer than in-core operable length). Now as for an example of 4×4 school boy method, let consider the simple case of 2×2. That corresponds to the multiplication of two integers, both with a length of 3×2^{25} and in-core operable length of 3×2^{24}. Without optimization, each $2^{24} \times 1{,}000$ decimal based multiplication needs eight FFA for input and four FFS for output as follows;

$$(A_1 \times BASE + A_2) \times (B_1 \times BASE + B_2)$$

$$= (FFS\ of\ (FFA\ of\ A_1 \cdot FFA\ of\ B_1)) \times BASE^2$$

$$+((FFS\ of\ (FFA\ of\ A_1 \cdot FFA\ of\ B_2)) + (FFS\ of\ (FFA\ of\ A_2 \cdot FFA\ of\ B_1))) \times BASE$$

$+(FFS\ of\ (FFA\ of\ A_2 \cdot FFA\ of\ B_2))$,

where $\cdot$ is the convolution product for the Fourier transformed data, + is the vector-wise addition and BASE is $2^{24} \times 1{,}000$.

There are very many FFA and FFS operations. These operations can be reduced a lot by the following schemes:

1) First apply FFA operations to A_1, A_2, B_1 and B_2. Let the results be A_1', A_2', B_1' and B_2', respectively.
2) Now $(A_1' \cdot B_2' + A_2' \cdot B_1')$ preserve information in the sense explained in section 2.4.
3) Then, the linearity of FFS operation satisfies the following equation:

$$(A_1 \times BASE + A_2) \times (B_1 \times BASE + B_2)$$
$$= (FFS\ of\ (A_1' \cdot B_1')) \times BASE^2 + (FFS\ of\ (A_1' \cdot B_2' + A_2' \cdot B_1')) \times BASE$$
$$+ (FFS\ of\ (A_2' \cdot B_2')) .$$

According to the results of CPU time profile analysis for 33 million decimal places calculation, multiplication occupies 90% of CPU time. (Here, CPU time does not contain the time for input and output operations.) Thus, this optimization is rather efficient when the length of multiplicand becomes long. (In this case, FFA operations was reduced from 8 to 4 and FFS operations was reduced from 4 to 3.)

3.2.2. Reuse of Internal Iteration Results

Newton's iteration for square roots and reciprocals has second order convergence nature. This implies that internal iteration results at the calculation of the half-length precision of π can be the initial value for the iteration at the calculation of the full-length precision of π. (And this selection is the best selection for reducing CPU time.) According to measurements of CPU time, this scheme reduced the CPU time 10-20% for the 16 or 33 million decimal places calculation.

If we utilize this fact, we can reduce the computation time probably by 10-20%. To do so, however, we had to prepare the permanent storage of about 1.5 Gb. This was completely impossible at the time of actual run. For the history of π calculation, we had saved the internal iteration results from the calculation of 16 million decimal places (data size is around 100 Mb). That data helped for the calculation of 32 million decimal places of π substantially. And also for the calculation of 16 million decimal places of π, we had utilized the internal iteration results from the calculation of 8 million decimal places.

4. Statistical Analysis of 200,000,000 Decimals of π

In order to analyze the statistics of 200,000,000 decimals of π, we used the same statistical tests as Pathria[13], except that the number of digits expanded from 100,000 to 200,000,000. In this section we briefly show the statistical data and some interesting figures from the 200,000,000 decimals of π. Now, the 200,000,000-th decimal number of $\pi - 3$ is 9.

4.1. Results of Statistical tests

Five kinds of statistical tests were performed. These are the frequency test, the serial tests, the Poker hand test, the gap test and the five-digit sum test. We have reproduced only the results of frequency test in the Table 3. The other results are to be published through the reference[7].

4.2. Some Interesting Figures

Analysis of digit sequences for 200,000,000 decimal places of $\pi - 3$ gives some interesting figures;

1) A longest descending sequence of 2109876543 appears (from 26,160,634) only once. The next longest descending sequence is 876543210 (from 2,747,956) only. The next longest descending sequence of length 8 appears 9 times.
2) The longest ascending sequence is 901234567 which appears from 197,090,144. The next longest ascending sequence is 23456789 (from 995,998), 89012345 (from 33,064,267, 39,202,678, 62,632,993 and 78,340,559), 90123456 (from 35,105,378, 44,994,887, 98,647,533 and 127,883,114), 56789012 (from 100,800,812 and 139,825,562), 67890123 (from 102,197,548, 135,721,079 and 178,278,161), 01234567 (from 112,099,767), 78901234 (from

119,172,322, 122,016,838, 182,288,028 and 195,692,744), 12345678 (from 186,557,266) and 45678901 (from 194,981,709). The next longest ascending sequence of length 7 appears 170 times.

3) A sequence of maximum multiplicity (of 9) appears 3 times. These are 7 (from 24,658,601), 6 (from 45,681,781) and 8 (from 46,663,520). The next longest sequence of multiplicity (of 8) appears 16 times.

4) The longest sequence of 27182818 appears from 73,154,827, 143,361,474 and 183,026,622. The next longest sequence of 2718281 appears 22 times.

5) The longest sequence of 14142135 appears from 52,638, 10,505,872 and 143,965,527. The next longest sequence of 1414213 appears from 13,816,189, 40,122,589, 72,670,122, 87,067,359, 104,717,213, 115,301,872, 145,035,762, 147,685,125, 155,299,021, 165,871,476, 166,005,277, 166,491,213 and 191,208,533. The next longest sequence of 141421 appears 169 times.

6) The longest sequence of 31415926 appears 2 times. These are from 50,366,472 and 157,060,182. The next longest sequence of 3141592 appears 7 times.

5. Conclusion

Details of 201,326,000 decimal digits of π calculation and some results of statistical tests on 200,000,000 decimals of π are presented. The original program is written in FORTRAN 77 and heavily utilizes floating point operations. Multiple-precision add, subtract and constant multiplication programs were also vectorized through linear recurrence special instruction. Programs do not depend on the scheme of round-off and cut-off to the results for the floating point operations. Then, not only round-off machine, e.g. supercomputers of CRAY Inc. and ETA systems, but also cut-off machine, e.g. Japanese supercomputers, can generate correct π digits. The program calls few system specific subroutines, e.g. CLOCK, TIME, but adaptation to the new machine is easy. In order to get speed, available main memory size is crucial and shorten elapsed time, availability of high speed I/O devices is also crucial. Thus, the π calculation program based on the scheme explained here can be a good benchmark program for the supercomputer.

We have programmed the fast multiple-precision multiplication through a *floating point real* FFT package, which was available as one of the program libraries. This means that half a hundred lines of code is sufficient for the fast multiple-precision multiplication. (A few lines of array declaration, one line for calling the FFA routine, a few lines of convolution products, one line for calling the FFS routine and a few lines of normalization under a suitable base.) These schemes would be of benefit to other high-precision constant and function calculations. And another scheme, high utilization of floating point operations in the processing of integers, is also favorable for the integer calculation, especially to number crunchers.

As the Table 1 shows, it took 12 years for extending the length of known π value from 100,000 to 1,000,000, 10 years from 1,000,000 to 10,000,000 and 4 years from 10,000,000 to the order of 100,000,000. When can we unveil the digits after 1,000,000,000, how and by whom?

6. Acknowledgments

The author would like to express my thanks to Mr. Yoshiaki Tamura for his collaboration with me and to Dr. Kazunori Miyoshi for utilizing his statistical analysis program for π. Thanks are also due to Mr. Akihiko Shibata for his information concerning the history of π calculation. I also thank to Hitachi Co. for giving me a chance to extend the length of π value.

Calculated by	Machine used	Date	Precision (calculated)	declared	correct	Time (check)	Formula (check)
Reitwiesner et al.	ENIAC	1949	(2040)		2037	~70h (~70h)	M(M)
Nicholson & Jeenel	NORC	1954	(3093)		3092	13m (13m)	M(M)
Felton	Pegasus	1957	(10021)		7480	33h (33h)	K(G)
Genuys	IBM 704	1958	(10000)		10000	1h 40m (1h 40m)	M(M)
Felton	Pegasus	"	(10021)		10020	33h (33h)	K(G)
Guilloud	IBM 704	1959	(16167)		16167	4.3h (4.3h)	M(M)
Shanks & Wrench	IBM 7090	1961	(100265)		100265	8h 43m (4h 22m)	S(G)
Guilloud & Filliatre	IBM 7030	1966	(250000)		250000	41h 55m (24h 35m)	G(S)
Guilloud & Dichampt	CDC 6600	1967	(500000)		500000	28h 10m (16h 35m)	G(S)
Guilloud & Bouyer	CDC 7600	1973	(1001250)		1001250	23h 18m (13h 40m)	G(S)
Miyoshi & Kanada	FACOM M-200	1981	(2000040)	2000000	2000036	137.3h (143.3h)	K(M)
Guilloud	x	1981-82	(x)	2000050	2000050	x (x)	x
Tamura	MELCOM 900II	1982	(2097152)		2097144	7h 14m (2h 21m)	L(L)
Tamura & Kanada	HITAC M-280H	"	(4194304)		4194288	2h 21m (6h 52m)	L(L)
Tamura & Kanada	"	"	(8388608)		8388576	6h 52m (< 30h)	L(L)
Kanada, Yoshino & Tamura	"	1983	(16777216)		16777206	< 30h (6h 36m)	L(L)
Ushiro & Kanada	HITAC S-810/20	1983 Oct.	(10013400)	10000000	10013395	< 24h (< 30h)	G(L)
Gosper	Symbolics 3670	1985 Oct.	(>=17526200)		17526200	x (28h)	R(B4)
Bailey	CRAY-2	1986 Jan.	(29360128)	29360000	29360111	28h (40h)	B4(B2)
Kanada & Tamura	HITAC S-810/20	1986 Sep.	(33554432)	33554000	33554414	6h 36m (23h)	L(L)
Kanada & Tamura	HITAC S-810/20	1986 Oct.	(67108864)		67108839	23h (35h 15m)	L(L)
Kanada, Tamura, Kubo etc.	NEC SX-2	1987 Jan.	(134214728)	133554000	134214700	35h 15m (48h 2m)	L(B4)
Kanada & Tamura	HITAC S-820/80	1988 Jan.	(201326572)	201326000	201326551	5h 57m (7h 30m)	L(B4)

Table 1. Historical records of the calculation of π performed on electronic computers. M, K, G, S, L, R, B4 and B2 are the formulae of Machin, Klingenstierna, Gauss, Störmer, and Gauss-Legendre, formula explained in the reference[3], Borwein's quartic convergent, Borwein's quadratic convergent, respectively. Symbol 'x' means 'unknown'. Check time means the additional time for the calculated value checking. This information was basically obtained from Mr. Shibata[18].

Type of operation	Brent et. al.	Borwein's quartic	Borwein's quadratic
$X^{1/2}$ *or* $X^{1/4}$	n	n	n
×	2n+2	5n	6n
reciprocal	1	n+1	2n
+,−	3n+1	6n	3n

Table 2. Comparison of basic, time consuming, operations in the three historically important algorithms for π calculation. Here n is a number of iterations. In order to compare with fare, both side columns numbers should be doubled.

Digit	0	1	2	3	4	5	6	7	8	9	χ^2
100	8	8	12	11	10	8	9	8	12	14	4.20
200	19	20	24	19	22	20	16	12	25	23	6.80
500	45	59	54	50	53	50	48	36	53	52	6.88
1k	93	116	103	102	93	97	94	95	101	106	4.74
2k	182	212	207	188	195	205	200	197	202	212	4.34
5k	466	532	496	459	508	525	513	488	492	521	10.77
10K	968	1026	1021	974	1012	1046	1021	970	948	1014	9.32
20K	1954	1997	1986	1986	2043	2082	2017	1953	1962	2020	7.72
50K	5033	5055	4867	4947	5011	5052	5018	4977	5030	5010	5.86
100K	9999	10137	9908	10025	9971	10026	10029	10025	9978	9902	4.09
200K	20104	20063	19892	20010	19874	20199	19898	20163	19956	19841	7.31
500K	49915	49984	49753	50000	50357	50235	49824	50230	49911	49791	7.73
1M	99959	99758	100026	100229	100230	100359	99548	99800	99985	100106	5.51
2M	199792	199535	200077	200141	200083	200521	199403	200310	199447	200691	9.00
4M	399419	399463	399822	399913	400792	400032	399032	400650	400183	400694	7.92
5M	499620	499898	499508	499933	500544	500025	498758	500880	499880	500954	7.88
8M	799111	800110	799788	800234	800202	800154	798885	800560	800638	800318	3.79
10M	999440	999333	1000306	999964	1001093	1000466	999337	1000207	999814	1000040	2.78
15M	1500081	1499675	1501044	1499917	1501166	1500417	1498447	1499584	1500435	1499234	4.07
20M	2001162	1999832	2001409	1999343	2001106	2000125	1999269	1998404	1999720	1999630	4.17
25M	2500496	2499915	2500707	2499313	2502826	2500139	2499603	2498290	2499189	2499522	5.28
30M	2999157	3000554	3000969	2999222	3002593	2999997	2999548	2998175	2999592	3000193	4.34
50M	4999632	5002220	5000573	4998630	5004009	4999797	4998017	4998895	4998494	4999733	6.17
80M	7998807	8002788	8001828	7997656	8003525	7996500	7998165	7999389	8000308	8001034	5.95
100M	9999922	10002475	10001092	9998442	10003863	9993478	9999417	9999610	10002180	9999521	7.27
150M	14998689	15001880	15001586	14999130	15003829	14993562	14998434	14999462	15001416	15002012	4.90
200M	19997437	20003774	20002185	20001410	19999846	19993031	19999161	20000287	20002307	20000562	4.13

Table 3. Summary of frequency for the first 200,000,000 digits of π - 3 and corresponding χ^2 values.

References

1. Bailey, D.H., "The Computation of πi to 29,360,000 Decimal Digits Using Borwein's Quartically Convergent Algorithm," *Math. Comp.*, vol. 50, no. 181, pp. 66-88, Jan. 1988.
2. Borwein, J.M. and Borwein, P.B., *Pi and the AGM - A study in Analytic Number Theory and Computational Complexity*, A Wiley-Interscience Publication, New York, 1987.
3. Borwein, J.M. and Borwein, P.B., "Ramanujan and Pi," *Scientific American*, vol. 258, pp. 66-73, Feb. 1988.
4. Brent, R.P., "Fast Multiple-Precision Evaluation of Elementary Functions," *J. ACM.*, vol. 23, no. 2, pp. 242-251, April 1976.
5. Brent, R.P., "Multiple-Precision Zero-Finding Methods and the Complexity of Elementary Function Evaluation," in *Analytic Computational Complexity*, ed. Traub, J.F., pp. 151-176, Academic Press, New York, 1976.
6. Gentleman, W.M. and Sande, G., "Fast Fourier Transforms - For Fun and Profit," *AFIPS Conf. Proc. FJCC*, vol. 29, pp. 563-578, Spartan Books, San Francisco, Calif., November 1966.
7. Kanada, Y., "Statistical Analysis of π up to 200,000,000 Decimal Places," CCUT-TR-88-02, in preparation, Computer Centre, University of Tokyo, Bunkyo-ku, Yayoi 2-11-16, Tokyo 113, Japan, Nov. 1988.
8. Tamura, Y., Yoshino, S., Ushiro, Y. and Kanada, Y., "Circular Constant - It's High Speed Calculation Method and Statistics -," in *Proceedings Programming Symposium, in Japanese*, Information Processing Society of Japan, Jan. 1984.
9. Tamura, Y. and Kanada, Y., "Calculation of π to 4,194,293 Decimals Based on the Gauss-Legendre Algorithm," CCUT-TR-83-01, Computer Centre, University of Tokyo, Bunkyo-ku, Yayoi 2-11-16, Tokyo 113, Japan, May 1983.
10. Kaneko, T. and Liu, B., "Accumulation of Round-Off Error in Fast Fourier Transforms," *J. ACM.*, vol. 17, no. 4, pp. 637-654, Oct. 1970.
11. Knuth, D.E., *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms*, Addison-Wesley, Reading, Mass., 1981.
12. Laasonen, P., "On the Iterative Solution of the Matrix Equation A X^2 - I = 0," *Math. Tables and Other Aids to Comp.*, vol. 12, pp. 109-116, 1958.
13. Pathria, R.K., "A Statistical Study of Randomness Among the First 10,000 Digits of π," *Math. Comp.*, vol. 16, pp. 188-197, 1962.
14. Salamin, E., "Computation of π Using Arithmetic-Geometric Mean," *Math. Comp.*, vol. 30, no. 135, pp. 565-570, July 1976.
15. Schönhage, A. and Strassen, V., "Schnelle Multiplikation Grosser Zahlen," *Computing*, vol. 7, pp. 281-292, 1971.
16. Schutz, G., "Iterative Berechnung der Reziproken Matrix," *Z. Angew. Math. Mech.*, vol. 13, pp. 57-59, 1933.
17. Shanks, D. and Wrench, Jr. W., "Calculation of π to 100,000 Decimals," *Math. Comp.*, vol. 16, pp. 76-79, 1962.
18. Shibata, A., 1982-1983. Private communications
19. Tamura, Y., "Calculation of π to 2 Million Decimals Using Legendre-Gauss' Relation," Proceeding of the International Latitude Observatory of Mizusawa No.22 (in Japanese) in press, Internal Latitude Observatory of Mizusawa, Hoshigaoka-cho 2-12, Mizusawa City, 023 Iwate Prefecture, Japan, 1983.
20. Ushiro, Y., "Calculation of Multiple-Precision Numbers through Vectorizing Processor," in *Numerical Calculation Workshop, in Japanese*, Information Processing Society of Japan, 26-28 May. 1983.
21. Kanada, Y., Tamura, Y., Yoshino, S. and Ushiro, Y., "Calculation of π to 10,013,395 Decimal Places Based on the Gauss-Legendre Algorithm and Gauss Arctangent Relation," CCUT-TR-84-01, Computer Centre, University of Tokyo, Bunkyo-ku, Yayoi 2-11-16, Tokyo 113, Japan, Dec 1983.

Ramanujan and Pi

Some 75 years ago an Indian mathematical genius developed ways of calculating pi with extraordinary efficiency. His approach is now incorporated in computer algorithms yielding millions of digits of pi

by Jonathan M. Borwein and Peter B. Borwein

Pi, the ratio of any circle's circumference to its diameter, was computed in 1987 to an unprecedented level of accuracy: more than 100 million decimal places. Last year also marked the centenary of the birth of Srinivasa Ramanujan, an enigmatic Indian mathematical genius who spent much of his short life in isolation and poor health. The two events are in fact closely linked, because the basic approach underlying the most recent computations of pi was anticipated by Ramanujan, although its implementation had to await the formulation of efficient algorithms (by various workers including us), modern supercomputers and new ways to multiply numbers.

Aside from providing an arena in which to set records of a kind, the quest to calculate the number to millions of decimal places may seem rather pointless. Thirty-nine places of pi suffice for computing the circumference of a circle girdling the known universe with an error no greater than the radius of a hydrogen atom. It is hard to imagine physical situations requiring more digits. Why are mathematicians and computer scientists not satisfied with, say, the first 50 digits of pi?

Several answers can be given. One is that the calculation of pi has become something of a benchmark computation: it serves as a measure of the sophistication and reliability of the computers that carry it out. In addition, the pursuit of ever more accurate values of pi leads mathematicians to intriguing and unexpected niches of number theory. Another and more ingenuous motivation is simply "because it's there." In fact, pi has been a fixture of mathematical culture for more than two and a half millenniums.

Furthermore, there is always the chance that such computations will shed light on some of the riddles surrounding pi, a universal constant that is not particularly well understood, in spite of its relatively elementary nature. For example, although it has been proved that pi cannot ever be exactly evaluated by subjecting positive integers to any combination of adding, subtracting, multiplying, dividing or extracting roots, no one has succeeded in proving that the digits of pi follow a random distribution (such that each number from 0 to 9 appears with equal frequency). It is possible, albeit highly unlikely, that after a while all the remaining digits of pi are 0's and 1's or exhibit some other regularity. Moreover, pi turns up in all kinds of unexpected places that have nothing to do with circles. If a number is picked at random from the set of integers, for instance, the probability that it will have no repeated prime divisors is six divided by the square of pi. No different from other eminent mathematicians, Ramanujan was prey to the fascinations of the number.

The ingredients of the recent approaches to calculating pi are among the mathematical treasures unearthed by renewed interest in Ramanujan's work. Much of what he did, however, is still inaccessible to investigators. The body of his work is contained in his "Notebooks," which are personal records written in his own nomenclature. To make matters more frustrating for mathematicians who have studied the "Notebooks," Ramanujan generally did not include formal proofs for his theorems. The task of deciphering and editing the "Notebooks" is only now nearing completion, by Bruce C. Berndt of the University of Illinois at Urbana-Champaign.

To our knowledge no mathematical redaction of this scope or difficulty has ever been attempted. The effort is certainly worthwhile. Ramanujan's legacy in the "Notebooks" promises not only to enrich pure mathematics but also to find application in various fields of mathematical physics. Rodney J. Baxter of the Australian National University, for example, acknowledges that Ramanujan's findings helped him to solve such problems in statistical mechanics as the so-called hard-hexagon model, which considers the behavior of a system of interacting particles laid out on a honeycomblike grid. Similarly, Carlos J. Moreno of the City University of New York and Freeman J. Dyson of the Institute for Advanced Study have pointed out that Ramanujan's work is beginning to be applied by physicists in superstring theory.

Ramanujan's stature as a mathematician is all the more astonishing when one considers his limited formal education. He was born on December 22, 1887, into a somewhat impoverished family of the Brahmin caste in the town of Erode in southern India and grew up in Kumbakonam, where his father was an accountant to a clothier. His mathematical precocity was recognized early, and at the age of seven he was given a scholarship to the Kumbakonam Town High School. He is said to have recited mathematical formulas to his schoolmates—including the value of pi to many places.

When he was 12, Ramanujan mastered the contents of S. L. Loney's rather comprehensive *Plane Trigonometry*, including its discussion of the sum and products of infinite sequences, which later were to figure prominently in his work. (An infinite sequence is an unending string of terms, often generated by a simple formula. In this context the interesting sequences are those whose terms can be added or multiplied to yield an identifiable, finite value. If the terms are added, the resulting expression is called a series; if they are multiplied, it is called a product.) Three years later he borrowed the *Synopsis of Elementary Results in Pure Mathematics*, a listing of some 6,000 theorems (most of them given without proof) compiled by G. S. Carr, a tutor at the University of Cambridge. Those two books were the basis of Ramanujan's mathematical training.

In 1903 Ramanujan was admitted to a local government college. Yet total absorption in his own mathematical diversions at the expense of everything else caused him to fail his examinations, a pattern repeated four years later at another college in Madras. Ramanujan did set his avocation aside—if only temporarily—to look for a job after his marriage in 1909. Fortunately in 1910 R. Ramachandra Rao, a well-to-do patron of mathematics, gave him a monthly stipend largely on the strength of favorable recommendations from various sympathetic Indian mathematicians and the findings he already had jotted down in the "Notebooks."

In 1912, wanting more conventional work, he took a clerical position in the Madras Port Trust, where the chairman was a British engineer, Sir Francis Spring, and the manager was V. Ramaswami Aiyar, the founder of the Indian Mathematical Society. They encouraged Ramanujan to communicate his results to three prominent British mathematicians. Two apparently did not respond; the one who did was G. H. Hardy of Cambridge, now regarded as the foremost British mathematician of the period.

Hardy, accustomed to receiving crank mail, was inclined to disregard Ramanujan's letter at first glance the day it arrived, January 16, 1913. But after dinner that night Hardy and a close colleague, John E. Littlewood, sat down to puzzle through a list of 120 formulas and theorems Ramanujan had appended to his letter. Some hours later they had reached a verdict: they were seeing the work of a genius and not a crackpot. (According to his own "pure-talent scale" of mathematicians, Hardy was later to rate Ramanujan a 100, Littlewood a 30 and himself a 25. The German mathematician David Hilbert, the most influential figure of the time, merited only an 80.) Hardy described the revelation and its consequences as the one romantic incident in his life. He wrote that some of Ramanujan's formulas defeated him completely, and yet "they must be true, because if they were not true, no one would have had the imagination to invent them."

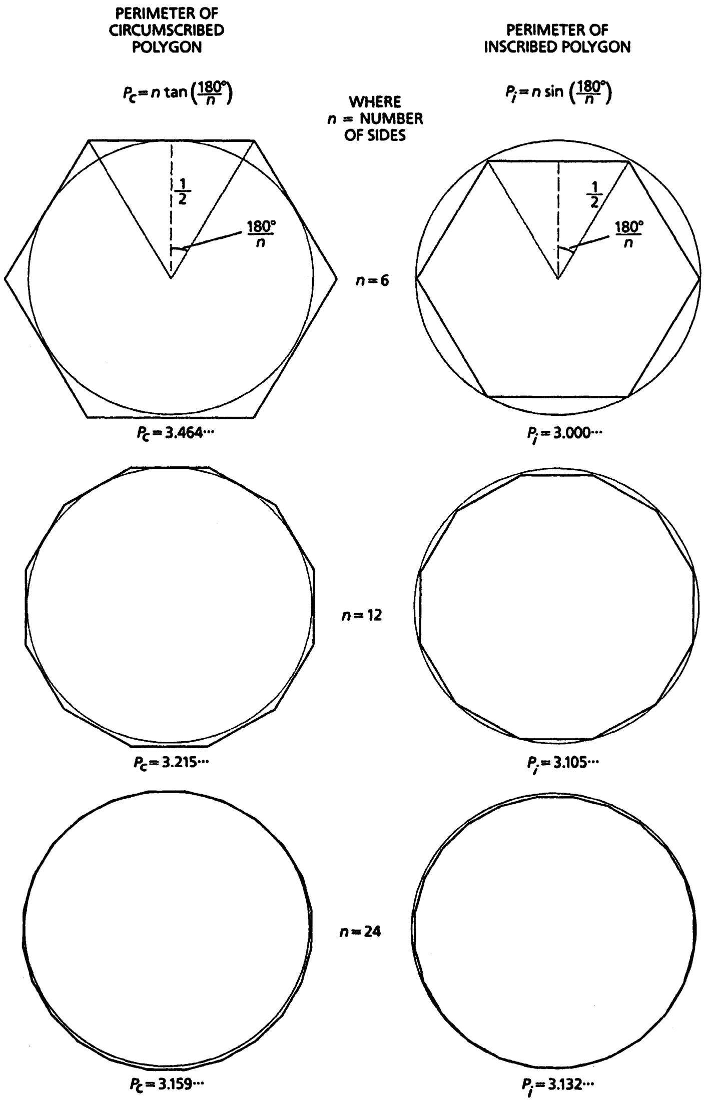
Hardy immediately invited Ramanujan to come to Cambridge. In spite of his mother's strong objections as well as his own reservations, Ramanujan set out for England in March of 1914. During the next five years Hardy and Ramanujan worked together at Trinity College. The blend of Hardy's technical expertise and Ramanujan's raw brilliance produced an unequaled collaboration. They published a series of seminal papers on the properties of various arithmetic functions, laying the groundwork for the answer to such questions as: How many prime divisors is a given number likely to have? How many ways can one express a number as a sum of smaller positive integers?

In 1917 Ramanujan was made a Fellow of the Royal Society of London and a Fellow of Trinity College—the first Indian to be awarded either honor. Yet as his prominence grew his health deteriorated sharply, a decline perhaps accelerated by the difficulty of maintaining a strict vegetarian diet in war-rationed England. Although Ramanujan was in and out of sanatoriums, he continued to pour forth new results. In 1919, when peace made travel abroad safe again, Ramanujan returned to India. Already an icon for young Indian intellectuals, the 32-year-old Ramanujan died on April 26, 1920, of what was then diagnosed as tuberculosis but now is thought to have been a severe vitamin deficiency. True to mathematics until the end, Ramanujan did not slow down during his last, pain-racked months, producing the re-

SRINIVASA RAMANUJAN, born in 1887 in India, managed in spite of limited formal education to reconstruct almost single-handedly much of the edifice of number theory and to go on to derive original theorems and formulas. Like many illustrious mathematicians before him, Ramanujan was fascinated by pi: the ratio of any circle's circumference to its diameter. Based on his investigation of modular equations (*see box on page 114*), he formulated exact expressions for pi and derived from them approximate values. As a result of the work of various investigators (including the authors), Ramanujan's methods are now better understood and have been implemented as algorithms.

markable work recorded in his so-called "Lost Notebook."

Ramanujan's work on pi grew in large part out of his investigation of modular equations, perhaps the most thoroughly treated subject in the "Notebooks." Roughly speaking, a modular equation is an algebraic relation between a function expressed in terms of a variable x—in mathematical notation, $f(x)$—and the same function expressed in terms of x raised to an integral power, for example $f(x^2)$, $f(x^3)$ or $f(x^4)$. The "order" of the modular equation is given by the integral power. The simplest modular equation is the second-order one: $f(x) = 2\sqrt{f(x^2)}\,/\,[1 + f(x^2)]$. Of course, not every function will satisfy a modular equation, but there is a class of functions, called modular functions, that do. These functions have various surprising symmetries that give them a special place in mathematics.



ARCHIMEDES' METHOD for estimating pi relied on inscribed and circumscribed regular polygons (polygons with sides of equal length) on a circle having a diameter of one unit (or a radius of half a unit). The perimeters of the inscribed and circumscribed polygons served respectively as lower and upper bounds for the value of pi. The sine and tangent functions can be used to calculate the polygons' perimeters, as is shown here, but Archimedes had to develop equivalent relations based on geometric constructions. Using 96-sided polygons, he determined that pi is greater than $3^{10}/_{71}$ and less than $3^{1}/_{7}$.

Ramanujan was unparalleled in his ability to come up with solutions to modular equations that also satisfy other conditions. Such solutions are called singular values. It turns out that solving for singular values in certain cases yields numbers whose natural logarithms coincide with pi (times a constant) to a surprising number of places [*see box on page 114*]. Applying this general approach with extraordinary virtuosity, Ramanujan produced many remarkable infinite series as well as single-term approximations for pi. Some of them are given in Ramanujan's one formal paper on the subject, *Modular Equations and Approximations to π*, published in 1914.

Ramanujan's attempts to approximate pi are part of a venerable tradition. The earliest Indo-European civilizations were aware that the area of a circle is proportional to the square of its radius and that the circumference of a circle is directly proportional to its diameter. Less clear, however, is when it was first realized that the ratio of any circle's circumference to its diameter and the ratio of any circle's area to the square of its radius are in fact the same constant, which today is designated by the symbol π. (The symbol, which gives the constant its name, is a latecomer in the history of mathematics, having been introduced in 1706 by the English mathematical writer William Jones and popularized by the Swiss mathematician Leonhard Euler in the 18th century.)

Archimedes of Syracuse, the greatest mathematician of antiquity, rigorously established the equivalence of the two ratios in his treatise *Measurement of a Circle.* He also calculated a value for pi based on mathematical principles rather than on direct measurement of a circle's circumference, area and diameter. What Archimedes did was to inscribe and circumscribe regular polygons (polygons whose sides are all the same length) on a circle assumed to have a diameter of one unit and to consider the polygons' respective perimeters as lower and upper bounds for possible values of the circumference of the circle, which is numerically equal to pi [*see illustration on opposite page*].

This method of approaching a value for pi was not novel: inscribing polygons of ever more sides in a circle had been proposed earlier by Antiphon, and Antiphon's contemporary, Bryson of Heraclea, had added circumscribed polygons to the procedure. What was novel was Archimedes' correct determination of the effect of doubling the number of sides on both the circumscribed and the inscribed polygons. He thereby developed a procedure that, when repeated enough times, enables one in principle to calculate pi to any number of digits. (It should be pointed out that the perimeter of a regular polygon can be readily calculated by means of simple trigonometric functions: the sine, cosine and tangent functions. But in Archimedes' time, the third century B.C., such functions were only partly understood. Archimedes therefore had to rely mainly on geometric constructions, which made the calculations considerably more demanding than they might appear today.)

Archimedes began with inscribed and circumscribed hexagons, which yield the inequality $3 < \pi < 2\sqrt{3}$. By doubling the number of sides four times, to 96, he narrowed the range of pi to between $3\frac{10}{71}$ and $3\frac{1}{7}$, obtaining the estimate $\pi \approx 3.14$. There is some evidence that the extant text of *Measurement of a Circle* is only a fragment of a larger work in which Archimedes described how, starting with decagons and doubling them six times, he got a five-digit estimate: $\pi \approx 3.1416$.

Archimedes' method is conceptually simple, but in the absence of a ready way to calculate trigonometric functions it requires the extraction of roots, which is rather time-consuming when done by hand. Moreover, the estimates converge slowly to pi: their error decreases by about a factor of four per iteration. Nevertheless, all European attempts to calculate pi before the mid-17th century relied in one way or another on the method. The 16th-century Dutch mathematician Ludolph van Ceulen dedicated much of his career to a computation of pi. Near the end of his life he obtained a 32-digit estimate by calculating the perimeter of inscribed and circumscribed polygons having 2^{62} (some 10^{18}) sides. His value for pi, called the Ludolphian number in parts of Europe, is said to have served as his epitaph.

The development of calculus, largely by Isaac Newton and Gottfried Wilhelm Leibniz, made it possible to calculate pi much more expeditiously. Calculus provides efficient techniques for computing a function's derivative (the rate of change in the function's value as its variables change) and its integral (the sum of the function's values over a range of variables). Applying the techniques, one can demonstrate that inverse trigonometric functions are given by integrals of quadratic functions that describe the curve of a circle. (The inverse of a trigonometric function gives the angle that corresponds to a particular value of the function. For example, the inverse tangent of 1 is 45 degrees or, equivalently, $\pi/4$ radians.)

(The underlying connection between trigonometric functions and algebraic expressions can be appreciated by considering a circle that has a radius of one unit and its center at the origin of a Cartesian x-y plane. The equation for the circle—whose area is numerically equal to pi—is $x^2 + y^2 = 1$, which is a restatement of the Pythagorean theorem for a right triangle with a hypotenuse equal to 1. Moreover, the sine and cosine of the angle between the positive x axis and any point on the circle are equal respectively to the point's coordinates, y and x; the angle's tangent is simply y/x.)

Of more importance for the purposes of calculating pi, however, is the fact that an inverse trigonometric function can be "expanded" as a series, the terms of which are computable from the derivatives of the function. Newton himself calculated pi to 15 places by adding the first few terms of a series that can be derived

WALLIS' PRODUCT (1665)

$$\frac{\pi}{2} = \frac{2\times2}{1\times3}\times\frac{4\times4}{3\times5}\times\frac{6\times6}{5\times7}\times\frac{8\times8}{7\times9}\times\cdots = \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}$$

GREGORY'S SERIES (1671)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

MACHIN'S FORMULA (1706)

$$\frac{\pi}{4} = 4\arctan(1/5) - \arctan(1/239), \qquad \text{where } \arctan X = X - \frac{X^3}{3} + \frac{X^5}{5} - \frac{X^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{X^{(2n+1)}}{2n+1}$$

RAMANUJAN (1914)

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9{,}801}\sum_{n=0}^{\infty}\frac{(4n)![1{,}103 + 26{,}390n]}{(n!)^4\,396^{4n}}, \quad \text{where } n! = n\times(n-1)\times(n-2)\times\cdots\times1 \text{ and } 0! = 1$$

BORWEIN AND BORWEIN (1987)

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)![212{,}175{,}710{,}912\sqrt{61} + 1{,}657{,}145{,}277{,}365 + n(13{,}773{,}980{,}892{,}672\sqrt{61} + 107{,}578{,}229{,}802{,}750)]}{(n)!^3(3n)![5{,}280(236{,}674 + 30{,}303\sqrt{61})]^{(3n+3/2)}}$$

TERMS OF MATHEMATICAL SEQUENCES can be summed or multiplied to yield values for pi (divided by a constant) or its reciprocal. The first two sequences, discovered respectively by the mathematicians John Wallis and James Gregory, are probably among the best-known, but they are practically useless for computational purposes. Not even 100 years of computing on a supercomputer programmed to add or multiply the terms of either sequence would yield 100 digits of pi. The formula discovered by John Machin made the calculation of pi feasible, since calculus allows the inverse tangent (arc tangent) of a number, *x*, to be expressed in terms of a sequence whose sum converges more rapidly to the value of the arc tangent the smaller *x* is. Virtually all calculations for pi from the beginning of the 18th century until the early 1970's have relied on variations of Machin's formula. The sum of Ramanujan's sequence converges to the true value of 1/π much faster: each successive term in the sequence adds roughly eight more correct digits. The last sequence, formulated by the authors, adds about 25 digits per term; the first term (for which *n* is 0) yields a number that agrees with pi to 24 digits.

as an expression for the inverse of the sine function. He later confessed to a colleague: "I am ashamed to tell you to how many figures I carried these calculations, having no other business at the time."

In 1674 Leibniz derived the formula $1 - 1/3 + 1/5 - 1/7 \ldots = \pi/4$, which is the inverse tangent of 1. (The general inverse-tangent series was originally discovered in 1671 by the Scottish mathematician James Gregory. Indeed, similar expressions appear to have been developed independently several centuries earlier in India.) The error of the approximation, defined as the difference between the sum of n terms and the exact value of $\pi/4$, is roughly equal to the $n+1$th term in the series. Since the denominator of each successive term increases by only 2, one must add approximately 50 terms to get two-digit accuracy, 500 terms for three-digit accuracy and so on. Summing the terms of the series to calculate a value for pi more than a few digits long is clearly prohibitive.

An observation made by John Machin, however, made it practicable to calculate pi by means of a series expansion for the inverse-tangent function. He noted that pi divided by 4 is equal to 4 times the inverse tangent of 1/5 minus the inverse tangent of 1/239. Because the inverse-tangent series for a given value converges more quickly the smaller the value is, Machin's formula greatly simplified the calculation. Coupling his formula with the series expansion for the inverse tangent, Machin computed 100 digits of pi in 1706. Indeed, his technique proved to be so powerful that all extended calculations of pi from the beginning of the 18th century until recently relied on variants of the method.

Two 19th-century calculations deserve special mention. In 1844 Johann Dase computed 205 digits of pi in a matter of months by calculating the values of three inverse tangents in a Machin-like formula. Dase was a calculating prodigy who could multiply 100-digit numbers entirely in his head—a feat that took him roughly eight hours. (He was perhaps the closest precursor of the modern supercomputer, at least in terms of memory capacity.) In 1853 William Shanks outdid Dase by publishing his computation of pi to 607 places, although the digits that followed the 527th place were wrong. Shank's task took years and was a rather routine, albeit laborious, application of Machin's formula. (In what must itself be some kind of record, 92 years passed before Shank's error was detected, in a comparison between his value and a 530-place approximation produced by D. F. Ferguson with the aid of a mechanical calculator.)

The advent of the digital computer saw a renewal of efforts to calculate ever more digits of pi, since the machine was ideally suited for lengthy, repetitive "number crunching." ENIAC, one of the first digital computers, was applied to the task in June, 1949, by John von Neumann and his colleagues. ENIAC produced 2,037 digits in 70 hours. In 1957 G. E. Felton attempted to compute 10,000 digits of pi, but owing to a machine error only the first 7,480 digits were correct. The 10,000-digit goal was reached by F. Genuys the following year on an IBM 704 computer. In 1961 Daniel Shanks and John W. Wrench, Jr., calculated 100,000 digits of pi in less than nine hours on an IBM 7090. The million-digit mark was passed in 1973 by Jean Guilloud and M. Bouyer, a feat that took just under a day of computation on a CDC 7600. (The computations done by Shanks and Wrench and by Guilloud and Bouyer were in fact carried out twice using different inverse-tangent identities for pi. Given the history of both human and machine error in these calculations, it is only after such verification that modern "digit hunters" consider a record officially set.)

Although an increase in the speed of computers was a major reason ever more accurate calculations for pi could be performed, it soon became clear that there were inescapable limits. Doubling the number of digits lengthens computing time by at least a factor of four, if one applies the traditional methods of performing arithmetic in computers. Hence even allowing for a hundredfold increase in computational speed, Guilloud and Bouyer's program would have required at least a quarter century to produce a billion-digit value for pi. From the perspective of the early 1970's such a computation did not seem realistically practicable.

Yet the task is now feasible, thanks

MODULAR FUNCTIONS AND APPROXIMATIONS TO PI

A modular function is a function, $\lambda(q)$, that can be related through an algebraic expression called a modular equation to the same function expressed in terms of the same variable, q, raised to an integral power: $\lambda(q^p)$. The integral power, p, determines the "order" of the modular equation. An example of a modular function is

$$\lambda(q) = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8 .$$

Its associated seventh-order modular equation, which relates $\lambda(q)$ to $\lambda(q^7)$, is given by

$$\sqrt[8]{\lambda(q)\lambda(q^7)} + \sqrt[8]{[1-\lambda(q)][1-\lambda(q^7)]} = 1 .$$

Singular values are solutions of modular equations that must also satisfy additional conditions. One class of singular values corresponds to computing a sequence of values, k_p, where

$$k_p = \sqrt{\lambda(e^{-\pi\sqrt{p}})}$$

and p takes integer values. These values have the curious property that the logarithmic expression

$$\frac{-2}{\sqrt{p}} \log\left(\frac{k_p}{4}\right)$$

coincides with many of the first digits of pi. The number of digits the expression has in common with pi increases with larger values of p.

Ramanujan was unparalleled in his ability to calculate these singular values. One of his most famous is the value when p equals 210, which was included in his original letter to G. H. Hardy. It is

$$k_{210} = (\sqrt{2}-1)^2(2-\sqrt{3})(\sqrt{7}-\sqrt{6})^2(8-3\sqrt{7})(\sqrt{10}-3)^2(\sqrt{15}-\sqrt{14})(4-\sqrt{15})^2(6-\sqrt{35}) .$$

This number, when plugged into the logarithmic expression, agrees with pi through the first 20 decimal places. In comparison, k_{240} yields a number that agrees with pi through more than one million digits.

Applying this general approach, Ramanujan constructed a number of remarkable series for pi, including the one shown in the illustration on the preceding page. The general approach also underlies the two-step, iterative algorithms in the top illustration on the opposite page. In each iteration the first step (calculating y_n) corresponds to computing one of a sequence of singular values by solving a modular equation of the appropriate order; the second step (calculating α_n) is tantamount to taking the logarithm of the singular value.

not only to faster computers but also to new, efficient methods for multiplying large numbers in computers. A third development was also crucial: the advent of iterative algorithms that quickly converge to pi. (An iterative algorithm can be expressed as a computer program that repeatedly performs the same arithmetic operations, taking the output of one cycle as the input for the next.) These algorithms, some of which we constructed, were in many respects anticipated by Ramanujan, although he knew nothing of computer programming. Indeed, computers not only have made it possible to apply Ramanujan's work but also have helped to unravel it. Sophisticated algebraic-manipulation software has allowed further exploration of the road Ramanujan traveled alone and unaided 75 years ago.

One of the interesting lessons of theoretical computer science is that many familiar algorithms, such as the way children are taught to multiply in grade school, are far from optimal. Computer scientists gauge the efficiency of an algorithm by determining its bit complexity: the number of times individual digits are added or multiplied in carrying out an algorithm. By this measure, adding two n-digit numbers in the normal way has a bit complexity that increases in step with n; multiplying two n-digit numbers in the normal way has a bit complexity that increases as n^2. By traditional methods, multiplication is much "harder" than addition in that it is much more time-consuming.

Yet, as was shown in 1971 by A. Schönhage and V. Strassen, the multiplication of two numbers can in theory have a bit complexity only a little greater than addition. One way to achieve this potential reduction in bit complexity is to implement so-called fast Fourier transforms (FFT's). FFT-based multiplication of two large numbers allows the intermediary computations among individual digits to be carefully orchestrated so that redundancy is avoided. Because division and root extraction can be reduced to a sequence of multiplications, they too can have a bit complexity just slightly greater than that of addition. The result is a tremendous saving in bit complexity and hence in computation time. For this reason all recent efforts to calculate pi rely on some variation of the FFT technique for multiplication.

Yet for hundreds of millions of dig-

(a) Let $y_0 = \frac{1}{\sqrt{2}}$ $\quad \alpha_0 = \frac{1}{2}$

and

$$y_{n+1} = \frac{1-\sqrt{1-y_n^2}}{1+\sqrt{1-y_n^2}} \qquad \alpha_{n+1} = [(1+y_{n+1})^2\alpha_n] - 2^{n+1}y_{n+1}$$

(b) Let $y_0 = \sqrt{2}-1$ $\quad \alpha_0 = 6 - 4\sqrt{2}$

and

$$y_{n+1} = \frac{1-\sqrt[4]{1-y_n^4}}{1+\sqrt[4]{1-y_n^4}}$$

$$\alpha_{n+1} = [(1+y_{n+1})^4\alpha_n] - 2^{2n+3}y_{n+1}(1+y_{n+1}+y_{n+1}^2)$$

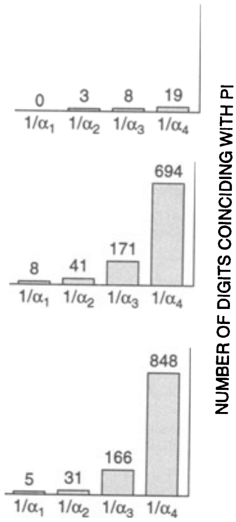
(c) Let $S_0 = 5(\sqrt{5}-2)$ $\quad \alpha_0 = \frac{1}{2}$

and

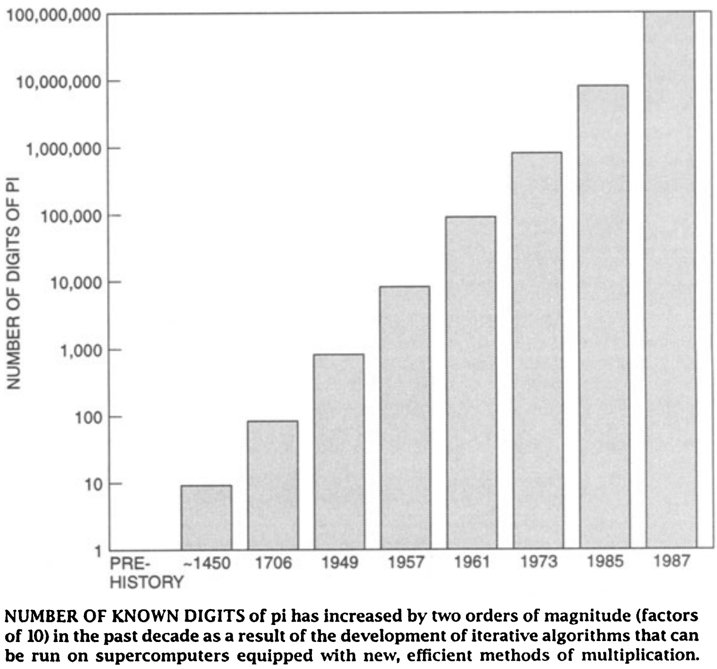
$$S_{n+1} = \frac{25}{S_n(Z + X/Z + 1)^2}, \text{ where } X = \frac{5}{S_n} - 1, \quad Y = (X-1)^2 + 7$$

$$\text{and } Z = \sqrt[5]{\frac{X(Y+\sqrt{Y^2-4X^3})}{2}}$$

$$\alpha_{n+1} = S_n^2\alpha_n - 5^n\left[\frac{S_n^2-5}{2} + \sqrt{S_n(S_n^2-2S_n+5)}\right]$$



ITERATIVE ALGORITHMS that yield extremely accurate values of pi were developed by the authors. (An iterative algorithm is a sequence of operations repeated in such a way that the ouput of one cycle is taken as the input for the next.) Algorithm *a* converges to $1/\pi$ quadratically: the number of correct digits given by α_n more than doubles each time n is increased by 1. Algorithm *b* converges quartically and algorithm *c* converges quintically, so that the number of coinciding digits given by each iteration increases respectively by more than a factor of four and by more than a factor of five. Algorithm *b* is possibly the most efficient known algorithm for calculating pi; it was run on supercomputers in the last three record-setting calculations. As the authors worked on the algorithms it became clear to them that Ramanujan had pursued similar methods in coming up with his approximations for pi. In fact, the computation of s_n in algorithm *c* rests on a remarkable fifth-order modular equation discovered by Ramanujan.



NUMBER OF KNOWN DIGITS of pi has increased by two orders of magnitude (factors of 10) in the past decade as a result of the development of iterative algorithms that can be run on supercomputers equipped with new, efficient methods of multiplication.

115

its of pi to be calculated practically a beautiful formula known a century and a half earlier to Carl Friedrich Gauss had to be rediscovered. In the mid-1970's Richard P. Brent and Eugene Salamin independently noted that the formula produced an algorithm for pi that converged quadratically, that is, the number of digits doubled with each iteration. Between 1983 and the present Yasumasa Kanada and his colleagues at the University of Tokyo have employed this algorithm to set several world records for the number of digits of pi.

We wondered what underlies the remarkably fast convergence to pi of the Gauss-Brent-Salamin algorithm, and in studying it we developed general techniques for the construction of similar algorithms that rapidly converge to pi as well as to other quantities. Building on a theory outlined by the German mathematician Karl Gustav Jacob Jacobi in 1829, we realized we could in principle arrive at a value for pi by evaluating integrals of a class called elliptic integrals, which can serve to calculate the perimeter of an ellipse. (A circle, the geometric setting of previous efforts to approximate pi, is simply an ellipse with axes of equal length.)

Elliptic integrals cannot generally be evaluated as integrals, but they can be easily approximated through iterative procedures that rely on modular equations. We found that the Gauss-Brent-Salamin algorithm is actually a specific case of our more general technique relying on a second-order modular equation. Quicker convergence to the value of the integral, and thus a faster algorithm for pi, is possible if higher-order modular equations are used, and so we have also constructed various algorithms based on modular equations of third, fourth and higher orders.

In January, 1986, David H. Bailey of the National Aeronautics and Space Administration's Ames Research Center produced 29,360,000 decimal places of pi by iterating one of our algorithms 12 times on a Cray-2 supercomputer. Because the algorithm is based on a fourth-order modular equation, it converges on pi quartically, more than quadrupling the number of digits with each iteration. A year later Kanada and his colleagues carried out one more iteration to attain 134,217,000 places on an NEC SX-2 supercomputer and thereby verified a similar computation they had done earlier using the Gauss-Brent-Salamin algorithm. (Iterating our algorithm twice more—a feat entirely feasible if one could somehow monopolize a supercomputer for a few weeks—would yield more than two billion digits of pi.)

HOW TO GET TWO BILLION DIGITS OF PI WITH A CALCULATOR*

Let

$$y_0 = \sqrt{2} - 1 \qquad \alpha_0 = 6 - 4\sqrt{2}$$

$$y_1 = [1 - \sqrt[4]{1-y_0^4}]/[1 + \sqrt[4]{1-y_0^4}] \qquad \alpha_1 = (1+y_1)^4\alpha_0 - 2^3 y_1(1+y_1+y_1^2)$$

$$y_2 = [1 - \sqrt[4]{1-y_1^4}]/[1 + \sqrt[4]{1-y_1^4}] \qquad \alpha_2 = (1+y_2)^4\alpha_1 - 2^5 y_2(1+y_2+y_2^2)$$

$$y_3 = [1 - \sqrt[4]{1-y_2^4}]/[1 + \sqrt[4]{1-y_2^4}] \qquad \alpha_3 = (1+y_3)^4\alpha_2 - 2^7 y_3(1+y_3+y_3^2)$$

$$y_4 = [1 - \sqrt[4]{1-y_3^4}]/[1 + \sqrt[4]{1-y_3^4}] \qquad \alpha_4 = (1+y_4)^4\alpha_3 - 2^9 y_4(1+y_4+y_4^2)$$

$$y_5 = [1 - \sqrt[4]{1-y_4^4}]/[1 + \sqrt[4]{1-y_4^4}] \qquad \alpha_5 = (1+y_5)^4\alpha_4 - 2^{11} y_5(1+y_5+y_5^2)$$

$$y_6 = [1 - \sqrt[4]{1-y_5^4}]/[1 + \sqrt[4]{1-y_5^4}] \qquad \alpha_6 = (1+y_6)^4\alpha_5 - 2^{13} y_6(1+y_6+y_6^2)$$

$$y_7 = [1 - \sqrt[4]{1-y_6^4}]/[1 + \sqrt[4]{1-y_6^4}] \qquad \alpha_7 = (1+y_7)^4\alpha_6 - 2^{15} y_7(1+y_7+y_7^2)$$

$$y_8 = [1 - \sqrt[4]{1-y_7^4}]/[1 + \sqrt[4]{1-y_7^4}] \qquad \alpha_8 = (1+y_8)^4\alpha_7 - 2^{17} y_8(1+y_8+y_8^2)$$

$$y_9 = [1 - \sqrt[4]{1-y_8^4}]/[1 + \sqrt[4]{1-y_8^4}] \qquad \alpha_9 = (1+y_9)^4\alpha_8 - 2^{19} y_9(1+y_9+y_9^2)$$

$$y_{10} = [1 - \sqrt[4]{1-y_9^4}]/[1 + \sqrt[4]{1-y_9^4}] \qquad \alpha_{10} = (1+y_{10})^4\alpha_9 - 2^{21} y_{10}(1+y_{10}+y_{10}^2)$$

$$y_{11} = [1 - \sqrt[4]{1-y_{10}^4}]/[1 + \sqrt[4]{1-y_{10}^4}] \qquad \alpha_{11} = (1+y_{11})^4\alpha_{10} - 2^{23} y_{11}(1+y_{11}+y_{11}^2)$$

$$y_{12} = [1 - \sqrt[4]{1-y_{11}^4}]/[1 + \sqrt[4]{1-y_{11}^4}] \qquad \alpha_{12} = (1+y_{12})^4\alpha_{11} - 2^{25} y_{12}(1+y_{12}+y_{12}^2)$$

$$y_{13} = [1 - \sqrt[4]{1-y_{12}^4}]/[1 + \sqrt[4]{1-y_{12}^4}] \qquad \alpha_{13} = (1+y_{13})^4\alpha_{12} - 2^{27} y_{13}(1+y_{13}+y_{13}^2)$$

$$y_{14} = [1 - \sqrt[4]{1-y_{13}^4}]/[1 + \sqrt[4]{1-y_{13}^4}] \qquad \alpha_{14} = (1+y_{14})^4\alpha_{13} - 2^{29} y_{14}(1+y_{14}+y_{14}^2)$$

$$y_{15} = [1 - \sqrt[4]{1-y_{14}^4}]/[1 + \sqrt[4]{1-y_{14}^4}] \qquad \alpha_{15} = (1+y_{15})^4\alpha_{14} - 2^{31} y_{15}(1+y_{15}+y_{15}^2)$$

$1/\alpha_{15}$ agress with π for more than two billion decimal digits

*Of course, the calculator needs to have a two-billion-digit display; on a pocket calculator the computation would not be very interesting after the second iteration.

EXPLICIT INSTRUCTIONS for executing algorithm *b* in the top illustration on the preceding page makes it possible in principle to compute the first two billion digits of pi in a matter of minutes. All one needs is a calculator that has two memory registers and the usual capacity to add, subtract, multiply, divide and extract roots. Unfortunately most calculators come with only an eight-digit display, which makes the computation moot.

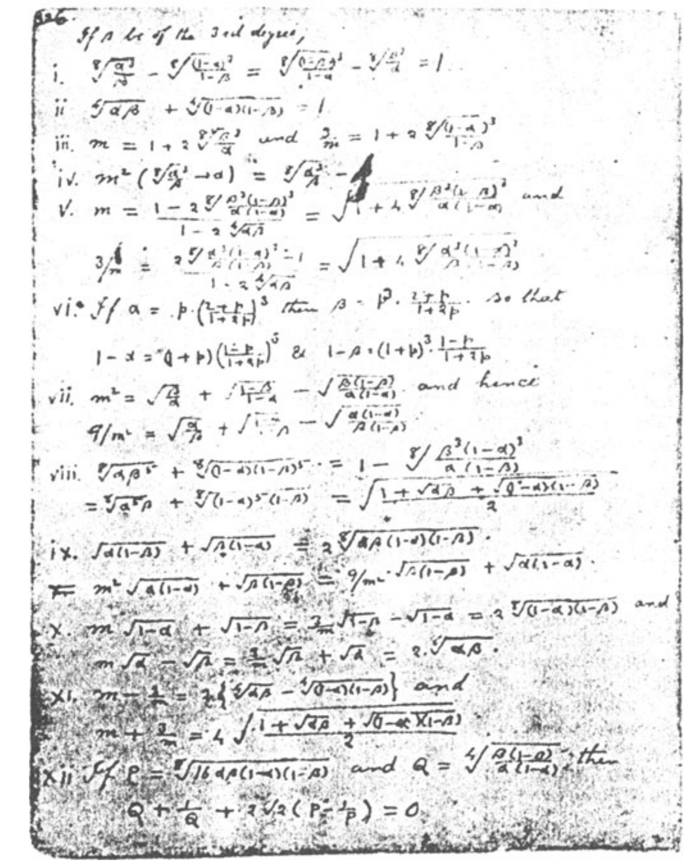
Iterative methods are best suited for calculating pi on a computer, and so it is not surprising that Ramanujan never bothered to pursue them. Yet the basic ingredients of the iterative algorithms for pi—modular equations in particular—are to be found in Ramanujan's work. Parts of his original derivation of infinite series and approximations for pi more than three-quarters of a century ago must have paralleled our own efforts to come up with algorithms for pi. Indeed, the formulas he lists in his paper on pi and in the "Notebooks" helped us greatly in the construction of some of our algorithms. For example, although we were able to prove that an 11th-order algorithm exists and knew its general formulation, it was not until we stumbled on Ramanujan's modular equations of the same order that we discovered its unexpectedly simple form.

Conversely, we were also able to derive all Ramanujan's series from the general formulas we had developed. The derivation of one, which converged to pi faster than any other series we knew at the time, came about with a little help from an unexpected source. We had justified all the quantities in the expression for the series except one: the coefficient 1,103, which appears in the numerator of the expression [*see illustration on page 113*]. We were convinced—as Ramanujan must have been—that 1,103 had to be correct. To prove it we had either to simplify a daunting equation containing variables raised to powers of several thousand or to delve considerably further into somewhat arcane number theory.

By coincidence R. William Gosper, Jr., of Symbolics, Inc., had decided in 1985 to exploit the same series of Ramanujan's for an extended-accuracy value for pi. When he carried out the calculation to more than 17 million digits (a record at the time), there was to his knowledge no proof that the sum of the series actually converged to pi. Of course, he knew that millions of digits of his value coincided with an earlier Gauss-Brent-Salamin calculation done by Kanada. Hence the possibility of error was vanishingly small.

As soon as Gosper had finished his calculation and verified it against Kanada's, however, we had what we needed to prove that 1,103 was the number needed to make the series true to within one part in $10^{10,000,000}$. In much the same way that a pair of integers differing by less than 1 must be equal, his result sufficed to specify the number: it is precisely 1,103. In effect, Gosper's computation became part of our proof. We knew that the series (and its associated algorithm) is so sensitive to slight inaccuracies that if Gosper had used any other value for the coefficient or, for that matter, if the computer had introduced a single-digit error during the calculation, he would have ended up with numerical nonsense instead of a value for pi.

Ramanujan-type algorithms for approximating pi can be shown to be very close to the best possible. If all the operations involved in the execution of the algorithms are totaled (assuming that the best techniques known for addition, multiplication and root extraction are applied), the bit complexity of computing n digits of pi is only marginally greater than that of multiplying two n-digit numbers. But multiplying two n-digit numbers by means of an FFT-based technique is only marginally more complicated than summing two n-digit numbers, which is the simplest of the arithmetic operations possible on a computer.



RAMANUJAN'S "NOTEBOOKS" were personal records in which he jotted down many of his formulas. The page shown contains various third-order modular equations—all in Ramanujan's nonstandard notation. Unfortunately Ramanujan did not bother to include formal proofs for the equations; others have had to compile, edit and prove them. The formulas in the "Notebooks" embody subtle relations among numbers and functions that can be applied in other fields of mathematics or even in theoretical physics.

Mathematics has probably not yet felt the full impact of Ramanujan's genius. There are many other wonderful formulas contained in the "Notebooks" that revolve around integrals, infinite series and continued fractions (a number plus a fraction, whose denominator can be expressed as a number plus a fraction, whose denominator can be expressed as a number plus a fraction, and so on). Unfortunately they are listed with little—if any—indication of the method by which Ramanujan proved them. Littlewood wrote: "If a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further."

The herculean task of editing the "Notebooks," initiated 60 years ago by the British analysts G. N. Watson and B. N. Wilson and now being completed by Bruce Berndt, requires providing a proof, a source or an occasional correction for each of many thousands of asserted theorems and identities. A single line in the "Notebooks" can easily elicit many pages of commentary. The task is made all the more difficult by the nonstandard mathematical notation in which the formulas are written. Hence a great deal of Ramanujan's work will not become accessible to the mathematical community until Berndt's project is finished.

Ramanujan's unique capacity for working intuitively with complicated formulas enabled him to plant seeds in a mathematical garden (to borrow a metaphor from Freeman Dyson) that is only now coming into bloom. Along with many other mathematicians, we look forward to seeing which of the seeds will germinate in future years and further beautify the garden.

Approximations and complex multiplication according to Ramanujan

by

D.V. Chudnovsky*), G.V. Chudnovsky*)

Department of Mathematics
Columbia University
New York, NY 10027

*)This work was supported in part by the N.S.F., U.S. Air Force and O.C.R.E.A.E. program.

ISBN 0-12-058560-X

Introduction

This talk revolves around two focuses: complex multiplications (for elliptic curves and Abelian varieties) in connection with algebraic period relations, and (diophantine) approximations to numbers related to these periods. Our starting point is Ramanujan's works [1], [2] on approximations to π via the theory of modular and hypergeometric functions. We describe in chapter 1 Ramanujan's original quadratic period--quasiperiod relations for elliptic curves with complex multiplication and their applications to representations of fractions of π and other logarithms in terms of rapidly convergent nearly integral (hypergeometric) series. These representations serve as a basis of our investigation of diophantine approximations to π and other related numbers. In Chapter 2 we look at period relations for arbitrary CM-varieties following Shimura and Deligne. Our main interest lies with modular (Shimura) curves arising from arithmetic Fuchsian groups acting on H. From these we choose arithmetic triangular groups, where period relations can be expressed in the form of hypergeometric function identities. Particular attention is devoted to two (commensurable) triangle groups, $(0,3;2,6,6)$ and $(0,3;2,4,6)$, arising from the quaternion algebra over $\mathbb{Q}$ with the discriminant $D = 2\cdot 3$. We also touch upon the algebraic independence problem for periods and quasiperiods of general CM-varieties and particularly CM-curves associated with the triangle groups (hypergeometric curves as we call them). The diophantine approximation problem for numbers connected with periods, particularly for multiples of π, is analyzed using Padé approximations to power series representing these numbers. We give a brief review of Padé approximations, their effective construction, and problems of analytic and arithmetic (p-adic) convergence of Padé approximants. Padé approximations to nearly integral power series (G-functions) are used in connection with Ramanujan-like representations of $1/\pi$ and other similar period constants. We discuss measures of irrationality for algebraic multiples of π and related numbers that follow from Padé approximation methods.

The problem of uniformization by nonarithmetic subgroups is discussed in connection with the Whittaker conjecture [11] on an explicit expression for accessory parameters in the (Schottkey-type) uniformization of hyperelliptic Riemann surfaces of genus $g \geq 2$ by Fuchsian groups. On the basis of numerical computations of monodromy groups of linear differential equations, we concluded that the conjecture [11] is generically incorrect. Moreover, it appears that accessory parameters in the uniformization problem of Riemann surfaces defined over $\bar{\mathbb{Q}}$ are nonalgebraic with the exception of uniformization by arithmetic subgroups and of cases when the differential equations are reduced to hypergeometric ones (the monodromy group is connected to one of the triangle groups). We briefly describe numerical and theoretical results on the transcendence of elements of the monodromy groups of linear differential equations over $\bar{\mathbb{Q}}(x)$.

We conclude the paper with a discussion of numerical approximations to solutions of algebraic and differential equations. We present generalizations of our previous results [12] on the complexity of approximations to solutions of linear differential equations. We describe a new, "bit-burst" method of evaluation of solutions of linear differential equations everywhere on their Riemann surface. In the worst case, an evaluation with n bits of precision at an n-bit point requires $O(M_{bit}(n)\log^3 n)$ boolean (bit) operations, where $M_{bit}(n)$ is the boolean complexity of an n-bit multiplication. For functions satisfying additional arithmetic conditions (e.g. E-functions and G-functions) the factor $\log^3 n$ could be further decreased to $\log^2 n$ or, even, $\log n$. We also describe the natural parallelizations of the presented algorithms that are well suited for practical implementation.

We want to thank the organizers of the Ramanujan conference and, particularly, B. Berndt and R. Askey. We want to thank the IBM Computer Algebra Group for their help and access to the SCRATCHPAD System. We thank SYMBOLICS Corporation for the use of their workstation 3645.

1. Complex multiplications and Ramanujan's period relations

Most of the material in this talk evolves around mathematics closely associated with one of the earliest Ramanujan papers "Modular equations and approximations to π" [1] published in 1914, which according to Hardy [2] "is mainly Indian work, done before he came to England." In that or another way the same kind of mathematics appears in later Ramanujan research, including his notebooks. Hardy interpreted many of Ramanujan's results and identities as connected mainly with "complex multiplication", and Ramanujan's interest in resolving modular equations in explicit radicals was later picked up in Watson's outstanding series [3] of "Singular Moduli" papers. Singular moduli themselves, and general modular equations relating automorphic functions with respect to congruence subgroup of a full modular group are traditional subjects of late $\overline{\underline{\text{XIX}}}$ century mathematics, whose importance is clearly realized in modern number theory and algebraic geometry, particularly in diophantine geometry in connection with arithmetic theory of elliptic curves and rational points on them. Also modular equations turned up as a convenient tool of fast operational complexity algorithms of computation of π and of values of elementary functions (see corresponding chapters in Borweins' book [5]).

Instead of complex multiplication as merely a subject of "singular moduli" of elliptic functions we will touch upon the complex multiplication in a slightly more general context: from the point of view of nontrivial endomorphisms of certain classes of Jacobians of particular algebraic curves. (We are not going to discuss a variety of complex "complex multiplication" subjects on L-functions and Abelian varieties, though we'll have to borrow particular consequences of vast theories developed in general by Shimura, Deligne and others.)

The choice of curves is clearly determined by Ramanujan's interest: these are curves with 4 critical points, whose Abelian periods are expressed via hypergeometric integrals (for simplicity one can call those curves hypergeometric ones), see [10].

Transforming one of the 4 critical points into infinity and normalizing, one recovers the Gauss hypergeometric function integrals representative of these periods. We display these well-known expressions:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-xz)^{-a}dx,$$

$(c > b > 0)$, with the expansion near $z = 0$:

$$F(a,b;c;z) = 1 + \frac{a\cdot b}{c\cdot 1} \cdot z + \frac{a(a+1)\cdot b(b+1)}{c\cdot(c+1)\cdot 1\cdot 2} \cdot z^2 + \dots .$$

Our primary interest in complex multiplication is not arithmetical but transcendental; rather than to study the number-theoretical functions associated with complex multiplication invariants, we want to know the basic facts about the values of these invariants: are these numbers transcendental (over $\mathbb{Z}$)? If algebraic relations do exist, over what fields of definition do they exist? These basic questions of transcendence, algebraic independence and linear independence are the subject of diophantine approximations. With these questions come their qualitative counterparts: when numbers are irrational (transcendental), how well are they approximated by rationals? Can these best approximations be determined effectively and/or explicitly? (Usually one asks in this context: can one determine the continued fraction expansion of the number?)

The class of numbers we are interested in is generated over $\bar{\mathbb{Q}}$ by periods and quasiperiods of Abelian varieties, i.e. by integrals

$$\int_\gamma \omega \quad \text{and} \quad \int_\gamma \eta$$

for differentials ω and η of the first and the second kind, respectively, from $H^1_{DR}(A)$, and $\gamma \in H_1(A,\mathbb{Z})$, for an Abelian variety A defined over $\bar{\mathbb{Q}}$.

In particular, when A is a Jacobian $J(\Gamma)$ of a nonsingular curve Γ over $\bar{\mathbb{Q}}$, we are looking at periods and quasiperiods forming the full Riemann matrix of Γ-total of $2g \times 2g$ elements, where g denotes the genus of Γ.

In this context, complex multiplications, understood as nontrivial endomorphisms of A, are usually expressed as nontrivial algebraic relations among the elements of Riemann's original $2g \times g$ pure period matrix π of A. These "purely period relations" are well known, and are mainly algebraic in nature, and in one-dimensional case ($g = 1$), give pairs of periods whose ratio is a "singular module". An interesting thing discovered by Ramanujan in this classical (even in his time) field was the existence of new quasiperiods relations. In the Weierstrass-like notation, commonly accepted nowdays, period and quasiperiod relations in the elliptic curve case can be described as follows.

One starts with an elliptic curve over $\bar{\mathbb{Q}}$ with a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ ($g_2, g_3 \in \bar{\mathbb{Q}}$), having the fundamental periods ω_1, ω_2 (with $\mathrm{Im}\frac{\omega_2}{\omega_1} > 0$) and the corresponding quasi-periods η_1, η_2. The only relation between ω_i and η_j that always holds is the Legendre identity:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$

Thus π belongs to the field generated by periods and quasiperiods over $\bar{\mathbb{Q}}$. In the complex multiplication case $\tau = \frac{\omega_2}{\omega_1}$ is an imaginary quadratic number.

Whenever $\tau \in \mathbb{Q}(\sqrt{-d})$ for $d > 0$, invariants g_2 and g_3 can be chosen from the Hilbert class field of $\mathbb{Q}(\sqrt{-d})$, and this field is the minimal extension with this property.

A priori complex multiplication means only a single relation between ω_i.

It seems that until Ramanujan's paper nobody explicitly stated the existence of the second relation between periods and quasiperiods. This relation is the following one:

Whenever τ is a quadratic number: $A\tau^2 + B\tau + C = 0$, the four numbers: $\omega_1, \omega_2, \eta_1, \eta_2$ are linearly dependent over $\bar{\mathbb{Q}}$ only on two of them.

Explicitly:

$$\begin{aligned} &\omega_2 = \tau\omega_1, \\ &A\tau\eta_2 - C\eta_1 + \alpha\omega_1 = 0 \end{aligned} \tag{1.1}$$

for $\alpha \in \bar{\mathbb{Q}}$ ($\alpha \in \mathbb{Q}(\tau, g_2, g_3)$).

The relations (1.1) are not entirely original; Legendre's investigation of the lemmiscate case provides with (1.1) in two cases, where τ is equivalent to i or to ρ under $SL_2(\mathbb{Z})$; moreover, these cases were clearly known to Euler, who evaluated the appropriate complete elliptic integrals. However, those two particular cases are "wrong ones": in these cases the importance of the relation (1.1) is lost because $\alpha = 0$, and it is hard to understand the need for its appearance. In a few other special singular moduli cases, (1.1) appears in the classical literature, see [4].

Of course, Ramanujan did not use the Weierstrass equations and preferred the Legendre ones, where one can see the hypergeometric functions instantly.

In order to pass to Legendre notations [5], one puts $4x^3 - g_2x - g_3 = 4(x-e_1)(x-e_2)(x-e_3)$, and looks at the modular function

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}$$

(an automorphic function $k^2 = k^2(\tau)$ with respect to the principal congruence subgroup $\Gamma(2)$ of $\Gamma(1) = SL_2(\mathbb{Z})$ in the variable τ in $H = \{\tilde{z}: \operatorname{Im} \tilde{z} > 0\}$). Then the periods and quasiperiods are expressed through the complete elliptic integrals of the first and second kind:

$$K(k) = \frac{\pi}{2}\cdot F(\tfrac{1}{2},\tfrac{1}{2};1;k^2),$$

$$E(k) = \frac{\pi}{2}\cdot F(-\tfrac{1}{2},\tfrac{1}{2};1;k^2).$$

We also look at $K(k')$, $E(k')$ for $k'^2 = 1 - k^2$; then ω_1, η_1 are expressed in terms of $K = K(k)$, $E = E(k)$, while ω_2, η_2 are expressed in terms of iK' and iE':

$$\omega_1 = \frac{K}{\sqrt{e_1-e_3}}, \quad \omega_2 = \frac{iK'}{\sqrt{e_1-e_3}};$$

$$\eta_1 = \sqrt{e_1-e_3}\cdot E - e_1\omega_1, \quad \eta_2 = -\sqrt{e_1-e_3}\; iE' - e_3\omega_2.$$

Invariant α in (1.1)--a nontrivial part of the Ramanujan quasiperiod relation--is easily recognized as one of the simplest values of "nonholomorphic Eisenstein series". Weil's treatise [6] creates a clear impression that this quantity and its algebraicity had been known to Kronecker (or even Eisenstein). It seems to us that though similar and more general functions were carefully examined, this particular connection had been reconstructed by Weil, and cannot be easily separated from his own work on period relations. The "nonholomorphic" Eisenstein series are too important to be ignored.

The usual Eisenstein series associated with the lattice $\mathcal{L}$ of periods: $\mathcal{L} = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ are

$$G_k(\omega_1,\omega_2) = \sum_{\substack{\omega\in\mathcal{L}\\ \omega\neq 0}} \omega^{-k} \quad \text{for } k = 4,6,\ldots .$$

The corresponding normalized inhomogeneous series $E_k(\tau)$ are defined as

$$G_k(\omega_1,\omega_2) = \left(\frac{2\pi i}{\omega_2}\right)^k \cdot \frac{-B_k}{k!} \cdot E_k(\tau),$$

or

$$E_k(\tau) = 1 - \frac{2k}{B_k} \cdot \sum_{n=1}^{\infty} \sigma_{k-1}(n) \cdot q^n$$

for $\sigma_{k-1}(n) = \Sigma_{d|n} d^{k-1}$, and $q = e^{2\pi i\tau}$.

These q-series were subject of a variety of Ramanujan's studies [1-2] with his preferred notations a P,Q,R for $E_k(\tau)$ with $k = 2,4,6$, respectively.

For $k = 2$ the proper definition of $G_k(\omega_1,\omega_2)$ is a non-holomorphic one arising from

$$H(s,z) = \sum_{\omega\in\mathcal{L}} (\bar{z}+\bar{\omega})\,|z+\omega|^{-2s}$$

as:

$$G_2(\omega_1,\omega_2) = \lim_{s\to\infty^+} \sum_{\omega\in\mathcal{L}}{}^{*} \omega^{-2} \cdot |\omega|^{-2s}.$$

In the $E_k(\tau)$ notations, the quasiperiod relation is expressed by means of the function

$$s_2(\tau) \overset{\text{def}}{=} \frac{E_4(\tau)}{E_6(\tau)} \cdot \left(E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}\right), \tag{1.2}$$

which is invariant under the action of $\Gamma(1)$ but nonholomorphic. It is this object that was studied by Ramanujan in connection with α in (1.1). Ramanujan actually proved in [1] that this function admits algebraic values whenever τ is imaginary quadratic. Moreover, Ramanujan [1] presented a variety of algebraic expressions for this function, differentiating modular equations.

His work, or Weil's, shows that the function in (1.2) has values from the Hilbert class field $\mathbb{Q}(\tau, j(\tau))$ of $\mathbb{Q}(\tau)$ for quadratic τ. The relation of (1.2) to α is simple: for $\beta = s_2(\tau)$ from (1.2), $\alpha = (B + 2A\tau)\beta \cdot g_3/g_2$.

Amazingly, Mordell in his notes [1] on Ramanujan paper missed the true importance of (1.1) or (1.2), stating merely "... Ramanujan's method of obtaining purely algebraical approximations appears to be new." These relations were rediscovered in the 70's (among the rediscovers was Siegel [7]), see particularly [9], and stimulated search for multidimensional generalizations of the period relations promoted by Weil [8]. We'll talk about generalizations of elliptic period relations later, but meanwhile let us look on relations (1.1) once more. One can combine (1.1) with the Legendre relation to arrive to a phenomenally looking "quadratic relation" derived by Ramanujan, that expresses π in terms of squares of ω_1 and η_1 only (no ω_2 and η_2!) All this is interesting, as an algebraic identity, but Ramanujan transforms these quadratic relations into rapidly convergent generalized hypergeometric representation of simple algebraic multiples of $1/\pi$. To do this he needed not only modular functions but also hypergeometric function identities.

We reproduce first Ramanujan's own favorite [1], which was used by Gosper in 1985 in his $17.5\cdot 10^6$ decimal digit computation of π :

$$\frac{9801}{2\sqrt{2}\pi} = \sum_{n=0}^{\infty} \{1103 + 26390n\} \frac{(4n)!}{n!^4 \cdot (4\cdot 99)^{4n}} \qquad (44).$$

(Numeration here is temporarily borrowed from [1].)

The reason for this pretty representation of $1/\pi$ lies in the representation of $(K(k)/\pi)^2$ as a ${}_3F_2$-hypergeometric function. Apparently there are four classes of such representations all of which were determined by Ramanujan: these are <u>four</u> distinct classes of ${}_3F_2$-representation of $1/\pi$, all based on special cases of Clausen identity (and all presented by Ramanujan [1]):

$$F(a,b;a+b+\tfrac{1}{2};z)^2 = {}_3F_2\left(\begin{matrix}2a,a+b,2b\\ a+b+\frac{1}{2},2a+2b\end{matrix}\Big| z\right). \qquad (1.3)$$

Unfortunately, the Clausen identity is a unique one--no other nontrivial relation between parameters makes a product of hypergeometric functions a generalized hypergeometric function.

We display the basis for the Ramanujan's series representation for $1/\pi$. We'll discuss them later in connection with arithmetic triangle groups. Meanwhile, what is good in these identities for diophantine approximations? First of all, Ramanujan approximations to π are indeed remarkably fast numerical schemes of evaluation of π. Unlike some other numerical schemes these are series schemes, that can be accelerated into Padé approximation schemes. These Padé approximations schemes are better numerically, but more important, they are nontrivial arithmetically good rational approximations to algebraic multiples of π, that provide nontrivial measures of diophantine approximations.

(Deviating momentarily, we want to compare Padé approximations vs. power series approximations in numerical evaluation of functions and constants. Remarkably, asymptotically there is no significant difference between Boolean complexities of evaluation of Padé approximations to solutions of linear differential equations and of power series approximations within a given precision. Unfortunately, asymptotically there is no gain in the degree of approximations either; moreover there is a significant difference in storage requirements. Padé approximations require more storage. Even in cases when explicit Padé approximations are known, gains of using them can be visible only in about hundreds of digits of precision; not below or above. That is why unless special circumstances call for (like uniform approximations with a minimal storage in ordinary precision range), Padé approximations and continued fraction expansion techniques should not be used for numerical evaluation.

However, in diophantine approximations we have no choice. Only Padé approximations and a vast army of their generalizations are capable of approximating functions and constants and tell something of their arithmetic nature, of their irrationalities and transcendences, measures of approximation, etc.).

All Ramanujan's quadratic period relations (four types) can be deduced from one series by modular transformations, and we choose the series as the one associated with the modular invariant $J = J(\tau)$. In the place of Ramanujan's nonholomorphic function we take, as above in (1.2):

$$s_2(\tau) = \frac{E_4(\tau)}{E_6(\tau)}\left(E_2(\tau) - \frac{3}{\pi \operatorname{Im}(\tau)}\right).$$

Then the Clausen identity gives the following ${}_3F_2$-representation for an algebraic multiple of $1/\pi$:

$$\sum_{n=0}^{\infty}\left\{\frac{1}{6}(1-s_2(\tau))+n\right\}\cdot\frac{(6n)!}{(3n)!n!^3}\cdot\frac{1}{J(\tau)^n} = \frac{(-J(\tau))^{1/2}}{\pi}\cdot\frac{1}{(d(1728-J(\tau)))^{1/2}}. \tag{1.4}$$

Here if $\tau = (1+\sqrt{-d})/2$. If $h(-d) = 1$ the second factor in the right hand side is a rational number. The largest one class discriminat $-d = -163$ gives the most rapidly convergent series (though coefficients are slightly strange):

$$\sum_{n=0}^{\infty}\{c_1+n\}\cdot\frac{(6n)!}{(3n)!n!^3}\,\frac{(-1)^n}{(640{,}320)^n} = \frac{(640{,}320)^{3/2}}{163\cdot 8\cdot 27\cdot 7\cdot 11\cdot 19\cdot 127}\cdot\frac{1}{\pi}. \tag{1.5}$$

Here

$$c_1 = \frac{13,591,409}{163\cdot 2\cdot 9\cdot 7\cdot 11\cdot 19\cdot 127}$$

(and, of course, $J(\frac{1+\sqrt{-163}}{2}) = -(640,320)^3$).

Ramanujan [1] provides instead of this a variety of other formulas connected mainly with the three other triangle groups commensurable with $\Gamma(1)$.

Four classes of ${}_3F_2$ representations of algebraic multiples of $1/\pi$ correspond to four ${}_3F_2$ hypergeometric functions (that are squares of ${}_2F_1$ representations of complete elliptic integrals via the Clausen identity). These are

$$ {}_3F_2\left({1/2\ ,\ 1/6\ ,\ 5/6 \atop 1\ ,\ 1}\middle| x\right) = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!n!^3}\left(\frac{x}{12^3}\right)^n \tag{1.6}$$

$$ {}_3F_2\left({1/4\ ,\ 3/4\ ,\ 1/2 \atop 1\ ,\ 1}\middle| x\right) = \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4}\left(\frac{x}{4^4}\right)^n \tag{1.7}$$

$$ {}_3F_2\left({1/2\ ,\ 1/2\ ,\ 1/2 \atop 1\ ,\ 1}\middle| x\right) = \sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6}\left(\frac{x}{2^6}\right)^n \tag{1.8}$$

$$ {}_3F_2\left({1/3\ ,\ 2/3\ ,\ 1/2 \atop 1\ ,\ 1}\middle| x\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\cdot\frac{(2n)!}{n!^2}\left(\frac{x}{3^3\cdot 2^2}\right)^n. \tag{1.9}$$

The first function is the one arising in (1.4) with $x = 12^3/J(\tau)$. Other three functions correspond to modular transformations of $J(\tau)$. This means that appropriate $x = x(\tau)$ is a modular function of higher level (e.g. in (1.8), $x = 4k^2(1-k^2)$ for $k^2 = k^2(\tau)$), and series (1.7)-(1.9) for the same τ have slower convergence rates than the series in (1.6).

Representations similar to (1.5) can be derived for any of the series (1.6)-(1.9) for any singular moduli $\tau \in \mathbb{Q}(\sqrt{-d})$

and for any class number $h(-d)$, thus extending Ramanujan list [1] ad infinum. There is a simple recipe to generate these new identities, instead of elaborate procedure proposed in [1] (see also [5]) based on differentiating of modular equations. To derive these identities one needs the explicit expressions of $x_j = x(\tau_j)$ with the representatives τ_j in H of algebraically conjugate values of automorphic function $x = x(\tau)$ for $\tau \in \mathbb{Q}(\sqrt{-d})$ (say $\tau = \sqrt{-d}$ or $\tau = \frac{1+\sqrt{-d}}{2}$). E.g. for $x(\tau) = 12^3/J(\tau)$, τ_j: $j = 1,\ldots,h(-d)$ corresponds to the class number of $\mathbb{Q}(\sqrt{-d})$. The necessary values of $s_2(\tau_j)$ are easy to compute from q-series representation of $E_k(\tau)$, if to use the formula (1.2). These computations can be carried out in bounded precision, because, as we know, $s_2(\tau_j)$ lies in the Hilbert class field of $\mathbb{Q}(\sqrt{-d})$ and because, whenever $J(\tau_j)$ is algebraically conjugate to $J(\tau_i)$, numbers $s_2(\tau_j)$ and $s_2(\tau_i)$ are also algebraically conjugate. This allows us to express $s_2(\tau)$ in terms of $J(\tau)$ and $\sqrt{-d}$ explicitly using only finite precision approximations to all $s_2(\tau_j)$. This way one obtains rapidly convergent ${}_3F_2$ representations of algebraic multiples of π by nearly integral power series. When $h(-d) > 1$, these series contain nonrational numbers, making the series (1.5) the fastest convergent series representing a multiple of $1/\pi$, and having rational number entries only.

Even before Ramanujan's remarkable approximations to π, singular moduli evaluations were used to approximate multiples of π by logarithms of algebraic numbers (usually the values of modular invariants). One of the first series of such approximations belongs to Hermite [13]. Of course, by now it is reproduced in hundreds of papers and we have to give a customary example. One is looking here at the expansion of the modular invariant near infinity:

$$J(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots$$

for $q = e^{2\pi i\tau}$. The elementary theory of complex multiplication shows that for $\tau = (1+\sqrt{-d})/2$, $q^{-1} = e^{-\pi\sqrt{d}}$ is very close to an algebraic integer $J(\tau)$ of degree $h(-d)$. Usual examples (see description in [14]) involve the largest one class discriminant -163, $d = 163$, when:

$$e^{\pi\sqrt{163}} = -262537412640768743.9999999999992\ldots\ .$$

There is a variety of these and similar approximations of π by logarithms of other classical automorphic functions. One of the most popular, studied by Shanks et. al. [15], has a simple q-expansion:

For $f = f_1(\sqrt{-d})^{-24} = (k/4k'^2)^2$ at $\tau = \sqrt{-d}$,

$$\log f + \sqrt{d}\cdot\pi = 24 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} q^k(1-q^k)^{-1}.$$

It is also known that the right side can be expanded in powers of f:

$$\log f + \sqrt{d}\cdot\pi = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a_n}{n}\cdot f^n.$$

Here a_n are integers. Shanks [15] looks at specialization of this formula for $-d$ with class groups of special structure for relatively large d. These approximations are not technically approximations to π, but rather to a linear form in π and in another logarithm. All of them are natural consequences of Schwarz theory and the representation of the function inverse to the automorphic one (say $J(\tau)$) as a ratio of two solutions of a hypergeometric equation. One such formula is

$$\pi i\cdot\tau = \ell n(k^2) - \ell n(16) + \frac{G(\frac{1}{2},\frac{1}{2};1;k^2)}{F(\frac{1}{2},\frac{1}{2};1;k^2)}, \tag{1.10}$$

and another (our favorite) is Fricke's [16]

$$2\pi i\cdot\tau = \ell n\ J + \frac{G(\frac{1}{12},\frac{5}{12};1;\frac{12^3}{J})}{F(\frac{1}{12},\frac{5}{12};1;\frac{12^3}{J})}. \tag{1.11}$$

Here $G(a,b;c;x) = \Sigma_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}\cdot\{\Sigma_{j=0}^{n-1}(\frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{c+j})\}$ is the hypergeometric function (of the second kind) in the exceptional case, when there are logarithmic terms.

Perhaps the most popular approximations to linear forms in π and in another logarithm are associated with Stark-Baker solution to one-class and two-class problems (cf. [17]). Stark's approach [18] is based on Kronecker's limit formula, and in a way, similar to approximations given above, one represents

$$L(1,\chi)\cdot L(1,\chi\chi') - \frac{\pi^2}{6}\prod_{p|k}(1-\frac{1}{p^2})\sum_f \frac{\chi(a)}{a}$$

for $\chi(\cdot) = (\frac{k}{\cdot})$, $\chi'(\cdot) = (\frac{-d}{\cdot})$, $k > 0$ as a rapidly convergent

$q^{1/k}$-series (here $f = ax^2 + bxy + cy^2$ runs through a complete set of unequivalent quadratic forms with the discriminant $-d$). Using Dirichlet's class number formula one obtains an exceptional approximation (as above) to the linear form in two logarithms:

$$|h(-kd)\cdot\log\varepsilon_{\sqrt{k}} - q\pi\sqrt{d}| < e^{-\pi O(\sqrt{d}/k)}$$

for an arbitrary fundamental unit $\varepsilon_{\sqrt{k}}$ of $\mathbb{Q}(\sqrt{k})$ and for one-class discriminant $-d$. These remarkable linear forms were generalized by Stark to three-term linear forms in the class number two case.

(While these unusually good approximations can be used in the class number problem--approximations are so good that they are impossible for large d--these linear forms cannot be used for the analysis of arithmetic properties of the individual logarithms, like π, entering the linear form. Moreover, as the class number grows, the number of the terms in the linear form grows.)

Important developments initiated by Ramanujan in his truly algebraic approximations to $1/\pi$ can be extended to the analysis of linear forms in logarithms presented above. In fact, each term in these linear forms can be separately represented by a rapidly convergent series in $1/J$ with nearly integral coefficients.

For this one takes an automorphic function $\varphi(\tau)$ with respect to one of the congruence subgroups of $\Gamma(1)$ and expand functions like $F(\frac{1}{12},\frac{5}{12};1;12^3/J)$, $G(\frac{1}{12},\frac{5}{12};1,;12^3/J)$ in powers of $\varphi(\tau)$ instead of $J(\tau)$. Whenever $\varphi(\tau)$ is automorphic with respect to a classical triangle group, we arrive to the corresponding usual hypergeometric functions.

Other logarithms, like π, can be represented as values of convergent series satisfying Fuchsian linear differential equations. This is particularly clear for log $\epsilon_{\sqrt{k}}$ of a fundamental unit $\epsilon_{\sqrt{k}}$ of a real quadratic field $\mathbb{Q}(\sqrt{k})$. To represent this number as a convergent series (in, say, $1/J(\tau)$) one uses Kronecker's limit formula expressing this logarithm log $\epsilon_{\sqrt{k}}$ in terms of products of values of Dedekind's Δ-function ("Jugandtraum", see [6]). Such an expression of log $\epsilon_{\sqrt{k}}$ in terms of power series in $1/J(\tau)$ for $\tau = (1+\sqrt{-d})/2$ for any $d \equiv 3(8)$, depends, unfortunately, on k, because k is related to the level of the appropriate modular form $\varphi = \varphi_k(\tau)$.

For $k = 5$ Siegel [19] made an explicit computation that expresses $J(\tau)$ in terms of the resolvent $\varphi_5(\tau)$ of 5-th degree modular equation known from the classical theory of 5-th degree equations. His relations [19] were:

$$(\varphi-\epsilon^3)((\varphi-\epsilon)(\varphi^2+\epsilon^{-1}\varphi+\epsilon^{-2}))^3 + (\varphi/\sqrt{5})^5 J = 0$$

and

$$\varphi(\tau)(= \varphi_5(\tau)) = \epsilon^{h(-5p)/2}$$

for $\tau = (1+\sqrt{-p})/2$ and $\epsilon = \epsilon_{\sqrt{5}}$. Here one has $p \equiv 3(5)$; if $p \equiv 2(5)$ and replaces ϵ by ϵ^{-1} in the expression of $J = J(\tau)$.

This, in combination with Ramanujan's approximation to π, allows one to express $\log \varepsilon$ (its multiple) as a convergent series in $1/J$ (or in $1/\omega$).

References

[1] S. Ramanujan, Collected Papers, Cambridge, 1927, 23-39.

[2] G.H. Hardy, Ramanujan, Cambridge, 1940.

[3] G.N. Watson, Some singular moduli (I);(II);(III);(IV); Quart. J. Math. Oxford, 3 (1932), 81-98; 189-212; Proc. London Math. Soc. 40 (1936), 83-142; Acta Arithmetica, 1 (1936), 284-323.

[4] J.M. Borwein, P.B. Borwein, Pi and the AGM, Wiley, 1987.

[5] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, 4 ed., Cambridge, 1927.

[6] A. Weil, Elliptic Functions According to Eisenstein and Kronecker, Springer, 1976.

[7] C.L. Siegel, Bestimmung der elliptischen Modulfunktionen durch eine Transformations gleichung, Abh. Math. Sem. Univ. Hamburg, 27 (1964), 32-38.

[8] A. Weil, Sur les Périodes des Intégrales Abéliennes, Comm. Pure Appl. Math., 29 (1976), 813-819.

[9] D. Masser, Elliptic Functions and Transcendence, Lecture Notes Math., v. 437, Springer, 1975.

[10] S. Lefschetz, On certain numerical invariants of algebraic varieties with application to Abelian varieties, Trans. Amer. Math. Soc., 22 (1921), 327-482.

[11] E.T. Whittaker, On hyperlemniscate functions, a family of automorphic functions, J. London Math. Soc., 4 (1929), 274-278.

[12] D.V. Chudnovsky, G.V. Chudnovsky, Computer assisted number theory, Lecture Notes Math., Springer, 1240, 1987, 1-68.

[13] Ch. Hermite, Sur la Théorie des Équations Modulaires, C.R. Acad. Sci. Paris., 48 (1859), 940-1079-1097; 49 (1859), 16-110-141.

[14] H.M. Stark, Class-numbers of complex quadratic fields, Lecture Notes Math., Springer, v. 320, 1973, 153-174.

[15] D. Shanks, Dihedral quartic approximation and series for π, J. Number Theory, 14 (1982), 397-423.

[16] R. Fricke, Die Elliptischen Funktionen und ihre Anwendungen, v. 1, Teubner, 1916.

[17] A. Baker, Transcendental Number Theory, Cambridge, 1979.

[18] H.M. Stark, A transcendence theorem for class number problems, I; II; Ann. Math. 94 (1971), 153-173; 96(1972), 174-209.

[19] C.L. Siegel, Zum Beweise des Starkschen Satzes, Invent. Math., 5 (1968), 180-191.

[20] G. Shimura, Automorphic forms and the periods of Abelian varieties, J. Math. Soc. Japan, 31 (1979), 561-59.

[21] G. Shimura, The arithmetic of certain zeta functions and automorphic forms on orthogonal groups, Ann. of Math., 111 (1980), 313-375.

[22] P. Deligne, Valeurs de fonctions L et périodes d'integrales, Proc. Symp. Pure Math., v. 33, Part 2, Amer. Math. Soc., Providence, R.I., 313-346.

[23] P. Deligne, Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes, Soc. Math. de France, Memoire, N° 2, 1980, 23-33.

[24] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Forms, Princeton, University Press, 1971.

[25] G. Shimura, Y. Taniyama, Complex Multiplication of Abelian Varieties and Its Applications to Number Theory, Publications of the Mathematical Society of Japan, N° 6, 1961.

[26] G.V. Chudnovsky, Algebraic independence of values of exponential and elliptic functions. Proceedings of the International Congress of Mathematicians, Helsinki, 1979, Acad. Sci. Tennice, Helsinki, 1980, v. 1, 339-350.

[27] B.H. Gross, N. Koblitz, Gauss sums and the p-adic Γ-function, Ann. of Math., 109 (1979), 569-581.

[28] R. Fricke, F. Klein, Vorlesungen über die Theorie der Automorphen Functionen, bd. 2, Tenbner, 1926.

[29] R. Morris, On the automorphic functions of the group $(0,3;\ell_1,\ell_2,\ell_3)$, Trans. Amer. Math. Soc., 7 (1906), 425-448.

[30] K. Takeuchi, Arithmetic triangle groups, J. Math. Soc. Japan, 29 (1977), 91-106.

[31] H.P.F. Swinnerton-Dyer, Arithmetic groups in Discrete Groups and Automorphic Functions, Academic Press, 1977, 377-401.

[32] J.I. Hutchinson, On the automorphic functions of the group (0,3;2,6,6), Trans. Amer. Math. Soc., 5 (1904), 447-460.

[33] D.V. Chudnovsky, G.V. Chudnovsky, Note on Eisenstein's system of differential equations, in Classical and Quantum Models and Arithmetic Problems, M. Dekker, 1984, 99-116.

[34] D.V. Chudnovsky, G.V. Chudnovsky, The use of computer algebra for diophantine and differential equations, in Computer Algebra as a Tool for Research in Mathematics and Physics, Proceeding of the International Conference, 1984, New York University, M. Dekker, 1987.

[35] D.V. Chudnovsky, G.V. Chudnovsky, Applications of Padé approximations to diophantine inequalities in values of G-functions, Lecture Notes Math., v. 1135, Springer, 1985, 9-50.

[36] K. Mahler, Perfect systems, Compositio Math., 19 (1968), 95-166.

[37] D.V. Chudnovsky, G.V. Chudnovsky, Applications of Padé approximations to the Grothendieck conjecture on linear differential equations, Lecture Notes Math., v. 1135, Springer, 1985, 51-100.

[38] C.L. Siegel, Über einige Anwendungen diophantischer Approximationen, Abh. Preuss. Akad. Wiss. Phys. Math. Kl. 1, 1929.

[39] G.V. Chudnovsky, Contributions to the Theory of Transcendental Numbers, Mathematical Surveys and Monographs, v. 19, Amer. Math. Soc., Providence, R.I., 1984.

[40] G.V. Chudnovsky, Padé approximation and the Riemann monodromy problem, in Bifurcation Phenomena in Mathematical Physics and Related Topics, D. Reidel, Boston, 1980, 448-510.

[41] G.V. Chudnovsky, Rational and Padé approximation to solutions of linear differential equations and the monodromy theory. Lecture Notes Physics, v. 126, Springer, 1980, 136-169.

[42] G.V. Chudnovsky, Padé approximations to the generalized hypergeometric functions. I., J. Math. Pures et Appliques, Paris, 58 (1979), 445-476.

[43] G.V. Chudnovsky, On the method of Thue-Siegel, Ann. of Math., 117 (1983), 325-382.

[44] C.L. Siegel, Transcendental Numbers, Princeton University Press, 1949.

[45] G.V. Chudnovsky, Rational approximations to linear forms of exponentials and binomials, Proc. Nat'l Acad. Sci. U.S.A., 80 (1983), 3138-3141.

[46] D.V. Chudnovsky, G.V. Chudnovsky, A random walk in higher arithmetic, Adv. Appl. Math., 7 (1986), 101-122.

[47] H. Poincare, Sur les groupes des équations lineaires, Acta. Math., 4 (1884) 201-312.

[48] D.A. Hejhel, Monodromy groups and Poincaré series, Bull. Amer. Math. Soc., 84 (1978), 339-376.

[49] E.T. Whittaker, On the connexion of algebraic functions with automorphic functions, Phil. Trans., 122 A(1898), 1-32.

[50] R.A. Rankin, The differential equations associated with the uniformization of certain algebraic curves, Proc. Roy. Soc. Edinburgh, 65 (1958) 35-62.

[51] A. Schonhage, V. Strassen, Schnelle Multiplikation grosser Zahlen, Computing, 7 (1971), 281-292.

[52] D.V. Chudnovsky, G.V. Chudnovsky, Algebraic complexities and algebraic curves over finite fields, Proc. Natl. Acad. Sci. USA, 84 (1987), 1739-1743.

[53] A. Schonhage, Equation solving in terms of computational complexity, Proc. International Congress of Mathematicians, Berkeley, 1986.

[54] R.P. Brent, Multiple-precision zero-finding methods and the complexity of elementary function evaluation, in Analytic Computational Complexity, J.F. Traub, Ed., Academic Press, 1975, 151-176.

[55] R.P. Brent, The complexity of multiple-precision arithmetic, in Complexity of Computational Problem Solving, R.S. Anderson and R.P. Brent, Eds. Univ. of Queensland Press, Brisbane, Australia, 1975, 126-165.

[56] D.V. Chudnovsky, G.V. Chudnovsky, On expansions of algebraic functions in power and Puiseux series, I; II; J. of Complexity, 2 (1986), 271-294; 3 (1987), 1-25.

[57] O. Perron, Die Lehre von den Kettenbrüchen, Teubner, 1929.

[58] D. Bini, V. Pan, Polynomial division and its computational complexity, J. of Complexity, 2 (1986), 179-203.

[59] D. A. Hejhal, A classical approach to a well-known spectral correspondence on quaternion groups, Lecture Notes Math., v. 1135, Springer, 1985, 127-196.

Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. 96, No. 3, March 1989

Ramanujan, Modular Equations, and Approximations to Pi or How to Compute One Billion Digits of Pi

J. M. BORWEIN AND P. B. BORWEIN
Mathematics Department, Dalhousie University, Halifax, N.S. B3H 3J5 Canada

and

D. H. BAILEY
NASA Ames Research Center, Moffett Field, CA 94035

Preface. The year 1987 was the centenary of Ramanujan's birth. He died in 1920 Had he not died so young, his presence in modern mathematics might be more immediately felt. Had he lived to have access to powerful algebraic manipulation software, such as MACSYMA, who knows how much more spectacular his already astonishing career might have been.

This article will follow up one small thread of Ramanujan's work which has found a modern computational context, namely, one of his approaches to approximating pi. Our experience has been that as we have come to understand these pieces of Ramanujan's work, as they have become mathematically demystified, and as we have come to realize the intrinsic complexity of these results, we have come to realize how truly singular his abilities were. This article attempts to present a considerable amount of material and, of necessity, little is presented in detail. We have, however, given much more detail than Ramanujan provided. Our intention is that the circle of ideas will become apparent and that the finer points may be pursued through the indicated references.

1. Introduction. There is a close and beautiful connection between the transformation theory for elliptic integrals and the very rapid approximation of pi. This connection was first made explicit by Ramanujan in his 1914 paper "Modular Equations and Approximations to π" **[26]**. We might emphasize that Algorithms 1 and 2 are not to be found in Ramanujan's work, indeed no recursive approximation of π is considered, but as we shall see they are intimately related to his analysis. Three central examples are:

Sum 1. (Ramanujan)

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{[1103 + 26390n]}{396^{4n}}.$$

Algorithm 1. Let $\alpha_0 := 6 - 4\sqrt{2}$ and $y_0 := \sqrt{2} - 1$.
Let

$$y_{n+1} := \frac{1 - (1 - y_n^4)^{1/4}}{1 + (1 - y_n^4)^{1/4}}$$

and

$$\alpha_{n+1} := (1 + y_{n+1})^4\alpha_n - 2^{2n+3}y_{n+1}(1 + y_{n+1} + y_{n+1}^2).$$

Then

$$0 < \alpha_n - 1/\pi < 16 \cdot 4^n e^{-2 \cdot 4^n \pi}$$

and α_n converges to $1/\pi$ *quartically* (that is, with order four).

Algorithm 2. Let $s_0 := 5(\sqrt{5} - 2)$ and $\alpha_0 := 1/2$.
Let

$$s_{n+1} := \frac{25}{(z + x/z + 1)^2 s_n},$$

where

$$x := 5/s_n - 1 \qquad y := (x - 1)^2 + 7$$

and

$$z := \left[\frac{1}{2}x\left(y + \sqrt{y^2 - 4x^3}\right)\right]^{1/5}.$$

Let

$$\alpha_{n+1} := s_n^2 \alpha_n - 5^n \left\{ \frac{s_n^2 - 5}{2} + \sqrt{s_n\left(s_n^2 - 2s_n + 5\right)} \right\}.$$

Then

$$0 < \alpha_n - \frac{1}{\pi} < 16 \cdot 5^n e^{-5^n \pi}$$

and α_n converges to $1/\pi$ *quintically* (that is, with order five).

Each additional term in Sum 1 adds roughly eight digits, each additional iteration of Algorithm 1 quadruples the number of correct digits, while each additional iteration of Algorithm 2 quintuples the number of correct digits. Thus a mere thirteen iterations of Algorithm 2 provide in excess of one billion decimal digits of pi. In general, for us, pth-order convergence of a sequence $\{\alpha_n)$ to α means that α_n tends to α and that

$$|\alpha_{n+1} - \alpha| \leq C|\alpha_n - \alpha|^p$$

for some constant $C > 0$. Algorithm 1 is arguably the most efficient algorithm currently known for the extended precision calculation of pi. While the rates of convergence are impressive, it is the subtle and thoroughly nontransparent nature of these results and the beauty of the underlying mathematics that intrigue us most.

Watson [37], commenting on certain formulae of Ramanujan, talks of

> *a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing "Day," "Night," "Evening," and "Dawn" which Michelangelo has set over the tomb of Giuliano de'Medici and Lorenzo de'Medici.*

Sum 1 is directly due to Ramanujan and appears in [26]. It rests on a modular identity of order 58 and, like much of Ramanujan's work, appears without proof and with only scanty motivation. The first complete derivation we know of appears in [11]. Algorithms 1 and 2 are based on modular identities of orders 4 and 5, respectively. The underlying quintic modular identity in Algorithm 2 (the relation for s_n) is also due to Ramanujan, though the first proof is due to Berndt and will appear in [7].

One intention in writing this article is to explain the genesis of Sum 1 and of Algorithms 1 and 2. It is not possible to give a short self-contained account without assuming an unusual degree of familiarity with modular function theory. Also, parts of the derivation involve considerable algebraic calculation and may most easily be done with the aid of a symbol manipulation package (MACSYMA, MAPLE, REDUCE, etc.). We hope however to give a taste of methods involved. The full details are available in [11].

A second intention is very briefly to describe the role of these and related approximations in the recent extended precision calculations of pi. In part this entails a short discussion of the complexity and implementation of such calculations. This centers on a discussion of multiplication by fast Fourier transform methods. Of considerable related interest is the fact that these algorithms for π are provably close to the theoretical optimum.

2. The State of Our Current Ignorance. Pi is almost certainly the most natural of the transcendental numbers, arising as the circumference of a circle of unit diameter. Thus, it is not surprising that its properties have been studied for some twenty-five hundred years. What is surprising is how little we actually know.

We know that π is irrational, and have known this since Lambert's proof of 1771 (see [5]). We have known that π is transcendental since Lindemann's proof of 1882 [23]. We also know that π is not a Liouville number. Mahler proved this in 1953. An irrational number β is *Liouville* if, for any n, there exist integers p and q so that

$$0 < \left|\beta - \frac{p}{q}\right| < \frac{1}{q^n}.$$

Liouville showed these numbers are all transcendental. In fact we know that

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{14.65}} \tag{2.1}$$

for p, q integral with q sufficiently large. This *irrationality estimate*, due to Chudnovsky and Chudnovsky [16] is certainly not best possible. It is likely that 14.65 should be replaced by $2 + \varepsilon$ for any $\varepsilon > 0$. Almost all transcendental numbers satisfy such an inequality. We know a few related results for the rate of algebraic approximation. The results may be pursued in [4] and [11].

We know that e^π is transcendental. This follows by noting that $e^\pi = (-1)^{-i}$ and applying the Gelfond-Schneider theorem [4]. We know that $\pi + \log 2 + \sqrt{2}\log 3$ is transcendental. This result is a consequence of the work that won Baker a Fields Medal in 1970. And we know a few more than the first two hundred million digits of the decimal expansion for π (Kanada, see Section 3).

The state of our ignorance is more profound. We do not know whether such basic constants as $\pi + e$, π/e, or $\log \pi$ are irrational, let alone transcendental. The best we can say about these three particular constants is that they cannot satisfy any polynomial of degree eight or less with integer coefficients of average size less than 10^9 [3]. This is a consequence of some recent computations employing the

Ferguson-Forcade algorithm [17]. We don't know anything of consequence about the single continued fraction of pi, except (numerically) the first 17 million terms, which Gosper computed in 1985 using Sum 1. Likewise, apart from listing the first many millions of digits of π, we know virtually nothing about the decimal expansion of π. It is possible, albeit not a good bet, that all but finitely many of the decimal digits of pi are in fact 0's and 1's. Carl Sagan's recent novel *Contact* rests on a similar possibility. Questions concerning the normality of or the distribution of digits of particular transcendentals such as π appear completely beyond the scope of current mathematical techniques. The evidence from analysis of the first thirty million digits is that they are very uniformly distributed [2]. The next one hundred and seventy million digits apparently contain no surprises.

In part we perhaps settle for computing digits of π because there is little else we can currently do. We would be amiss, however, if we did not emphasize that the extended precision calculation of pi has substantial application as a test of the "global integrity" of a supercomputer. The extended precision calculations described in Section 3 uncovered hardware errors which had to be corrected before those calculations could be successfully run. Such calculations, implemented as in Section 4, are apparently now used routinely to check supercomputers before they leave the factory. A large-scale calculation of pi is entirely unforgiving; it soaks into all parts of the machine and a single bit awry leaves detectable consequences.

3. Matters Computational

I am ashamed to tell you to how many figures I carried these calculations, having no other business at the time.

Isaac Newton

Newton's embarrassment at having computed 15 digits, which he did using the arcsinlike formula

$$\begin{aligned}\pi &= \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \cdots\right)\\ &= \frac{3\sqrt{3}}{4} + 24\int_0^{\frac{1}{4}}\sqrt{x-x^2}\,dx,\end{aligned}$$

is indicative both of the spirit in which people calculate digits and the fact that a surprising number of people have succumbed to the temptation [5].

The history of efforts to determine an accurate value for the constant we now know as π is almost as long as the history of civilization itself. By 2000 B.C. both the Babylonians and the Egyptians knew π to nearly two decimal places. The Babylonians used, among others, the value 3 1/8 and the Egyptians used 3 13/81. Not all ancient societies were as accurate, however—nearly 1500 years later the Hebrews were perhaps still content to use the value 3, as the following quote suggests.

Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about.

Old Testament, 1 Kings 7:23

Despite the long pedigree of the problem, all nonempirical calculations have employed, up to minor variations, only three techniques.

i) The first technique due to Archimedes of Syracuse (287–212 B.C.) is, recursively, to calculate the length of circumscribed and inscribed regular $6 \cdot 2^n$-gons about a circle of diameter 1. Call these quantities a_n and b_n, respectively. Then $a_0 := 2\sqrt{3}$, $b_0 := 3$ and, as Gauss's teacher Pfaff discovered in 1800,

$$a_{n+1} := \frac{2a_n b_n}{a_n + b_n} \quad \text{and} \quad b_{n+1} := \sqrt{a_{n+1} b_n}.$$

Archimedes, with $n = 4$, obtained

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}.$$

While hardly better than estimates one could get with a ruler, this is the first method that can be used to generate an arbitrary number of digits, and to a nonnumerical mathematician perhaps the problem ends here. Variations on this theme provided the basis for virtually all calculations of π for the next 1800 years, culminating with a 34 digit calculation due to Ludolph van Ceulen (1540–1610). This demands polygons with about 2^{60} sides and so is extraordinarily time consuming.

ii) Calculus provides the basis for the second technique. The underlying method relies on Gregory's series of 1671

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad |x| \le 1$$

coupled with a formula which allows small x to be used, like

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

This particular formula is due to Machin and was employed by him to compute 100 digits of π in 1706. Variations on this second theme are the basis of all the calculations done until the 1970's including William Shanks' monumental hand-calculation of 527 digits. In the introduction to his book [32], which presents this calculation, Shanks writes:

> *Towards the close of the year 1850 the Author first formed the design of rectifying the circle to upwards of 300 places of decimals. He was fully aware at that time, that the accomplishment of his purpose would add little or nothing to his fame as a Mathematician though it might as a Computer; nor would it be productive of anything in the shape of pecuniary recompense.*

Shanks actually attempted to hand-calculate 707 digits but a mistake crept in at the 527th digit. This went unnoticed until 1945, when D. Ferguson, in one of the last "nondigital" calculations, computed 530 digits. Even with machine calculations mistakes occur, so most record-setting calculations are done twice—by sufficiently different methods.

The advent of computers has greatly increased the scope and decreased the toil of such calculations. Metropolis, Reitwieser, and von Neumann computed and analyzed 2037 digits using Machin's formula on ENIAC in 1949. In 1961, Dan Shanks and Wrench calculated 100,000 digits on an IBM 7090 [31]. By 1973, still using Machin-like arctan expansions, the million digit mark was passed by Guillard and Bouyer on a CDC 7600.

iii) The third technique, based on the transformation theory of elliptic integrals, provides the algorithms for the most recent set of computations. The most recent records are due separately to Gosper, Bailey, and Kanada. Gosper in 1985 calculated over 17 million digits (in fact over 17 million terms of the continued fraction) using a carefully orchestrated evaluation of Sum 1.

Bailey in January 1986 computed over 29 million digits using Algorithm 1 on a Cray 2 [2]. Kanada, using a related quadratic algorithm (due in basis to Gauss and made explicit by Brent [12] and Salamin [27]) and using Algorithm 1 for a check, verified 33,554,000 digits. This employed a HITACHI S-810/20, took roughly eight hours, and was completed in September of 1986. In January 1987 Kanada extended his computation to 2^{27} decimal places of π and the hundred million digit mark had been passed. The calculation took roughly a day and a half on a NEC SX2 machine. Kanada's most recent feat (Jan. 1988) was to compute 201,326,000 digits, which required only six hours on a new Hitachi S-820 supercomputer. Within the next few years many hundreds of millions of digits will no doubt have been similarly computed. Further discussion of the history of the computation of pi may be found in [5] and [9].

4. Complexity Concerns. One of the interesting morals from theoretical computer science is that many familiar algorithms are far from optimal. In order to be more precise we introduce the notion of *bit complexity*. Bit complexity counts the number of single operations required to complete an algorithm. The single-digit operations we count are $+, -, \times$. (We could, if we wished, introduce storage and logical comparison into the count. This, however, doesn't affect the order of growth of the algorithms in which we are interested.) This is a good measure of time on a serial machine. Thus, addition of two n-digit integers by the usual method has bit complexity $O(n)$, and straightforward uniqueness considerations show this to be asymptotically best possible.

Multiplication is a different story. Usual multiplication of two n-digit integers has bit complexity $O(n^2)$ and no better. However, it is possible to multiply two n-digit integers with complexity $O(n(\log n)(\log\log n))$. This result is due to Schönhage and Strassen and dates from 1971 [29]. It provides the best bound known for multiplication. No multiplication can have speed better than $O(n)$. Unhappily, more exact results aren't available.

The original observation that a faster than $O(n^2)$ multiplication is possible was due to Karatsuba in 1962. Observe that

$$(a + b10^n)(c + d10^n) = ac + [(a - b)(c - d) - ac - bd]10^n + bd10^{2n},$$

and thus multiplication of two $2n$-digit integers can be reduced to three multiplications of n-digit integers and a few extra additions. (Of course multiplication by 10^n is just a shift of the decimal point.) If one now proceeds recursively one produces a multiplication with bit complexity

$$O(n^{\log_2 3}).$$

Note that $\log_2 3 = 1.58\ldots < 2$.

We denote by $M(n)$ the bit complexity of multiplying two n-digit integers together by any method that is at least as fast as usual multiplication.

The trick to implementing high precision arithmetic is to get the multiplication right. Division and root extraction piggyback off multiplication using Newton's method. One may use the iteration

$$x_{k+1} = 2x_k - x_k^2 y$$

to compute $1/y$ and the iteration

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{y}{x_k}\right)$$

to compute $\sqrt{y}$. One may also compute $1/\sqrt{y}$ from

$$x_{k+1} = \frac{x_k(3 - yx_k^2)}{2}$$

and so avoid divisions in the computation of $\sqrt{y}$. Not only do these iterations converge quadratically but, because Newton's method is self-correcting (a slight perturbation in x_k does not change the limit), it is possible at the kth stage to work only to precision 2^k. If division and root extraction are so implemented, they both have bit complexity $O(M(n))$, in the sense that n-digit input produces n-digit accuracy in a time bounded by a constant times the speed of multiplication. This extends in the obvious way to the solution of any algebraic equation, with the startling conclusion that every algebraic number can be computed (to n-digit accuracy) with bit complexity $O(M(n))$. Writing down n-digits of $\sqrt{2}$ or $3\sqrt{7}$ is (up to a constant) no more complicated than multiplication.

The Schönhage-Strassen multiplication is hard to implement. However, a multiplication with complexity $O((\log n)^{2+\epsilon}n)$ based on an ordinary complex (floating point) fast Fourier transform is reasonably straightforward. This is Kanada's approach, and the recent records all rely critically on some variations of this technique.

To see how the fast Fourier transform may be used to accelerate multiplication, let $x := (x_0, x_1, x_2, \ldots, x_{n-1})$ and $y := (y_0, y_1, y_2, \ldots, y_{n-1})$ be the representations of two high-precision numbers in some radix b. The radix b is usually selected to be some power of 2 or 10 whose square is less than the largest integer exactly representable as an ordinary floating-point number on the computer being used. Then, except for releasing each "carry," the product $z := (z_0, z_1, z_2, \ldots, z_{2n-1})$ of x and y may be written as

$$\begin{aligned}
z_0 &= x_0 y_0 \\
z_1 &= x_0 y_1 + x_1 y_0 \\
z_2 &= x_0 y_2 + x_1 y_1 + x_2 y_0 \\
&\vdots \\
z_{n-1} &= x_0 y_{n-1} + x_1 y_{n-2} + \cdots + x_{n-1} y_0 \\
&\vdots \\
z_{2n-3} &= x_{n-1} y_{n-2} + x_{n-2} y_{n-1} \\
z_{2n-2} &= x_{n-1} y_{n-1} \\
z_{2n-1} &= 0.
\end{aligned}$$

Now consider x and y to have n zeros appended, so that x, y, and z all have length $N = 2n$. Then a key observation may be made: the product sequence z is precisely the discrete convolution $C(x, y)$:

$$z_k = C_k(x, y) = \sum_{j=0}^{N-1} x_j y_{k-j},$$

where the subscript $k - j$ is to be interpreted as $k - j + N$ if $k - j$ is negative.

Now a well-known result of Fourier analysis may be applied. Let $F(x)$ denote the *discrete Fourier transform* of the sequence x, and let $F^{-1}(x)$ denote the inverse discrete Fourier transform of x:

$$F_k(x) := \sum_{j=0}^{N-1} x_j e^{-2\pi ijk/N}$$

$$F_k^{-1}(x) := \frac{1}{N}\sum_{j=0}^{N-1} x_j e^{2\pi ijk/N}.$$

Then the "convolution theorem," whose proof is a straightforward exercise, states that

$$F[C(x, y)] = F(x)F(y)$$

or, expressed another way,

$$C(x, y) = F^{-1}[F(x)F(y)].$$

Thus the entire multiplication pyramid z can be obtained by performing two forward discrete Fourier transforms, one vector complex multiplication and one inverse transform, each of length $N = 2n$. Once the real parts of the resulting complex numbers have been rounded to the nearest integer, the final multiprecision product may be obtained by merely releasing the carries modulo b. This may be done by starting at the end of the z vector and working backward, as in elementary school arithmetic, or by applying other schemes suitable for vector processing on more sophisticated computers.

A straightforward implementation of the above procedure would not result in any computational savings—in fact, it would be several times more costly than the usual "schoolboy" scheme. The reason this scheme is used is that the discrete Fourier transform may be computed much more rapidly using some variation of the well-known "fast Fourier transform" (FFT) algorithm [13]. In particular, if $N = 2^m$, then the discrete Fourier transform may be evaluated in only $5m2^m$ arithmetic operations using an FFT. Direct application of the definition of the discrete Fourier transform would require 2^{2m+3} floating-point arithmetic operations, even if it is assumed that all powers of $e^{-2\pi i/N}$ have been precalculated.

This is the basic scheme for high-speed multiprecision multiplication. Many details of efficient implementations have been omitted. For example, it is possible to take advantage of the fact that the input sequences x and y and the output sequence z are all purely real numbers, and thereby sharply reduce the operation count. Also, it is possible to dispense with complex numbers altogether in favor of performing computations in fields of integers modulo large prime numbers. Interested readers are referred to [2], [8], [13], and [22].

When the costs of all the constituent operations, using the best known techniques, are totalled both Algorithms 1 and 2 compute n digits of π with bit complexity $O(M(n)\log n)$, and use $O(\log n)$ full precision operations.

The bit complexity for Sum 1, or for π using any of the arctan expansions, is between $O((\log n)^2 M(n))$ and $O(nM(n))$ depending on implementation. In each case, one is required to sum $O(n)$ terms of the appropriate series. Done naively, one obtains the latter bound. If the calculation is carefully orchestrated so that the terms are grouped to grow evenly in size (as rational numbers) then one can achieve the former bound, but with no corresponding reduction in the number of operations.

The Archimedean iteration of section 2 converges like $1/4^n$ so in excess of n iterations are needed for n-digit accuracy, and the bit complexity is $O(nM(n))$.

Almost any familiar transcendental number such as e, γ, $\zeta(3)$, or Catalan's constant (presuming the last three to be nonalgebraic) can be computed with bit complexity $O((\log n)M(n))$ or $O((\log n)^2 M(n))$. None of these numbers is known to be computable essentially any faster than this. In light of the previous observation that algebraic numbers are all computable with bit complexity $O(M(n))$, a proof that π cannot be computed with this speed would imply the transcendence of π. It would, in fact, imply more, as there are transcendental numbers which have complexity $O(M(n))$. An example is $0.10100100001\ldots$.

It is also reasonable to speculate that computing the nth digit of π is not very much easier than computing all the first n digits. We think it very probable that computing the nth digit of π cannot be $O(n)$.

5. The Miracle of Theta Functions

When I was a student, abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics, and each of us was ambitious to make progress in this field. And now? The younger generation hardly knows abelian functions.

Felix Klein **[21]**

Felix Klein's lament from a hundred years ago has an uncomfortable timelessness to it. Sadly, it is now possible never to see what Bochner referred to as "the miracle of the theta functions" in an entire university mathematics program. A small piece of this miracle is required here [**6**], [**11**], [**28**]. First some standard notations. The *complete elliptic integrals of the first and second kind*, respectively,

$$K(k) := \int_0^{\pi k} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \tag{5.1}$$

and

$$E(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt. \tag{5.2}$$

The second integral arises in the rectification of the ellipse, hence the name elliptic integrals. The *complementary modulus* is

$$k' := \sqrt{1 - k^2}$$

and the *complementary integrals* K' and E' are defined by

$$K'(k) := K(k') \quad \text{and} \quad E'(k) := E(k').$$

The first remarkable identity is *Legendre's relation* namely

$$E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \frac{\pi}{2} \tag{5.3}$$

(for $0 < k < 1$), which is pivotal in relating these quantities to pi. We also need to define two *Jacobian theta functions*

$$\Theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \tag{5.4}$$

and

$$\Theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{5.5}$$

These are in fact specializations with ($t = 0$) of the general theta functions. More generally

$$\Theta_3(t, q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{imt} \qquad (\operatorname{im} t > 0)$$

with similar extensions of Θ_2. In Jacobi's approach these general theta functions provide the basic building blocks for elliptic functions, as functions of t (see **[11]**, [39]).

The complete elliptic integrals and the special theta functions are related as follows. For $|q| < 1$

$$K(k) = \frac{\pi}{2}\Theta_3^2(q) \tag{5.6}$$

and

$$E(k) = (k')^2\left[K(k) + k\frac{dK(k)}{dk}\right], \tag{5.7}$$

where

$$k := k(q) = \frac{\Theta_2^2(q)}{\Theta_3^2(q)}, \qquad k' := k'(q) = \frac{\Theta_3^2(-q)}{\Theta_3^2(q)} \tag{5.8}$$

and

$$q = e^{-\pi K'(k)/K(k)}. \tag{5.9}$$

The *modular function* λ is defined by

$$\lambda(t) := \lambda(q) := k^2(q) := \left[\frac{\Theta_2(q)}{\Theta_3(q)}\right]^4, \tag{5.10}$$

where

$$q := e^{i\pi t}.$$

We wish to make a few comments about modular functions in general before restricting our attention to the particular modular function λ. *Modular functions* are functions which are meromorphic in H, the upper half of the complex plane, and which are invariant under a group of linear fractional transformations, G, in the sense that

$$f(g(z)) = f(z) \qquad \forall g \in G.$$

[Additional growth conditions on f at certain points of the associated fundamental region (see below) are also demanded.] We restrict G to be a subgroup of the *modular group* Γ where Γ is the set of all transformations w of the form

$$w(t) = \frac{at+b}{ct+d},$$

with a, b, c, d integers and $ad - bc = 1$. Observe that Γ is a group under composition. A *fundamental region* F_G is a set in H with the property that any element in H is uniquely the image of some element in F_G under the action of G. Thus the behaviour of a modular function is uniquely determined by its behaviour on a fundamental region.

Modular functions are, in a sense, an extension of elliptic (or doubly periodic) functions—functions such as sn which are invariant under linear transformations and which arise naturally in the inversion of elliptic integrals.

The definitions we have given above are not complete. We will be more precise in our discussion of λ. One might bear in mind that much of the theory for λ holds in considerably greater generality.

The *fundamental region* F we associate with λ is the set of complex numbers

$$F := \{\operatorname{im} t \geq 0\} \cap \big[\{|\operatorname{re} t| < 1 \text{ and } |2t \pm 1| > 1\} \cup \{\operatorname{re} t = -1\} \cup \{|2t+1| = 1\}\big].$$

The λ-*group* (or theta-subgroup) is the set of linear fractional transformations w satisfying

$$w(t) := \frac{at+b}{ct+d},$$

where a, b, c, d are integers and $ad - bc = 1$, while in addition a and d are odd and b and c are even. Thus the corresponding matrices are unimodular. What makes λ a λ-modular function is the fact that λ is meromorphic in $\{\operatorname{im} t > 0\}$ and that

$$\lambda(w(t)) := \lambda(t)$$

for all w in the λ-group, plus the fact that λ tends to a definite limit (possibly infinite) as t tends to a vertex of the fundamental region (one of the three points $(0, -1)$, $(0, 0)$, (i, ∞)). Here we only allow convergence from within the fundamental region.

Now some of the miracle of modular functions can be described. Largely because every point in the upper half plane is the image of a point in F under an element of the λ-group, one can deduce that any λ-modular function that is bounded on F is constant. Slightly further into the theory, but relying on the above, is the result that any two modular functions are algebraically related, and resting on this, but further again into the field, is the following remarkable result. Recall that q is given by (5.9).

THEOREM 1. *Let z be a primitive pth root of unity for p an odd prime. Consider the pth order modular equation for λ as defined by*

$$W_p(x, \lambda) := (x - \lambda_0)(x - \lambda_1) \cdots (x - \lambda_p), \tag{5.11}$$

where

$$\lambda_i := \lambda(z^i q^{1/p}) \qquad i < p$$

and

$$\lambda_p := \lambda(q^p).$$

*Then the function W_p is a polynomial in x and λ (**independent** of q), which has integer coefficients and is of degree $p+1$ in both x and λ.*

The modular equation for λ usually has a simpler form in the associated variables $u := x^{1/8}$ and $v := \lambda^{1/8}$. In this form the 5th-order modular equation is given by

$$\Omega_5(u,v) := u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4). \tag{5.12}$$

In particular

$$\frac{\Theta_2(q^p)}{\Theta_3(q^p)} = v^2 \quad \text{and} \quad \frac{\Theta_2(q)}{\Theta_3(q)} = u^2$$

are related by an algebraic equation of degree $p+1$.

The miracle is not over. The *pth-order multiplier* (for λ) is defined by

$$M_p(k(q), k(q^p)) := \frac{K(k(q^p))}{K(k(q))} = \left[\frac{\Theta_3(q^p)}{\Theta_3(q)}\right]^2 \tag{5.13}$$

and turns out to be a rational function of $k(q^p)$ and $k(q)$.

One is now in possession of a pth-order algorithm for K/π, namely: Let $k_i := k(q^{p^i})$. Then

$$\frac{2K(k_0)}{\pi} = M_p^{-1}(k_0, k_1) M_p^{-1}(k_1, k_2) M_p^{-1}(k_2, k_3) \cdots.$$

This is an entirely algebraic algorithm. One needs to know the pth-order modular equation for λ to compute k_{i+1} from k_i and one needs to know the rational multiplier M_p. The speed of convergence ($O(c^{p^i})$, for some $c < 1$) is easily deduced from (5.13) and (5.9).

The function $\lambda(t)$ is 1-1 on F and has a well-defined inverse, λ^{-1}, with branch points only at 0, 1 and ∞. This can be used to provide a one line proof of the "big" Picard theorem that a nonconstant entire function misses at most one value (as does exp). Indeed, suppose g is an entire function and that it is never zero or one; then $\exp(\lambda^{-1}(g(z)))$ is a bounded entire function and is hence constant.

Littlewood suggested that, at the right point in history, the above would have been a strong candidate for a 'one line doctoral thesis'.

6. Ramanujan's Solvable Modular Equations. Hardy [19] commenting on Ramanujan's work on elliptic and modular functions says

It is here that both the profundity and limitations of Ramanujan's knowledge stand out most sharply.

We present only one of Ramanujan's modular equations.

THEOREM 2.

$$\frac{5\Theta_3(q^{25})}{\Theta_3(q)} = 1 + r_1^{1/5} + r_2^{1/5}, \tag{6.1}$$

where for $i = 1$ *and* 2

$$r_i := \tfrac{1}{2}x\left(y \pm \sqrt{y^2 - 4x^3}\right)$$

with

$$x := \frac{5\Theta_3(q^5)}{\Theta_3(q)} - 1 \quad \text{and} \quad y := (x-1)^2 + 7.$$

This is a slightly rewritten form of entry 12(iii) of Chapter 19 of Ramanujan's *Second Notebook* (see [7], where Berndt's proofs may be studied). One can think of Ramanujan's quintic modular equation as an equation in the multiplier M_p of (5.13). The initial surprise is that it is solvable. The quintic modular relation for λ, W_5, and the related equation for $\lambda^{1/8}$, Ω_5, are both nonsolvable. The Galois group of the sixth-degree equation Ω_5 (see (5.12)) over $\mathbf{Q}(v)$ is A_5 and is nonsolvable. Indeed both Hermite and Kronecker showed, in the middle of the last century, that the solution of a general quintic may be effected in terms of the solution of the 5th-order modular equation (5.12) and the roots may thus be given in terms of the theta functions.

In fact, in general, the Galois group for W_p of (5.11) has order $p(p+1)(p-1)$ and is never solvable for $p \geq 5$. The group is quite easy to compute, it is generated by two permutations. If

$$q := e^{i\pi\tau}, \quad \text{then} \quad \tau \to \tau + 2 \quad \text{and} \quad \tau \to \frac{\tau}{(2\tau + 1)}$$

are both elements of the λ-group and induce permutations on the λ_i of Theorem 1. For any fixed p, one can use the q-expansion of (5.10) to compute the effect of these transformations on the λ_i, and can thus easily write down the Galois group.

While W_p is not solvable over $\mathbf{Q}(\lambda)$, it is solvable over $\mathbf{Q}(\lambda, \lambda_0)$. Note that λ_0 is a root of W_p. It is of degree $p + 1$ because W_p is irreducible. Thus the Galois group for W_p over $\mathbf{Q}(\lambda, \lambda_0)$ has order $p(p-1)$. For $p = 5$, 7, and 11 this gives groups of order 20, 42, and 110, respectively, which are obviously solvable and, in fact, for general primes, the construction always produces a solvable group.

From (5.8) and (5.10) one sees that Ramanujan's modular equation can be rewritten to give λ_5 solvable in terms of λ_0 and λ. Thus, we can hope to find an explicit solvable relation for λ_p in terms of λ and λ_0. For $p = 3$, W_p is of degree 4 and is, of course, solvable. For $p = 7$, Ramanujan again helps us out, by providing a solvable seventh-order modular identity for the closely related *eta function* defined by

$$\eta(q) := q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}).$$

The first interesting prime for which an explicit solvable form is not known is the "endecadic" ($p = 11$) case. We consider only prime values because for nonprime values the modular equation factors.

This leads to the interesting problem of mechanically constructing these equations. In principle, and to some extent in practice, this is a purely computational problem. Modular equations can be computed fairly easily from (5.11) and even more easily in the associated variables u and v. Because one knows a priori bounds on the size of the (integer) coefficients of the equations one can perform these calculations exactly. The coefficients of the equation, in the variables u and v, grow at most like 2^n. (See [11].) Computing the solvable forms and the associated computational problems are a little more intricate—though still in principle entirely mechanical. A word of caution however: in the variables u and v the endecadic modular equation has largest coefficient 165, a three digit integer. The endecadic modular equation for the intimately related function J (Klein's *absolute invariant*) has coefficients as large as

$$27090964785531389931563200281035226311929052227303 \times 2^{92}3^{19}5^{20}11^{2}53.$$

It is, therefore, one thing to solve these equations, it is entirely another matter to present them with the economy of Ramanujan.

The paucity of Ramanujan's background in complex analysis and group theory leaves open to speculation Ramanujan's methods. The proofs given by Berndt are difficult. In the seventh-order case, Berndt was aided by MACSYMA—a sophisticated algebraic manipulation package. Berndt comments after giving the proof of various seventh-order modular identities:

> *Of course, the proof that we have given is quite unsatisfactory because it is a verification that could not have been achieved without knowledge of the result. Ramanujan obviously possessed a more natural, transparent, and ingenious proof.*

7. Modular Equations and Pi. We wish to connect the modular equations of Theorem 1 to pi. This we contrive via the function *alpha* defined by:

$$\alpha(r) := \frac{E'(k)}{K(k)} - \frac{\pi}{(2K(k))^2}, \tag{7.1}$$

where

$$k := k(q) \quad \text{and} \quad q := e^{-\pi\sqrt{r}}.$$

This allows one to rewrite Legendre's equation (5.3) in a one-sided form without the conjugate variable as

$$\frac{\pi}{4} = K\left[\sqrt{r}E - (\sqrt{r} - \alpha(r))K\right]. \tag{7.2}$$

We have suppressed, and will continue to suppress, the k variable. With (5.6) and (5.7) at hand we can write a q-expansion for α, namely,

$$\alpha(r) = \frac{\dfrac{1}{\pi} - \sqrt{r}\,4\dfrac{\sum_{n=-\infty}^{\infty} n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty} (-q)^{n^2}}}{\left[\sum_{n=-\infty}^{\infty} q^{n^2}\right]^4}, \tag{7.3}$$

and we can see that as r tends to infinity $q = e^{-\pi\sqrt{r}}$ tends to zero and $\alpha(r)$ tends to $1/\pi$. In fact

$$\alpha(r) - \frac{1}{\pi} \approx 8\left(\sqrt{r} - \frac{1}{\pi}\right)e^{-\pi\sqrt{r}}. \tag{7.4}$$

The key now is iteratively to calculate α. This is the content of the next theorem.

THEOREM 3. *Let* $k_0 := k(q)$, $k_1 := k(q^p)$ *and* $M_p := M_p(k_0, k_1)$ *as in* (5.13). *Then*

$$\alpha(p^2r) = \frac{\alpha(r)}{M_p^2} - \sqrt{r}\left[\frac{k_0^2}{M_p^2} - pk_1^2 + \frac{pk_1'^2k_1\dot{M}_p}{M_p}\right],$$

where $\dot{}$ *represents the full derivative of* M_p *with respect to* k_0. *In particular,* α *is algebraic for rational arguments.*

We know that $K(k_1)$ is related via M_p to $K(k)$ and we know that $E(k)$ is related via differentiation to K. (See (5.7) and (5.13).) Note that $q \to q^p$ corresponds to $r \to p^2r$. Thus from (7.2) some relation like that of the above theorem must exist. The actual derivation requires some careful algebraic manipulation. (See **[11]**, where it has also been made entirely explicit for $p := 2, 3, 4, 5$, and 7, and where numerous algebraic values are determined for $\alpha(r)$.) In the case $p := 5$ we can specialize with some considerable knowledge of quintic modular equations to get:

THEOREM 4. *Let* $s := 1/M_5(k_0, k_1)$. *Then*

$$\alpha(25r) = s^2\alpha(r) - \sqrt{r}\left[\frac{(s^2-5)}{2} + \sqrt{s(s^2-2s+5)}\right].$$

This couples with Ramanujan's quintic modular equation to provide a derivation of Algorithm 2.

Algorithm 1 results from specializing Theorem 3 with $p := 4$ and coupling it with a quartic modular equation. The quartic equation in question is just two steps of the corresponding quadratic equation which is Legendre's form of the *arithmetic geometric mean iteration*, namely:

$$k_1 = \frac{2\sqrt{k}}{1+k}.$$

An algebraic pth-order algorithm for π is derived from coupling Theorem 3 with a pth-order modular equation. The substantial details which are skirted here are available in **[11]**.

8. Ramanujan's sum. This amazing sum,

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{[1103 + 26390n]}{396^{4n}}$$

is a specialization ($N = 58$) of the following result, which gives reciprocal series for π in terms of our function alpha and related modular quantities.

THEOREM 5.

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n d_n(N)}{(n!)^3} x_N^{2n+1}, \tag{8.1}$$

where,

$$x_N := \frac{4k_N(k'_N)^2}{(1+k_N^2)^2} := \left(\frac{g_N^{12}+g_N^{-12}}{2}\right)^{-1},$$

with

$$d_n(N) = \left[\frac{\alpha(N)x_N^{-1}}{1+k_N^2} - \frac{\sqrt{N}}{4}g_N^{-12}\right] + n\sqrt{N}\left(\frac{g_N^{12}-g_N^{-12}}{2}\right)$$

and

$$k_N := k(e^{-\pi\sqrt{N}}), \qquad g_N^{12} = (k'_N)^2/(2k_N).$$

Here $(c)_n$ *is the rising factorial*: $(c)_n := c(c+1)(c+2)\cdots(c+n-1)$.

Some of the ingredients for the proof of Theorem 5, which are detailed in [11], are the following. Our first step is to write (7.2) as a sum after replacing the E by K and dK/dk using (5.7). One then uses an identity of Clausen's which allows one to write the square of a hypergeometric function ${}_2F_1$ in terms of a generalized hypergeometric ${}_3F_2$, namely, for all k one has

$$\begin{aligned}(1+k^2)\left[\frac{2K(k)}{\pi}\right]^2 &= {}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2},1,1:\left(\frac{2}{g^{12}+g^{-12}}\right)^2\right)\\ &= \sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n\left(\frac{1}{2}\right)_n}{(1)_n(1)_n}\frac{\left(\frac{2}{g^{12}+g^{-12}}\right)^{2n}}{n!}.\end{aligned}$$

Here g is related to k by

$$\frac{4k(k')^2}{(1+k^2)^2} = \left(\frac{g^{12}+g^{-12}}{2}\right)^{-1}$$

as required in Theorem 5. We have actually done more than just use Clausen's identity, we have also transformed it once using a standard hypergeometric substitution due to Kummer. Incidentally, Clausen was a nineteenth-century mathematician who, among other things, computed 250 digits of π in 1847 using Machin's formula. The desired formula (8.1) is obtained on combining these pieces.

Even with Theorem 5, our work is not complete. We still have to compute

$$k_{58} := k(e^{-\pi\sqrt{58}}) \quad \text{and} \quad \alpha_{58} := \alpha(58).$$

In fact

$$g_{58}^2 = \left(\frac{\sqrt{29}+5}{2}\right)$$

is a well-known *invariant* related to the fundamental solution to Pell's equation for 29 and it turns out that

$$\alpha_{58} = \left(\frac{\sqrt{29}+5}{2}\right)^6 (99\sqrt{29} - 444)(99\sqrt{2} - 70 - 13\sqrt{29}).$$

One can, in principle, and for $N := 58$, probably in practice, solve for k_N by directly solving the Nth-order equation

$$W_N(k_N^2, 1 - k_N^2) = 0.$$

For $N = 58$, given that Ramanujan [26] and Weber [38] have calculated g_{58} for us, verification by this method is somewhat easier though it still requires a tractable form of W_{58}. Actually, more sophisticated number-theoretic techniques exist for computing k_N (these numbers are called *singular moduli*). A description of such techniques, including a reconstruction of how Ramanujan might have computed the various singular moduli he presents in [26]; is presented by Watson in a long series of papers commencing with [36]; and some more recent derivations are given in [11] and [30]. An inspection of Theorem 5 shows that all the constants in Series 1 are determined from g_{58}. Knowing α is equivalent to determining that the number 1103 is correct.

It is less clear how one explicitly calculates α_{58} in algebraic form, except by brute force, and a considerable amount of brute force is required; but a numerical calculation to any reasonable accuracy is easily obtained from (7.3) and 1103 appears! The reader is encouraged to try this to, say, 16 digits. This presumably is what Ramanujan observed. Ironically, when Gosper computed 17 million digits of π using Sum 1, he had no mathematical proof that Sum 1 actually converged to $1/\pi$. He compared ten million digits of the calculation to a previous calculation of Kanada et al. This verification that Sum 1 is correct to ten million places also provided the first complete proof that α_{58} is as advertised above. A nice touch—that the calculation of the sum should prove itself as it goes.

Roughly this works as follows. One knows enough about the exact algebraic nature of the components of $d_n(N)$ and x_N to know that if the purported sum (of positive terms) were incorrect, that before one reached 3 million digits, this sum must have ceased to agree with $1/\pi$. Notice that the components of Sum 1 are related to the solution of an equation of degree 58, but virtually no irrationality remains in the final packaging. Once again, there are very good number-theoretic reasons, presumably unknown to Ramanujan, why this must be so (58 is at least a good candidate number for such a reduction). Ramanujan's insight into this marvellous simplification remains obscure.

Ramanujan [26] gives 14 other series for $1/\pi$, some others almost as spectacular as Sum 1—and one can indeed derive some even more spectacular related series.* He gives almost no explanation as to their genesis, saying only that there are "corresponding theories" to the standard theory (as sketched in section 5) from which they follow. Hardy, quoting Mordell, observed that "it is unfortunate that Ramanujan has not developed the corresponding theories." By methods analogous

*(Added in proof) Many related series due to Borwein and Borwein and to Chudnovsky and Chudnovsky appear in papers in *Ramanujan Revisited*, Academic Press, 1988.

to those used above, all his series can be derived from the classical theory [11]. Again it is unclear what passage Ramanujan took to them, but it must in some part have diverged from ours.

We conclude by writing down another extraordinary series of Ramanujan's, which also derives from the same general body of theory,

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{2^{12n+4}}.$$

This series is composed of fractions whose numerators grow like 2^{6n} and whose denominators are exactly $16 \cdot 2^{12n}$. In particular this can be used to calculate the second block of n binary digits of π without calculating the first n binary digits. This beautiful observation, due to Holloway, results, disappointingly, in no intrinsic reduction in complexity.

9. Sources. References [7], [11], [19], [26], [36], and [37] relate directly to Ramanujan's work. References [2], [8], [9], [10], [12], [22], [24], [27], [29], and [31] discuss the computational concerns of the paper.

Material on modular functions and special functions may be pursued in [1], [6], [9], [14], [15], [18], [20], [28], [34], [38], and [39]. Some of the number-theoretic concerns are touched on in [3], [6], [9], [11], [16], [23], and [35].

Finally, details of all derivations are given in [11].

REFERENCES

1. M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, Dover, New York, 1964.
2. D. H. Bailey, The Computation of π to 29,360,000 decimal digits using Borweins' quartically convergent algorithm, *Math. Comput.*, 50 (1988) 283–96.
3. ______, Numerical results on the transcendence of constants involving π, e, and Euler's constant, *Math. Comput.*, 50 (1988) 275–81.
4. A. Baker, Transcendental Number Theory, Cambridge Univ. Press, London, 1975.
5. P. Beckmann, A History of Pi, 4th ed., Golem Press, Boulder, CO, 1977.
6. R. Bellman, A Brief Introduction to Theta Functions, Holt, Reinhart and Winston, New York, 1961.
7. B. C. Berndt, Modular Equations of Degrees 3, 5, and 7 and Associated Theta Functions Identities, chapter 19 of Ramanujan's Second Notebook, Springer—to be published.
8. A. Borodin and I. Munro, The Computational Complexity of Algebraic and Numeric Problems, American Elsevier, New York, 1975.
9. J. M. Borwein and P. B. Borwein, The arithmetic-geometric mean and fast computation of elementary functions, *SIAM Rev.*, 26 (1984), 351–365.
10. ______, An explicit cubic iteration for pi, *BIT*, 26 (1986) 123–126.
11. ______, Pi and the AGM—A Study in Analytic Number Theory and Computational Complexity, Wiley, N.Y., 1987.
12. R. P. Brent, Fast multiple-precision evaluation of elementary functions, *J. ACM*, 23 (1976) 242–251.
13. E. O. Brigham, The Fast Fourier Transform, Prentice-Hall, Englewood Cliffs, N.J., 1974.
14. A. Cayley, An Elementary Treatise on Elliptic Functions, Bell and Sons, 1885; reprint Dover, 1961.
15. A. Cayley, A memoir on the transformation of elliptic functions, *Phil. Trans. T.*, 164 (1874) 397–456.
16. D. V. Chudnovsky and G. V. Chudnovsky, Padé and Rational Approximation to Systems of Functions and Their Arithmetic Applications, Lecture Notes in Mathematics 1052, Springer, Berlin, 1984.
17. H. R. P. Ferguson and R. W. Forcade, Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two, *Bull. AMS*, 1 (1979) 912–914.
18. C. F. Gauss, Werke, Göttingen 1866–1933, Bd 3, pp. 361–403.

19. G. H. Hardy, Ramanujan, Cambridge Univ. Press, London, 1940.
20. L. V. King, On The Direct Numerical Calculation of Elliptic Functions and Integrals, Cambridge Univ. Press, 1924.
21. F. Klein, Development of Mathematics in the 19th Century, 1928, Trans Math Sci. Press, R. Hermann ed., Brookline, MA, 1979.
22. D. Knuth, The Art of Computer Programming, vol. 2: Seminumerical Algorithms, Addison-Wesley, Reading, MA, 1981.
23. F. Lindemann, Über die Zahl π, *Math. Ann.*, 20 (1882) 213–225.
24. G. Miel, On calculations past and present: the Archimedean algorithm, *Amer. Math. Monthly*, 90 (1983) 17–35.
25. D. J. Newman, Rational Approximation Versus Fast Computer Methods, in Lectures on Approximation and Value Distribution, Presses de l'Université de Montreal, 1982, pp. 149–174.
26. S. Ramanujan, Modular equations and approximations to π, *Quart. J. Math*, 45 (1914) 350–72.
27. E. Salamin, Computation of π using arithmetic-geometric mean, *Math. Comput.*, 30 (1976) 565–570.
28. B. Schoenberg, Elliptic Modular Functions, Springer, Berlin, 1976.
29. A. Schönhage and V. Strassen, Schnelle Multiplikation Grosser Zahlen, *Computing*, 7 (1971) 281–292.
30. D. Shanks, Dihedral quartic approximations and series for π, *J. Number Theory*, 14 (1982) 397–423.
31. D. Shanks and J. W. Wrench, Calculation of π to 100,000 decimals, *Math Comput.*, 16 (1962) 76–79.
32. W. Shanks, Contributions to Mathematics Comprising Chiefly of the Rectification of the Circle to 607 Places of Decimals, G. Bell, London, 1853.
33. Y. Tamura and Y. Kanada, Calculation of π to 4,196,393 decimals based on Gauss-Legendre algorithm, preprint (1983).
34. J. Tannery and J. Molk, Fonctions Elliptiques, vols. 1 and 2, 1893; reprint Chelsea, New York, 1972.
35. S. Wagon, Is π normal?, *The Math Intelligencer*, 7 (1985) 65–67.
36. G. N. Watson, Some singular moduli (1), *Quart. J. Math.*, 3 (1932) 81–98.
37. ______, The final problem: an account of the mock theta functions, *J. London Math. Soc.*, 11 (1936) 55–80.
38. H. Weber, Lehrbuch der Algebra, Vol. 3, 1908; reprint Chelsea, New York, 1980.
39. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, 4th ed, Cambridge Univ. Press, London, 1927.

Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. 96, No. 8, October 1989

Pi, Euler Numbers, and Asymptotic Expansions

J. M. BORWEIN, P. B. BORWEIN, and K. DILCHER[1], Dalhousie University, Halifax, Canada

JONATHAN M. BORWEIN was an Ontario Rhodes Scholar (1971) at Jesus College, Oxford, where he completed a D. Phil. (1974) with Michael Dempster. Since 1974 he has worked at Dalhousie University where he is professor of mathematics. He has also been on faculty at Carnegie-Mellon University (1980–82). He was the 1987 Coxeter-James lecturer of the Canadian Mathematical Society and was awarded the Atlantic Provinces Council on the Sciences 1988 Gold Medal for Research. His research interests include functional analysis, classical analysis, and optimization theory.

PETER B. BORWEIN obtained a Ph.D. (1979) from the University of British Columbia, under the supervision of David Boyd. He spent 1979–80 as a NATO research fellow in Oxford. Since then he has been on faculty at Dalhousie (except for a sabbatical year at the University of Toronto) and is now Associate Professor of Mathematics. His research interests include approximation theory, classical analysis, and complexity theory.

KARL DILCHER received his undergraduate education and a Dipl. Math. degree at Technische Universität Clausthal in West Germany. He completed his Ph.D (1983) at Queen's University in Kingston, Ontario with Paulo Ribenboim. Since 1984 he has been teaching at Dalhousie, where he is now Assistant Professor. His research interests include Bernoulli numbers and polynomials, and classical complex analysis.

1. Introduction. Gregory's series for π, truncated at 500,000 terms, gives to forty places

$$4\sum_{k=1}^{500,000}\frac{(-1)^{k-1}}{2k-1}=3.14159\underline{0}65358979324\underline{0}46264338326\underline{9}502884197.$$

The number on the right is not π to forty places. As one would expect, the 6th digit after the decimal point is wrong. The surprise is that the next 10 digits are correct. In fact, only the 4 underlined digits aren't correct. This intriguing observation was sent to us by R. D. North [10] of Colorado Springs with a request for an explanation. The point of this article is to provide that explanation. Two related

[1]Research of the authors supported in part by NSERC of Canada.

examples, to fifty digits, are

$$\frac{\pi}{2} \doteq 2 \sum_{k=1}^{50,000} \frac{(-1)^{k-1}}{2k-1}$$
$$= 1.5707\underset{1}{\underline{8}}632679489\underset{-1}{\underline{7}}619231321\underset{5}{\underline{1}}9163975\underset{-61}{\underline{205}}209858\underline{33147388}$$

and

$$\log 2 \doteq \sum_{k=1}^{50,000} \frac{(-1)^{k+1}}{k}$$
$$= .6931\underset{1}{\underline{3}}7180\underset{-1}{\underline{6}}5994530\underset{2}{\underline{39}}72321214\underset{-16}{\underline{74}}1765680\underset{272}{\underline{483}}00134\underline{46572},$$

where all but the underlined digits are correct. The numbers under the underlined digits are the numbers that must be added to correct these. The numbers 1, -1, 5, -61 are the first four *Euler numbers* while 1, -1, 2, -16, 272 are the first five *tangent numbers*. Our process of discovery consisted of generating these sequences and then identifying them with the aid of Sloane's *Handbook of Integer Sequences* [**11**]. What one is observing, in each case, is an asymptotic expansion of the error in Euler summation. The amusing detail is that the coefficients of the expansion are integers. All of this is explained by Theorem 1.

The standard facts we need about the *Euler numbers* $\{E_i\}$, the *tangent numbers* $\{T_i\}$, and the *Bernoulli numbers* $\{B_i\}$, may all be found in [1] or in [6]. The numbers are defined as the coefficients of the power series

$$\sec z = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n} z^{2n}}{(2n)!}, \tag{1.1}$$

$$\tan z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{T_{2n+1} z^{2n+1}}{(2n+1)!} \qquad \text{and } T_0 = 1, \tag{1.2}$$

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}. \tag{1.3}$$

They satisfy the relations

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0, \qquad E_{2n+1} = 0, \tag{1.4}$$

$$B_n = \frac{-nT_{n-1}}{2^n(2^n - 1)} \qquad n \geqslant 1, \tag{1.5}$$

and

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0. \tag{1.6}$$

These three identities allow for the easy generation of $\{E_n\}$, $\{T_n\}$, and $\{B_n\}$. The first few values are recorded below.

n	0	1	2	3	4	5	6	7	8
E_n	1	0	-1	0	5	0	-61	0	1365
T_n	1	-1	0	2	0	-16	0	272	0
B_n	1	$\frac{-1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$

It is clear from (1.4) that the Euler numbers are integral. From (1.5) and (1.6) it follows that the tangent numbers are integers. Also,

$$|E_{2n}| \sim \frac{4^{n+1}(2n)!}{\pi^{2n+1}} \quad \text{and} \quad |B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}}$$

as follows from (5.1) and (5.2) below. The main content of this note is the following theorem. The simple proof we offer relies on the Boole Summation Formula, which is a pretty but less well-known analogue of Euler summation. The details are contained in Sections 2 and 3 (except for c] which is a straightforward application of Euler summation). More complicated developments can be based directly on Euler summation or on results in [9].

THEOREM 1. *The following asymptotic expansions hold*:

a] $$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k-1}}{2k-1} \sim \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}} = \frac{1}{N} - \frac{1}{N^3} + \frac{5}{N^5} - \frac{61}{N^7} + \cdots$$

b] $$\log 2 - \sum_{k=1}^{N/2} \frac{(-1)^{k-1}}{k} \sim \frac{1}{N} + \sum_{m=1}^{\infty} \frac{T_{2m-1}}{N^{2m}} = \frac{1}{N} - \frac{1}{N^2} + \frac{2}{N^4} - \frac{16}{N^6} + \frac{272}{N^8} - \cdots$$

and

c] $$\frac{\pi^2}{6} - \sum_{k=1}^{N-1} \frac{1}{k^2} \sim \frac{1}{2N^2} + \sum_{m=0}^{\infty} \frac{B_{2m}}{N^{2m+1}} = \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{6N^3} - \frac{1}{30N^5} + \frac{1}{42N^7} \cdots .$$

From the asymptotics of $\{E_n\}$ and $\{B_n\}$ and (1.5) we see that each of the above infinite series is everywhere divergent; the correct interpretation of their asymptotics is

a'] $$\sum_{m=1}^{\infty} \frac{E_{2m}}{N^{2m}} = \sum_{m=1}^{K} \frac{E_{2m}}{N^{2m}} + 0\left(\frac{(2K+1)!}{(\pi N)^{2K+1}}\right)$$

b'] $$\sum_{m=1}^{\infty} \frac{T_{2m-1}}{N^{2m}} = \sum_{m=1}^{K} \frac{T_{2m-1}}{N^{2m}} + 0\left(\frac{(2K+1)!}{(\pi N)^{2K-1}}\right)$$

c'] $$\sum_{m=1}^{\infty} \frac{B_{2m}}{N^{2m+1}} = \sum_{m=1}^{K} \frac{B_{2m}}{N^{2m+1}} + 0\left(\frac{(2K+1)!}{(2\pi N)^{2K+1}}\right),$$

where in each case the constant concealed by the order symbol is independent of N and K. In fact, the constant 10 works in all cases.

2. The Boole Summation Formula. The Euler polynomials $E_n(x)$ can be defined by the generating function

$$\frac{2e^{tx}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad (|t| < \pi); \tag{2.1}$$

(see [1, p. 804]). Each $E_n(x)$ is a polynomial of degree n with leading coefficient 1. We also define the periodic Euler function $\bar{E}_n(x)$ by

$$\bar{E}_n(x+1) = -\bar{E}_n(x)$$

for all x, and

$$\bar{E}_n(x) = E_n(x) \quad \text{for} \quad 0 \leqslant x < 1.$$

It can be shown that $\bar{E}_n(x)$ has continuous derivatives up to the $(n-1)$st order.

The following is known as **Boole's summation formula** (see, for example, [9, p. 34]).

LEMMA 1. *Let $f(t)$ be a function with m continuous derivatives, defined on the interval $x \leqslant t \leqslant x + \omega$. Then for $0 \leqslant h \leqslant 1$*

$$f(x+h\omega) = \sum_{k=0}^{m-1} \frac{\omega^k}{k!}E_k(h) \cdot \frac{1}{2}\left(f^{(k)}(x+\omega) + f^{(k)}(x)\right) + R_m,$$

where

$$R_m = \frac{1}{2}\omega^m \int_0^1 \frac{\bar{E}_{m-1}(h-t)}{(m-1)!} f^{(m)}(x+\omega t)\,dt.$$

This summation formula is easy to establish by repeated integration by parts of the above integral. It is remarked in [9, p. 26] that this formula was known to Euler, for polynomial f and without the remainder term. Also note that Lemma 1 turns into Taylor's formula with Lagrange's remainder term if we replace h by h/ω and let ω approach zero.

To derive a convenient version of Lemma 1 for the applications we have in mind, we set $\omega = 1$ and impose further restrictions on f.

LEMMA 2. *Let f be a function with m continuous derivatives, defined on $t \geqslant x$. Suppose that $f^{(k)}(t) \to 0$ as $t \to \infty$ for all $k = 0, 1, \dots, m$. Then for $0 \leqslant h \leqslant 1$*

$$\sum_{v=0}^{\infty} (-1)^v f(x+h+v) = \sum_{k=0}^{m-1} \frac{E_k(h)}{2k!} f^{(k)}(x) + R_m,$$

where

$$R_m = \frac{1}{2}\int_0^{\infty} \frac{\bar{E}_{m-1}(h-t)}{(m-1)!} f^{(m)}(x+t)\,dt.$$

3. The Remainder for Gregory's Series. The Euler numbers E_n may also be defined by the generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{3.1}$$

Comparing (3.1) with (2.1), we see that

$$E_n = 2^n E_n\left(\frac{1}{2}\right). \tag{3.2}$$

The phenomenon mentioned in the introduction is entirely explained by the next proposition—if we set $n = 500{,}000$. It is also clear that we will get similar patterns for $n = 10^m/2$ with any positive integer m.

PROPOSITION 1. *For positive integers n and M we have*

$$4\sum_{k=n}^{\infty} \frac{(-1)^k}{2k+1} = (-1)^n \sum_{k=0}^{M} \frac{2E_{2k}}{(2n)^{2k+1}} + R_1(M), \tag{3.3}$$

where

$$|R_1(M)| \leqslant \frac{2|E_{2M}|}{(2n)^{2M+1}}.$$

Proof. Apply Lemma 2 with $f(x) = 1/x$; then set $x = n$ and $h = 1/2$. We get

$$\sum_{v=0}^{\infty} \frac{(-1)^v}{n+v+1/2} = \sum_{k=0}^{m-1} \frac{E_k(1/2)}{2k!} \frac{(-1)^k k!}{n^{k+1}} + R_m, \tag{3.4}$$

with

$$R_m = \frac{1}{2}\int_0^{\infty} \frac{\bar{E}_{m-1}(h-t)}{(m-1)!} \frac{(-1)^m m!}{(x+t)^{m+1}} dt.$$

We multiply both sides of (3.4) by $2(-1)^n$. Then the left-hand side is seen to be identical with the left-hand side of (3.3). After replacing m by $2M + 1$ and taking into account (3.2) and the fact that odd-index Euler numbers vanish, we see that the first terms on the right-hand sides of (3.3) and (3.4) agree. To estimate the error term, we use the following inequality,

$$|E_{2M}(x)| \leqslant 2^{-2M}|E_{2M}| \quad \text{for } 0 \leqslant x \leqslant 1$$

(see, e.g., [1, p. 805]). Carrying out the integration now leads to the error estimate given in Proposition 1. □

4. An Analogue For log 2. Lemma 2 can also be used to derive a result similar to Proposition 1, concerning truncations of the series

$$\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}. \tag{4.1}$$

In this case the tangent numbers T_n will play the role of the E_n in Proposition 1. It follows from the identity

$$\tan z = \frac{1}{i}\left(\frac{2e^{2iz}}{e^{2iz}+1} - 1\right)$$

together with (1.2) and (2.1) that

$$T_n = (-1)^n 2^n E_n(1) \tag{4.2}$$

as in [9, p. 28]. The T_n can be computed using the recurrence relation $T_0 = 1$ and

$$\sum_{k=0}^{n} \binom{n}{k} 2^k T_{n-k} + T_n = 0 \quad \text{for } n \geqslant 1.$$

Other properties can be found, e.g., in [8] or [9, Ch. 2].

PROPOSITION 2. *For positive integers n and M we have*

$$\sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k} = (-1)^{n+1}\left\{\frac{1}{2n} + \sum_{k=1}^{M} \frac{T_{2k-1}}{(2n)^{2k}}\right\} + R_2(M), \tag{4.3}$$

where

$$|R_2(M)| \leqslant \frac{|E_{2M}|}{(2n)^{2M+1}}.$$

Proof. We proceed as in the proof of Proposition 1. Here we take $x = n$ and $h = 1$. Using (4.2) and the fact that $T_0 = 1$ and $T_{2k} = 0$ for $k \geqslant 1$, we get the summation on the right-hand side of (4.3). The remainder term is estimated as in the proof of Proposition 1. □

Using Proposition 2 with $n = 10^m/2$ one again gets many more correct digits of $\log 2$ than is suggested by the error term of the Taylor series.

5. Generalizations. Proposition 1 and 2 can be extended easily in two different directions.

i). The well-known infinite series (see, e.g., [1, p. 807])

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{|E_{2n}|}{2^{2n+2}(2n)!}\pi^{2n+1} \qquad (n = 0, 1, \ldots), \tag{5.1}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2n}} &= (1 - 2^{1-2n})\zeta(2n) \\ &= (2^{2n-1} - 1)\frac{|B_{2n}|}{(2n)!}\pi^{2n} \qquad (n = 1, 2, \ldots)\end{aligned} \tag{5.2}$$

can be considered as extensions of Gregory's series and of (4.1). These series admit exact analogues to Propositions 1 and 2; one only has to replace $f(x) = 1/x$ by $f(x) = x^{-(2n+1)}$, respectively $x^{-(2n)}$, in the proofs.

We note that the Euler-MacLaurin summation formula leads to similar results for

$$\sum_{k=1}^{\infty} k^{-2n} = \frac{|B_{2n}|2^{2n-1}}{(2n)!}\pi^{2n}, \tag{5.3}$$

where multiples of the Bernoulli numbers B_{2n} take the place of the E_n and T_n in Propositions 1 and 2.

ii). A generalization of the Euler-MacLaurin and Boole summation formulas was derived by Berndt [3]. This can be applied to character analogues of the series (5.1)–(5.3). The roles of the E_n and T_n in Proposition 1 and 2 are then played by generalized Bernoulli numbers or by related numbers.

6. Additional Comments. The phenomenon observed in the introduction results from taking N to be a power of ten; taking $N = 2 \cdot 10^m$ also leads to "clean" expressions. References [1], [5], [6], and [9] include the basic material on Bernoulli and Euler numbers, while [8] deals extensively with their calculation, and [2] describes an entertaining analogue of Pascal's triangle. Much on the calculation of pi and related matters may be found in [4]. Euler summation is treated in [5], [6], and [9], while Boole summation is treated in [9]. Related material on the computation and acceleration of alternating series is given in [7].

Added in Proof. A version of the phenomeon was observed by M. R. Powell and various explanations were offered (see *The Mathematical Gazette*, 66 (1982) 220–221, and 67(1983) 171–188).

REFERENCES

1. M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, Dover, N.Y., 1964.
2. M. D. Atkinson, How to compute the series expansions of sec x and tan x, *Amer. Math. Monthly*, 93 (1986) 387-388.
3. B. C. Berndt, Character analogues of the Poisson and Euler-MacLaurin summation formulas with applications, *J. Number Theory*, 7 (1975 413-445.
4. J. M. Borwein and P. B. Borwein, Pi and the AGM - A Study in Analytic Number Theory and Computational Complexity, Wiley, N.Y., 1987.
5. T. J. I'a Bromwich, An Introduction to the Theory of Infinite Series, 2nd ed., MacMillan, London, 1926.
6. R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematics, Addison Wesley, Reading, Mass., 1989.
7. R. Johnsonbaugh, Summing an alternating series, this MONTHLY, 86 (1979) 637-648.
8. D. E. Knuth and T. J. Buckholtz, Computation of Tangent, Euler, and Bernoulli numbers, *Math. Comput.*, 21 (1967) 663-688.
9. N. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer-Verlag, Berlin, 1924.
10. R. D. North, personal communications, 1988.
11. N. J. A. Sloane, A Handbook of Integer Sequences, Academic Press, New York, 1973.

An Alternative Proof of the Lindemann-Weierstrass Theorem

F. Beukers, J. P. Bézivin, P. Robba

Dedicated to the Memory of Philippe Robba

Frits Beukers: born in 1953. University education in Leiden, Netherlands. Currently teaching at the State University of Utrecht. Research interests are number theory and arithmetic properties of linear differential equations.

Jean-Paul Bézivin: educated at the École Normale Supérieure and the Universities of Paris VI and VII. He currently teaches at the University of Paris VI. Research interests are p-adic analysis, linear recurrent sequences and arithmetic properties of formal power series.

Philippe Robba: born in 1941, died on October 12, 1988. Doctorate degree at the University of Paris in 1972, and professor at the University Paris XI from 1975 until his untimely death. Domain of research: p-adic analysis, mainly p-adic differential equations with emphasis on p-adic cohomologies.

Introduction. In December 1987 J. P. Bézivin and Ph. Robba found a new proof of the Lindemann-Weierstrass theorem as a by-product of their criterion of rationality for solutions of differential equations. Let us recall the Lindemann-Weierstrass theorem, to which we shall refer as LW from now on.

Let $\alpha_1, \dots, \alpha_t, b_1, \dots, b_t$ be algebraic numbers such that the b_i are all nonzero and the α_i are mutually distinct. Then

$$b_1 e^{\alpha_1} + b_2 e^{\alpha_2} + \cdots + b_t e^{\alpha_t} \neq 0.$$

It is well known that the transcendence of π follows from LW in the following way. Suppose, on the contrary, that π is algebraic. Then so is $\pi\sqrt{-1}$ and LW now implies that $e^{\pi\sqrt{-1}} + 1 \neq 0$, which is certainly not true. Thus we conclude that π is transcendental.

The usual proof of LW is essentially due to Hilbert and has been polished by a number of authors. One such version can be found in [2, Ch. XI] or in [4, Ch. I], [3]. The new proof of Bézivin and Robba looks totally different. It can be considered as a direct consequence of their criterion of rationality for solutions of linear differential equations.

The Bézivin-Robba criterion is based on a theorem of Pólya-Bertrandias, which is far from easy and relies heavily on p- adic analysis. We refer the interested reader to [**1**]. In March 1988 F. Beukers found that the use of the Pólya-Bertrandias criterion is much too heavy and can be avoided in a very elementary way. The result is a new proof of LW which is elementary and can compete with the usual one in shortness and simplicity. There is always a possibility that the similarity between the proofs is stronger than one would expect at first sight. In fact, very soon after a first draft of this paper was written (May 1988) Yu. Nesterenko pointed out to us that the numbers $v_n(k)$ which we use are equal to the integrals

$$\sum_{j=1}^{t} b_j e^{\alpha_j} \int_{\alpha_j}^{\infty} e^{-x} x^{n-kt} (x-\alpha_1)^k \cdots (x-\alpha_t)^k \, dx$$

which are used in the Hilbert proof. In spite of such similarities we feel that the arguments of our proof are nice enough to present in front of a wider audience. We would like to thank the referee for several improvements upon our presentation.

THEOREM. *Let* $b_1, \ldots, b_t$, $\alpha_1, \ldots, \alpha_t \in \overline{\mathbb{Q}}$ *such that* $b_i \neq 0$ $\forall i$ *and the* α_i *are mutually distinct. Then*

$$b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} \neq 0.$$

Proof. Consider the Taylor series expansion

$$b_1 e^{\alpha_1 x} + \cdots + b_t e^{\alpha_t x} = \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n,$$

where, clearly,

$$u_n = \sum_{i=1}^{t} b_i \alpha_i^n. \tag{1}$$

Put $(X-\alpha_1)\cdots(X-\alpha_t) = X^t - a_1 X^{t-1} - \cdots - a_t$. Clearly, for any $i = 1, \ldots, t$ and any $n \in \mathbb{Z}_{\geq 0}$

$$\alpha_i^{t+n} = a_1 \alpha_i^{t+n-1} + \cdots + a_t \alpha_i^n.$$

By taking suitable linear combinations and using (1) it follows that

$$u_{n+t} = a_1 u_{n+t-1} + \cdots + a_t u_n. \tag{2}$$

Without loss of generality we may assume that $u_n \in \mathbb{Q}$, $\forall n$. If not, then consider the product

$$\prod_{\sigma} \left(\sigma(b_1) e^{\sigma(\alpha_1)x} + \cdots + \sigma(b_n) e^{\sigma(\alpha_t)x} \right)$$

taken over all $\sigma \in \mathrm{Gal}(\mathbb{Q}(b_1, \ldots, b_t, \alpha_1, \ldots, \alpha_t)/\mathbb{Q})$, which, after evaluation, again acquires the form

$$\sum_i b_i' e^{\alpha_i' x},$$

where now the sets $\{b_i'\}$, $\{\alpha_i'\}$ are Galois-stable. This implies that the corresponding numbers a_i' and u_n' are rational. So from now on we assume $u_n \in \mathbb{Q}$, $\forall n$ and

$a_i \in \mathbb{Q}$ $(i = 1, \ldots, t)$. Let D be a common denominator of the a_i. Put $A = \max(1, |\alpha_i|)$. After multiplication with a suitable integer, if necessary, we may assume $u_0, \ldots, u_{t-1} \in \mathbb{Z}$. Hence, using (2) recursively,

$$D^n u_n \in \mathbb{Z}, \tag{3}$$

and by (1),

$$|u_n| \leqslant c_1 A^n$$

for some $c_1 > 0$ and all $n \geqslant 0$. Now suppose that $b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} = 0$, or, equivalently,

$$\sum_{r=0}^{\infty} \frac{u_r}{r!} = 0.$$

Put

$$v_n = n! \sum_{r=0}^{n} \frac{u_r}{r!}$$

and notice

$$|v_n| = n! \left| \sum_{r=0}^{n} \frac{u_r}{r!} \right| = n! \left| \sum_{r=n+1}^{\infty} \frac{u_r}{r!} \right| \leqslant \frac{c_1}{n+1} \sum_{r=n+1}^{\infty} \frac{A^r}{(r = n+1)!} = c_2 \frac{a^{n+1}}{n+1}. \tag{4}$$

If we had $A = D = 1$, like in the high school proof of $e \notin \mathbb{Q}$, inequality (4) gives us a contradiction since we have both $v_n \in \mathbb{Z}$ and $|v_n| \leqslant c_2/(n+1)$, i.e. $v_n = 0$ for all sufficiently large n, in other words $\Sigma_0^{\infty} v_n X^n$ is a polynomial. In our general case a similar principle works.

Claim:

$$\sum_{n=0}^{\infty} v_n X^n \in \mathbb{Q}(X).$$

Assuming the claim we proceed with the proof. Define

$$v(X) = \sum_{n=0}^{\infty} v_n X^n.$$

Notice that

$$\frac{v_n}{n!} - \frac{v_{n-1}}{(n-1)!} = \frac{u_n}{n!} \quad \text{or} \quad v_n - n v_{n-1} = u_n.$$

So,

$$\sum_{n=0}^{\infty} (v_n - n v_{n-1}) X^n = \sum_{n=0}^{\infty} u_n X^n. \tag{5}$$

Using (1), the right-hand side of (5) is seen to be

$$\sum_{n=0}^{\infty} u_n X^n = \sum_{i=1}^{t} \frac{b_i}{1 - \alpha_i X},$$

whereas the left-hand side equals

$$v(X) - X \frac{d}{dX}(X v(X)) = (1 - X) v(X) - X^2 \frac{d}{dX} v(X).$$

So (5) becomes

$$\mathscr{L}v(X) = \sum_{i=1}^{t} \frac{b_i}{1-\alpha_i X}, \qquad \mathscr{L} = -X^2\frac{d}{dX} + (1-X). \tag{6}$$

By the claim we know that $v(X) \in \mathbf{Q}(X)$ and so the non-zero poles of $\mathscr{L}v(X)$ have order at least two. However, the right-hand side of (6) has only simple poles. This contradiction proves our theorem, since the assumption $b_1 e^{\alpha_1} + \cdots + b_t e^{\alpha_t} = 0$ has turned out to be untenable. □

It now remains to prove our claim. We first observe that, as v is a solution of the differential equation (6), if v is a rational function then its poles must be at the points $1/\alpha_i$. Therefore we expect that there exists an integer k such that $(1 - a_1 X - \cdots - a_t X^t)^k v(X)$ is a polynomial.

DEFINITION. *For any k, $n \in \mathbf{Z}_{\geqslant 0}$ we define $v_n(k)$ as coefficient in the formal power series*

$$\sum_{n=0}^{\infty} v_n(k) X^n = \left(1 - a_1 X - \cdots - a_t X^t\right)^k \sum_{n=0}^{\infty} v_n X^n.$$

For later use we also note that

$$v_n(k+1) = v_n(k) - a_1 v_{n-1}(k) - \cdots - a_t v_{n-t}(k) \quad \text{for all } n \geqslant t,\ k \geqslant 0. \tag{7}$$

LEMMA 1. *Let $C = 1 + |a_1| + \cdots + |a_t|$. For all $n \geqslant kt$ we have*
i) $|v_n(k)| \leqslant c_2 A^n C^k$
ii) $D^n v_n(k) \in \mathbf{Z}$
iii) *$k!$ divides $D^n v_n(k)$.*

Proof. The first two assertions follow easily by induction on k from (3), (4) and (7). The third assertion can be shown as follows. Write

$$v_n = u_n + n u_{n-1} + \cdots + n(n-1)\cdots(n-k+2) u_{n-k+1} + w_n,$$

where $w_n = n! \sum_{r=0}^{\infty} u_r / r!$. Notice that $D^{n-k} w_n \in \mathbf{Z}$ and $k!$ divides $D^{n-k} w_n$. Consider the power series

$$v(X) = \sum_{n=0}^{\infty} v_n X^n \qquad w(X) = \sum_{n=0}^{\infty} w_n X^n.$$

Then

$$v(X) - w(X) = \sum_{n=0}^{\infty} \{u_n + n u_{n-1} + \cdots + n(n-1)\cdots(n-k+2) u_{n-k+1}\} X^n.$$

For any $0 \leqslant r \leqslant k-1$ we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} n(n-1)\cdots(n-r+1) u_{n-r} X^n &= \sum_{i=1}^{t} \sum_{n=0}^{\infty} r!\binom{n}{r} b_i \alpha_i^n X^n \\ &= \sum_{i=1}^{t} r!\, b_i \alpha_i^r X^r \frac{1}{(1-\alpha_i X)^{r+1}} \\ &= \frac{P_r(X)}{\left(1 - a_1 X - \cdots - a_t X^t\right)^{r+1}},\end{aligned}$$

where $P_r(X)$ is a polynomial of degree $< t(r+1)$. Hence

$$v(X) - w(X) = \frac{P(X)}{\left(1 - a_1 X - \cdots - tX^t\right)^k},$$

where $P(X)$ is a polynomial of degree $< tk$. Hence $v_n(k) = w_n(k)\ \forall n \geqslant kt$, where

$$\sum_{n=0}^{\infty} w_n(k) X^n = \left(1 - a_1 X - \cdots - a_t X^t\right)^k w(X).$$

Since $k!$ divides $D^{n-k} w_n$, we see that $k!$ divides $D^n w_n(k) = D^n v_n(k)$ as asserted.

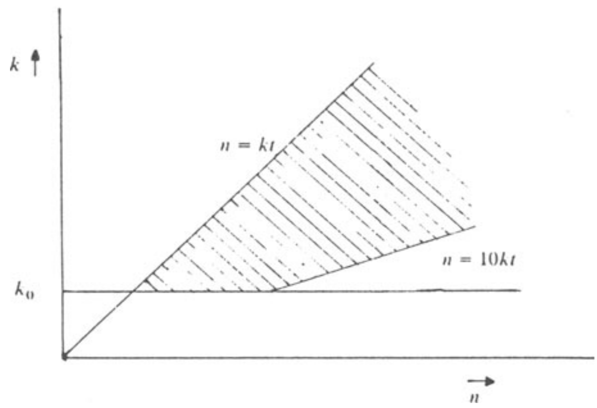
Proof of the Claim. It is sufficient to prove that $\sum_{n=0}^{\infty} v_n(k) X^n \in \mathbb{Q}[X]$ for some $k \in \mathbb{N}$. From the Lemma it follows that if $v_n(k) \neq 0$ and $n \geqslant kt$, then

$$k! \leqslant |D^n v_n(k)| \leqslant c_2 (AD)^n C^k.$$

Hence, if $k! > c_2(AD)^n C^k$ and $n \geqslant kt$ then $v_n(k) = 0$. Choose k_0 so large that $k! > c_2(AD)^{10kt} C^k\ \forall k \geqslant k_0$. Then

$$v_n(k) = 0 \quad \text{for all } k \geqslant k_0,\ kt \leqslant n \leqslant 10kt. \tag{8}$$

This situation can be pictured as follows: If a point (n, k) falls in the shaded region, we have automatically $v_n(k) = 0$ according to (8). So, finally, by (7) and induction on $n - 10kt$, it follows that $v_n(k) = 0$ for all (n, k) in the infinite triangular region $k_0 \leqslant k \leqslant n/10t$. Thus we conclude that $v_n(k_0) = 0\ \ \forall n \geqslant k_0 t$ and hence $\sum_{n=0}^{\infty} v_n(k) X^n \in \mathbb{Q}[X]$, which proves our claim.

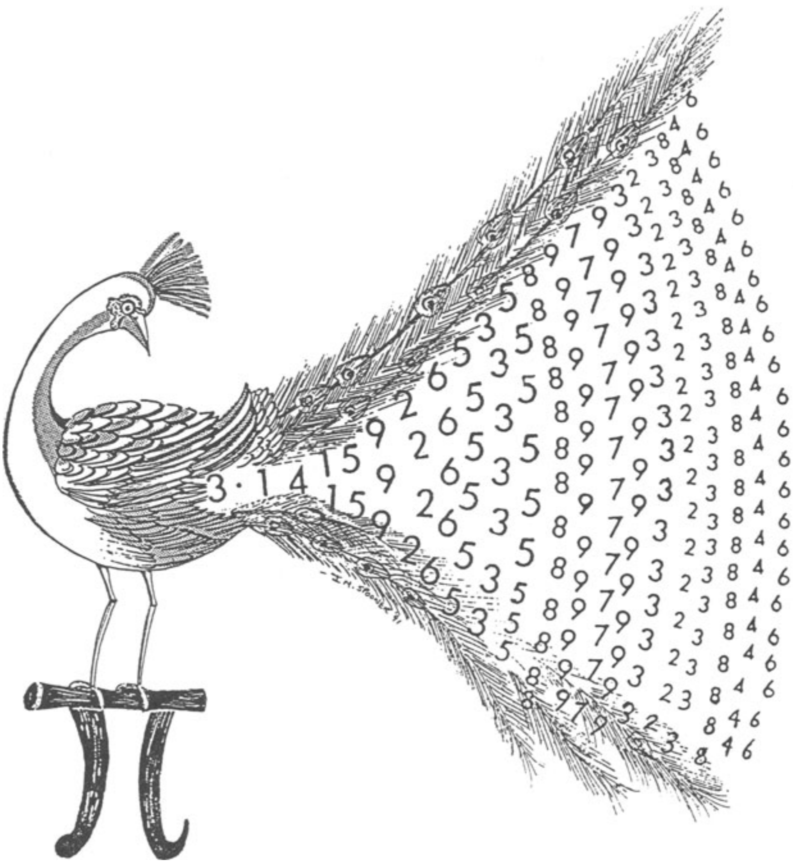


REFERENCES

1. J. P. Bézivin and P. Robba, Rational solution of linear differential operations, *J. Australian Math. Soc.* (to appear).
2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1959.
3. D. Hilbert, Gesammelte Abhandlungen, vol. I, Chelsea, 1965, pp. 1–4.
4. A. Baker, Transcendental Number Theory, Cambridge Univ. Press, 1975.

THE TAIL OF π

Roger Webster
THE UNIVERSITY OF SHEFFIELD



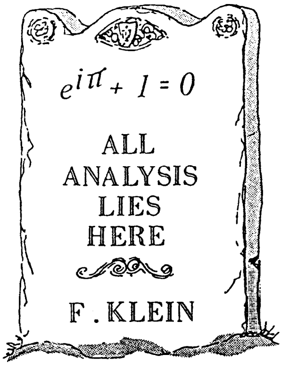
Fourteenth Annual IMO Lecture

at

The Royal Society

Tuesday 17th September 1991

2000	**Babylonians** use $\pi = 25/8$ and **Egyptians** use $\pi = 256/81$
900	**Bible, I Kings 7:23** implies $\pi = 3$
434	**Anaxagoras** attempts to *square the circle*
414	**Aristophanes** refers to *squaring the circle* in his comedy *The Birds*
240	**Archimedes** shows that $3\frac{10}{71} < \pi < 3\frac{1}{7}$ using his classical method
480	**Tsu Chung-chih** approximates π by 355/113
1429	**Al-Kashi** calculates π to 16 decimal places
1593	**Viète** expresses π as an infinite product using only 2s and $\sqrt{}$s
1610	**Ludolph van Ceulen** calculates π to 35 decimal places
1621	**Snell** refines Archimedes' classical method
1630	**Grienberger** uses Snell's refinement to calculate π to 39 decimal places
1655	**Wallis** shows that $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots$
1671	**James Gregory** finds that $\tan^{-1} x = x - x^3/3 + x^5/5 - \ldots$ for $\|x\| \leq 1$
1674	**Leibniz** shows that $\pi/4 = 1 - 1/3 + 1/5 - \ldots$
1699	**Sharp** uses Gregory's series with $x = \sqrt{3}$ to calculate π to 71 decimal places
1706	**Machin** finds π to 100 decimal places
1706	**William Jones** first uses π for the circle ratio
1736	**Euler** proves that $1/1^2 + 1/2^2 + 1/3^2 + \ldots = \pi^2/6$
1737	**Euler** uses the symbol π, thus establishing it as standard notation
1761	**Lambert** shows that π is irrational
1777	**Buffon** devises his needle problem
1794	**Legendre** shows that π^2 is irrational
1844	**Johann Dase**, a lightning calculator, finds π to 200 decimal places
1873	**Shanks** calculates π to 707? decimal places
1882	**Lindemann** shows that π is transcendental, so the circle cannot be squared
1945	**Ferguson** finds errors, starting with the 528th place, in Shanks' value for π
1948	**Ferguson** and **Wrench** publish corrected value of π to 808 decimal places
1949	**ENIAC** performs first electronic computation of π to $2,037$ decimal places
1973	**Guilloud** and **Bouyer** in Paris compute π to a million decimal places
1976	**Salamin** and **Brent** find an arithmetic-geometric mean algorithm for π
1989	**Chudnovsky** brothers in New York compute π to 1,011,196,691 decimal places
1989	**Kanada** in Tokyo finds π to $1,073,740,000$ decimal places



UBIQUITOUS π

This mysterious 3.141592 ..., which comes in at every door and window, and down every chimney.
De Morgan

Numbers may come and numbers may go, but π goes on for ever.
Anon

PUZZLE π: *Pi* makes frequent appearances in crosswords - a particularly fine clue is the following: 3.1421001000500, *blow it!* (7)

PRECISION π: Thirty-nine decimal places of π suffice to compute the circumference of a circle girdling the known universe to within an accuracy of the radius of a hydrogen atom.

PRIME π: Euler discovered the following formula for $\pi/2$ - the numerators are the odd primes, the denominators are even numbers *not* divisible by 4, differing by 1 from the numerators:

$$\frac{\pi}{2}=\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdots\ .$$

PROBABILISTIC π: The probability that a number chosen at random from the set of natural numbers has no repeated prime divisors is $6/\pi^2$, as so too is the probability that two natural numbers written down at random are prime to each other.

PARISIAN π: The *Palais de la Decouverte* in Paris houses a small circular gallery dedicated to π. In 1937 its cupola was adorned with a spiral of 707 large wooden numbers depicting Shanks' expansion of π. Imagine the embarrassment, and expense, caused to the Museum by Ferguson's discovery in 1945 that the last 180 numbers were all wrong!

PULCHRITUDINOUS π: In 1988 *Mathematical Intelligencer* asked its readers to rate 24 theorems, on a scale from 0 to 10, for beauty. Top, with an average score of 7.7, was $e^{i\pi}=-1$.

POETIC π: The first of the following two stanzas is a limerick composed by one Harvey L. Carter, the second is the cheer of California Institute of Technology 'Beavers'.

Tis a favorite project of mine
A new value of pi to assign.
I would fix it at 3
For it's simpler, you see
Than 3 point 14159.

Secant, cosine, tangent, sine
Logarithm, logarithm,
Hyperbolic sine,
3 point 14159
Slipstick, sliderule
TECH TECH TECH !

If π were 3, this sentence would look something like this.

PUZZLE π

EASY AS π: Father watched Junior in his playpen laying out blocks in a most neat row: CADAEIBF. Something about the sequence of letters astounded him. Was Junior a natural mathematician or was it only a coincidence ? When Junior added the ninth block, Father gasped. The sequence was mathematically perfect. *What was the ninth block, and what was the sequence ?*

CIRCULAR π: Write the letters of the English alphabet, in capitals, clockwise around a circle. Cross out those letters that have vertical symmetry, i.e. A,H,...,Y. How are the groups of letters that remain related to π ?

ANAGRAMMATICAL π: Try the following clue taken from the TIMES CROSSWORD PUZZLE NO 17,926 of 10th March 1989: *Enable π to be used as a common denominator* (8).

GCSE π: When is π equal to 29 ?

A-LEVEL π: Prove that

$$22/7 \; - \; \pi \; = \; \int_0^1 \frac{x^4(1-x)^4}{1+x^2} \; dx.$$

OLYMPIAD π: Show that there exists *precisely one* set of *three* single-digit positive integers a, b, c such that

$$\pi/4 \; = \; \tan^{-1} 1/a \; + \; \tan^{-1} 1/b \; + \; \tan^{-1} 1/c \, .$$

TWO DIGIT π: How is 11.00100100001111110110 1 ... related to π ?

TELEGRAPHIC π: Spot the error in the following sentence taken from the *Daily Telegraph* of 2nd January 1991: *The team includes David and Gregory Chudnovsky, who won a place in the Guinness Book of Records by computing pi, the ratio between the diameter and the radius of a circle, to more than a thousand million decimal places.*

eCCeNTRIC π: What is eccentric about the number $(\pi^4 + \pi^5)^{\frac{1}{6}}$?

GREEK π: When was π equal to 80 and ͵π equal to 80,000 ?

MANILA π: Find the missing term in the sequence P, PA, PAK, PE, — , PL, PY.

ROMAN π: Move *one* match in the diagram below to produce a correct? equation.

$$\mathrm{II} = \frac{\mathrm{XXII}}{\mathrm{VIII}}$$

The Deconstruction of Pi

That was when I saw the Pendulum.

The sphere, hanging from a long wire set into the ceiling of the choir, swayed back and forth with isochronal majesty.

I knew—but anyone could have sensed it in the magic of that serene breathing—that the period was governed by the square root of the length of the wire and by π, that number which however irrational to sublunar minds, through a higher rationality binds the circumference and diameter of all possible circles. The time it took the sphere to swing from end to end was determined by an arcane conspiracy between the most timeless of measures: the singularity of the point of suspension, the duality of the plane's dimensions, the triadic beginning of π, the secret quadratic nature of the root, and the unnumbered perfection of the circle itself.

Eco, Umberto, *Foucault's Pendulum*, Ballantine, 1988, p. 3.

Pi Mnemonics and the Art of Constrained Writing

Michael Keith
June 1996

Following the success of "Circle Digits", my prose mnemonic for the first 402 digits of π that was published in *The Mathematical Intelligencer* in 1986, I began work on several even more ambitions mnemonics. I realized that, in esssence, the construction of a literary π mnemonic is a (rather difficult) form of *constrained writing.* Constrained writing is the art of constructing a work of prose or poetry that obeys some artificially-imposed constraint. (For example, the French novel *La Disparition* by George Perec does not contain the letter *e*.)

In writing a π mnemonic the number of letters in each word is dictated by the digits of π. Not satisfied with the difficulty of this single constraint, I took up the challenge of rewriting a well-known English poem, attempting to preserve as much as possible the story, tone, and rhyme scheme of the original, while at the same time making the word lengths a mnemonic for π.

Here is the result.

Poe, E.
Near A Raven

Midnights so dreary, tired and weary.
   Silently pondering volumes extolling all by-now obsolete lore.
During my rather long nap - the weirdest tap!
   An ominous vibrating sound disturbing my chamber's antedoor.
      "This", I whispered quietly, "I ignore".

Perfectly, the intellect remembers: the ghostly fires, a glittering ember.
   Inflamed by lightning's outbursts, windows cast penumbras upon this floor.
Sorrowful, as one mistreated, unhappy thoughts I heeded:
   That inimitable lesson in elegance - Lenore -
      Is delighting, exciting...nevermore.

Ominously, curtains parted (my serenity outsmarted),
   And fear overcame my being - the fear of "forevermore".
Fearful foreboding abided, selfish sentiment confided,
   As I said, "Methinks mysterious traveler knocks afore.
      A man is visiting, of age threescore."

Taking little time, briskly addressing *something*: "Sir," (robustly)
   "Tell what source originates clamorous noise afore?
Disturbing sleep unkindly, is it you a-tapping, so slyly?
   Why, devil incarnate!--" Here completely unveiled I my antedoor;--
      Just darkness, I ascertained - nothing more.

While surrounded by darkness then, I persevered to clearly comprehend.
  I perceived the weirdest dream...of everlasting "nevermores".
Quite, quite, quick nocturnal doubts fled - such relief! - as my intellect said,
  (Desiring, imagining still) that perchance the apparition was uttering a whispered "Lenore".
    This only, as evermore.

Silently, I reinforced, remaining anxious, quite scared, afraid,
  While intrusive tap did then come thrice - O, so stronger than sounded afore.
"Surely" (said silently) "it was the banging, clanging window lattice."
  Glancing out, I quaked, upset by horrors hereinbefore,
    Perceiving: a "nevermore".

Completely disturbed, I said, "Utter, please, what prevails ahead.
  Repose, relief, cessation, or but more dreary 'nevermores'?"
The bird intruded thence - O, irritation ever since! -
  Then sat on Pallas' pallid bust, watching me (I sat not, therefore),
    And stated "nevermores".

Bemused by raven's dissonance, my soul exclaimed, "I seek intelligence;
  Explain thy purpose, or soon cease intoning forlorn 'nevermores'!"
"Nevermores", winged *corvus* proclaimed - thusly was a raven named?
  Actually maintain a surname, upon Pluvious seashore?
    I heard an oppressive "nevermore".

My sentiments extremely pained, to perceive an utterance so plain,
  Most interested, mystified, a meaning I hoped for.
"Surely," said the raven's watcher, "separate discourse is wiser.
  Therefore, liberation I'll obtain, retreating heretofore -
    Eliminating all the 'nevermores' ".

Still, the detestable raven just remained, unmoving, on sculptured bust.
  Always saying "never" (by a red chamber's door).
A poor, tender heartache maven - a sorrowful bird - a raven!
  O, I wished thoroughly, forthwith, that he'd fly heretofore.
    Still sitting, he recited "nevermores".

The raven's dirge induced alarm - "nevermore" quite wearisome.
  I meditated: "Might its utterances summarize of a calamity before?"
O, a sadness was manifest - a sorrowful cry of unrest;
  "O," I thought sincerely, "it's a melancholy great - furthermore,
    Removing doubt, this explains 'nevermores' ".

Seizing just that moment to sit - closely, carefully, advancing beside it,
  Sinking down, intrigued, where velvet cushion lay afore;
A creature, midnight-black, watched there - it studied my soul, unawares.
  Wherefore, explanations my insight entreated for.
    Silently, I pondered the "nevermores".

“Disentangle, nefarious bird! Disengage - I am disturbed!”
    Intently its eye burned, raising the cry within my core.
“That delectable Lenore - whose velvet pillow this was, heretofore,
    Departed thence, unsettling my consciousness therefore.
        She’s returning - that maiden - aye, nevermore.”

Since, to me, that thought was madness, I renounced continuing sadness.
    Continuing on, I soundly, adamantly forswore:
“Wretch,” (addressing blackbird only) “fly swiftly - emancipate me!”
    “Respite, respite, detestable raven - and discharge me, I implore!”
        A ghostly answer of: “nevermore”.

“ ‘Tis a prophet? Wraith? Strange devil? Or the ultimate evil?”
    “Answer, tempter-sent creature!”, I inquired, like before.
“Forlorn, though firmly undaunted, with ‘nevermores’ quite indoctrinated,
    Is everything depressing, generating great sorrow evermore?
        I am subdued!”, I then swore.

In answer, the raven turned - relentless distress it spurned.
    “Comfort, surcease, quiet, silence!” - pleaded I for.
“Will my (abusive raven!) sorrows persist unabated?
    Nevermore Lenore respondeth?”, adamantly I encored.
        The appeal was ignored.

“O, satanic inferno’s denizen -- go!”, I said boldly, standing then.
    “Take henceforth loathsome “nevermores” - O, to an ugly Plutonian shore!
Let nary one expression, O bird, remain still here, replacing mirth.
    Promptly leave and retreat!”, I resolutely swore.
        Blackbird’s riposte: “nevermore”.

So he sitteth, observing always, perching ominously on these doorways.
    Squatting on the stony bust so untroubled, O therefore.
Suffering stark raven’s conversings, so I am condemned, subserving,
    To a nightmare cursed, containing miseries galore.
        Thus henceforth, I’ll rise (from a darkness, a grave) -- nevermore!

-- Original: E. Poe
-- Redone by measuring circles.

A few closing remarks:

The poem encodes the first 740 digits of π, starting with the word “Poe” all the way through the final word of the signature lines, “circles”. The encoding rule is this: a word of n letters represents the digit n if $n<9$, the digit 0 if $n=10$, and two adjacent digits if $n>10$ (for example, a 12-letter word represents the digit ‘1’ followed by ‘2’). This encoding rule is both more elegant and easier to satisfy than the method I used in "Circle Digits" (which was to use non-full-stop punctuation marks to indicate a zero digit).

Due to the nature of the π-mnemonic constraint, one concession is made to the form of Poe's original. In the original there are six lines per stanza, with the fourth and fifth lines usually being very similar. Clearly, reproducing this feature exactly is not possible given the nature of π's digits, so the fifth line in each stanza is simply omitted.

However, most of the other features of Poe's poem have been retained, even to the point of using many of the same words. For the most part the story is followed at both the macro and micro level - for example, the bird enters and sits on the bust of Pallas on the next to last line of the seventh stanza, just as in the original.

Finally, note the use of the term "blackbird" a couple of times. Though not strictly correct (a raven is a black bird, not a blackbird), its appearance is quite appropriate, being a subtle nod to the e-less novel *La Disparition* by that master of constrained writing, George Perec. Perec's novel was recently translated into English by Gilbert Adair, and in translation contains another constrained version of "The Raven", in this case devoid of the letter *e*. The title of Adair's poem is...*Black Bird*!

ON THE RAPID COMPUTATION OF VARIOUS POLYLOGARITHMIC CONSTANTS

DAVID BAILEY, PETER BORWEIN[1] AND SIMON PLOUFFE

ABSTRACT.

We give algorithms for the computation of the d-th digit of certain transcendental numbers in various bases. These algorithms can be easily implemented (multiple precision arithmetic is not needed), require virtually no memory, and feature run times that scale nearly linearly with the order of the digit desired. They make it feasible to compute, for example, the billionth binary digit of $\log(2)$ or π on a modest work station in a few hours run time.

We demonstrate this technique by computing the ten billionth hexadecimal digit of π, the billionth hexadecimal digits of π^2, $\log(2)$ and $\log^2(2)$, and the ten billionth decimal digit of $\log(9/10)$.

These calculations rest on the observation that very special types of identities exist for certain numbers like π, π^2, $\log(2)$ and $\log^2(2)$. These are essentially polylogarithmic ladders in an integer base. A number of these identities that we derive in this work appear to be new, for example the critical identity for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right).$$

[1] Research supported in part by NSERC of Canada.
1991 *Mathematics Subject Classification.* 11A05 11Y16 68Q25.
Key words and phrases. Computation, digits, log, polylogarithms, SC, π, algorithm.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Introduction.

It is widely believed that computing just the d-th digit of a number like π is really no easier than computing all of the first d digits. From a bit complexity point of view this may well be true, although it is probably very hard to prove. What we will show is that it is possible to compute just the d-th digit of many transcendentals in (essentially) linear time and logarithmic space. So while this is not of fundamentally lower complexity than the best known algorithms (for say π or $\log(2)$), this makes such calculations feasible on modest workstations without needing to implement arbitrary precision arithmetic.

We illustrate this by computing the ten billionth hexadecimal digit of π, the billionth hexadecimal digits of π^2, $\log(2)$ and $\log^2(2)$, and the ten billionth decimal digit of $\log(9/10)$. Details are given in Section 4. A previous result in this same spirit is the Rabinowitz-Wagon "spigot" algorithm for π. In that scheme, however, the computation of the digit at position n depends on all digits preceding position n.

We are interested in computing in polynomially logarithmic space and polynomial time. This class is usually denoted SC (space $= \log^{O(1)}(d)$ and time $= d^{O(1)}$ where d is the place of the "digit" to be computed). Actually we are most interested in the space we will denote by SC* of polynomially logarithmic space and (almost) linear time (here we want the time $= O(d\log^{O(1)}(d))$). There is always a possible ambiguity when computing a digit string base b in distinguishing a sequence of digits $a(b-1)(b-1)(b-1)$ from $(a+1)000$. In this particular case we consider either representation as an acceptable computation. In practice this problem does not arise.

It is not known whether division is possible in SC, similarly it is not known whether base change is possible in SC. The situation is even worse in SC*, where it is not even known whether multiplication is possible. If two numbers are in SC* (in the same base) then their product computes in time $= O(d^2\log^{O(1)}(d))$ and is in SC but not obviously in SC*. The d^2 factor here is present because the logarithmic space requirement precludes the usage of advanced multiplication techniques, such as those based on FFTs.

We will not dwell on complexity issues except to point out that different algorithms are needed for different bases (at least given our current ignorance about base change) and very little closure exists on the class of numbers with d-th digit computable in SC. Various of the complexity related issues are discussed in [6,8,9,11,14].

As we will show in Section 3, the class of numbers we can compute in SC* in base b includes all numbers of the form

$$\sum_{k=1}^{\infty} \frac{p(k)}{b^{ck}q(k)}, \tag{1.1}$$

where p and q are polynomials with integer coefficients and c is a positive integer. Since addition is possible in SC*, integer linear combinations of such numbers are also feasible (provided the base is fixed).

The algorithm for the binary digits of π, which also shows that π is in SC* in base 2, rests on the following remarkable identity:

Theorem 1. *The following identity holds:*

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right). \tag{1.2}$$

This can also be written as:

$$\pi = \sum_{i=1}^{\infty} \frac{p_i}{16^{\lfloor \frac{i}{8} \rfloor} i}, \qquad [p_i] = [\overline{4,0,0,-2,-1,-1,0,0}] \tag{1.3}$$

where the overbar notation indicates that the sequence is periodic.

Proof. This identity is equivalent to:

$$\pi = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8} dx, \tag{1.4}$$

which on substituting $y := \sqrt{2}x$ becomes

$$\pi = \int_0^1 \frac{16\,y - 16}{y^4 - 2\,y^3 + 4\,y - 4} dy\,.$$

The equivalence of (1.2) and (1.4) is straightforward. It follows from the identity

$$\begin{aligned}\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx &= \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\,dx \\ &= \frac{1}{\sqrt{2}^k} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}\end{aligned}$$

That the integral (1.4) evaluates to π is an exercise in partial fractions most easily done in Maple or Mathematica. □

This proof entirely conceals the route to discovery. We found the identity (1.2) by a combination of inspired guessing and extensive searching using the PSLQ integer relation algorithm [3,12].

Shortly after the authors originally announced the result (1.2), several colleagues, including Helaman Ferguson, Tom Hales, Victor Adamchik, Stan Wagon, Donald Knuth and Robert Harley, pointed out to us other formulas for π of this type. One intriguing example is

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{2}{8i+1} + \frac{2}{4i+2} + \frac{1}{4i+3} - \frac{1/2}{4i+5} - \frac{1/2}{4i+6} - \frac{1/4}{4i+7}\right),$$

which can be written more compactly as

$$\pi = \sum_{i=0}^{\infty} \frac{(-1)^i}{4^i}\left(\frac{2}{4i+1} + \frac{2}{4i+2} + \frac{1}{4i+3}\right).$$

In [2], this and some related identities are derived using Mathematica.

As it turns out, these other formulas for π can all be written as formula (1.2) plus a rational multiple of the identity

$$0 = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{-8}{8i+1} + \frac{8}{8i+2} + \frac{4}{8i+3} + \frac{8}{8i+4} + \frac{2}{8i+5} + \frac{2}{8i+6} - \frac{1}{8i+7}\right).$$

The proof of this identity is similar to that of Theorem 1.

The identities of the next section and Section 5 show that, in base 2, π^2, $\log^2(2)$ and various other constants, including $\{\log(2), \log(3), \ldots, \log(22)\}$ are in SC*. (We don't know however if $\log(23)$ is even in SC.)

We will describe the algorithm in the Section 3. Complexity issues are discussed in [3,5,6,7,8,9,14,19,21] and algorithmic issues in [5,6,7,8,14]. The requisite special function theory may be found in [1,5,15,16,17,20].

2. Identities.

As usual, we define the m-th polylogarithm L_m by

$$L_m(z) := \sum_{i=1}^{\infty} \frac{z^i}{i^m}, \qquad |z| < 1\,. \tag{2.1}$$

The most basic identity is

$$-\log(1 - 2^{-n}) = L_1(1/2^n) \tag{2.2}$$

which shows that $\log(1 - 2^{-n})$ is in SC* base 2 for integer n. (See also section 5.)

Much less obvious are the identities

$$\pi^2 = 36L_2(1/2) - 36L_2(1/4) - 12L_2(1/8) + 6L_2(1/64) \tag{2.3}$$

and

$$\log^2(2) = 4L_2(1/2) - 6L_2(1/4) - 2L_2(1/8) + L_2(1/64)\,. \tag{2.4}$$

These can be written as

$$\pi^2 = 36\sum_{i=1}^{\infty} \frac{a_i}{2^i i^2}, \qquad [a_i] = \overline{[1, -3, -2, -3, 1, 0]} \tag{2.5}$$

$$(2.6). \qquad \log^2(2) = 2\sum_{i=1}^{\infty} \frac{b_i}{2^i i^2}, \qquad [b_i] = [\overline{2,-10,-7,-10,2,-1}].$$

Here the overline notation indicates that the sequences repeat. Thus we see that π^2 and $\log^2(2)$ are in SC* in base 2. These two formulas can alternately be written

$$\pi^2 = \frac{9}{8}\sum_{i=0}^{\infty} \frac{1}{64^i}\left(\frac{16}{(6i+1)^2} - \frac{24}{(6i+2)^2} - \frac{8}{(6i+3)^2} - \frac{6}{(6i+4)^2} + \frac{1}{(6i+5)^2}\right)$$

$$\log^2(2) = \frac{1}{8}\sum_{i=0}^{\infty} \frac{1}{64^i}\left(\frac{-16}{(6i)^2} + \frac{16}{(6i+1)^2} - \frac{40}{(6i+2)^2} - \frac{14}{(6i+3)^2} - \frac{10}{(6i+4)^2} + \frac{1}{(6i+5)^2}\right).$$

Identities (2.3)-(2-6) are examples of polylogarithmic ladders in the base 1/2 in the sense of [16]. As with (1.2) we found them by searching for identities of this type using an integer relation algorithm. We have not found them directly in print. However (2.5) follows from equation (4.70) of [15] with $\alpha = \pi/3, \beta = \pi/2$ and $\gamma = \pi/3$. Identity (2.6) now follows from the well known identity

$$(2.7) \qquad 12L_2(1/2) = \pi^2 - 6\log^2(2).$$

A distinct but similar formula that we have found for π^2 is

$$\pi^2 = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{16}{(8i+1)^2} - \frac{16}{(8i+2)^2} - \frac{8}{(8i+3)^2} - \frac{16}{(8i+4)^2} - \frac{4}{(8i+5)^2} - \frac{4}{(8i+6)^2} + \frac{2}{(8i+7)^2}\right.$$

which can be derived from the methods of section 1.

There are several ladder identities involving L_3:

$$(2.8) \qquad 35/2\zeta(3) - \pi^2\log(2) = 36L_3(1/2) - 18L_3(1/4) - 4L_3(1/8) + L_3(1/64),$$

$$(2.9) \qquad 2\log^3(2) - 7\zeta(3) = -24L_3(1/2) + 18L_3(1/4) + 4L_3(1/8) - L_3(1/64),$$

$$(2.10) \quad 10\log^3(2) - 2\pi^2\log(2) = -48L_3(1/2) + 54L_3(1/4) + 12L_3(1/8) - 3L_3(1/64).$$

The favored algorithms for π of the last centuries involved some variant of Machin's 1706 formula:

$$(2.11) \qquad \frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

There are many related formula [15,16,17,20] but to be useful to us all the arguments of the arctans have to be a power of a common base, and we have not discovered any such formula for π . One can however write

$$(2.12) \qquad \frac{\pi}{2} = 2\arctan\frac{1}{\sqrt{2}} + \arctan\frac{1}{\sqrt{8}}$$

This can be written as

$$\sqrt{2}\pi = 4f(1/2) + f(1/8) \quad \text{where} \quad f(x) := \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{2i+1} \tag{2.13}$$

and allows for the calculation of $\sqrt{2}\pi$ in SC*.

Another two identities involving Catalan's constant G, π and log(2) are:

$$G - \frac{\pi \log(2)}{8} = \sum_{i=1}^{\infty} \frac{c_i}{2^{\lfloor \frac{i+1}{2} \rfloor} i^2}, \qquad [c_i] = [\overline{1,1,1,0,-1,-1,-1,0}] \tag{2.14}$$

and

$$\frac{5}{96}\pi^2 - \frac{\log^2(2)}{8} = \sum_{i=1}^{\infty} \frac{d_i}{2^{\lfloor \frac{i+1}{2} \rfloor} i^2}, \qquad [d_i] = [\overline{1,0,-1,-1,-1,0,1,1}] \tag{2.15}$$

These may be found in [17 p. 105, p. 151]. Thus $8G - \pi\log(2)$ is also in SC* in base 2, but it is open and interesting as to whether G is itself in SC* in base 2.

A family of base 2 ladder identities exist:

$$\frac{L_m(1/64)}{6^{m-1}} - \frac{L_m(1/8)}{3^{m-1}} - \frac{2\ L_m(1/4)}{2^{m-1}} + \frac{4\ L_m(1/2)}{9} - \frac{5\ (-\log(2))^m}{9\,m!} \tag{2.16}$$

$$+ \frac{\pi^2\,(-\log(2))^{m-2}}{54\ (m-2)!} - \frac{\pi^4\,(-\log(2))^{m-4}}{486\ (m-4)!} - \frac{403\,\zeta(5)\,(-\log(2))^{m-5}}{1296\ (m-5)!} = 0.$$

The above identity holds for $1 \leq m \leq 5$; when the arguments to factorials are negative they are taken to be infinite so the corresponding terms disappear. See [16, p. 45].

As in the case of formula (1.2) for π, colleagues of the authors have subsequently pointed out several other formulas of this type for various constants. Three examples reported by Knuth, which are based on formulas in [13, p. 17, 18, 22, 47, 139], are

$$\sqrt{2}\ln(1+\sqrt{2}) = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{1}{8i+1} + \frac{1/2}{8i+3} + \frac{1/4}{8i+5} + \frac{1/8}{8i+7}\right)$$

$$\sqrt{2}\arctan(1/\sqrt{2}) = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{1}{8i+1} - \frac{1/2}{8i+3} + \frac{1/4}{8i+5} - \frac{1/8}{8i+7}\right)$$

$$\arctan(1/3) = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{1}{8i+1} - \frac{1}{8i+2} - \frac{1/2}{8i+4} - \frac{1/4}{8i+5}\right)$$

Thus these constants are also in class SC*. Some other examples can be found in [18].

3. The Algorithm.

Our algorithm to compute individual base-b digits of certain constants is based on the binary scheme for exponentiation, wherein one evaluates x^n rapidly by successive squaring and multiplication. This reduces the number of multiplications to less than $2\log_2(n)$. According to Knuth [14], where details are given, this trick goes back at least to 200 B.C. In our application, we need to perform exponentiation modulo a positive integer c, but the overall scheme is the same — one merely performs all operations modulo c. An efficient formulation of this algorithm is as follows.

To compute $r = b^n \bmod c$, first set t to be the largest power of two $\leq n$, and set $r = 1$. Then

A: if $n \geq t$ then $r \leftarrow br \bmod c$; $\quad n \leftarrow n - t$; $\quad$ endif

$t \leftarrow t/2$

if $t \geq 1$ then $r \leftarrow r^2 \bmod c$; $\quad$ go to A; $\quad$ endif

Here and in what follows, "mod" is used in the binary operator sense, namely as the binary function defined by $x \bmod y := x - \lfloor x/y \rfloor y$. Note that the above algorithm is entirely performed with positive integers that do not exceed c^2 in size. Thus it can be correctly performed, without round-off error, provided a numeric precision of at least $1 + 2\log_2 c$ bits is used.

Consider now a constant defined by a series of the form

$$S = \sum_{k=0}^{\infty} \frac{1}{b^{ck} p(k)},$$

where b and c are positive integers and $p(k)$ is a polynomial with integer coefficients. First observe that the digits in the base b expansion of S beginning at position $n+1$ can be obtained from the fractional part of $b^n S$. Thus we can write

$$b^n S \bmod 1 = \sum_{k=0}^{\infty} \frac{b^{n-ck}}{p(k)} \bmod 1 \tag{3.4}$$

$$= \sum_{k=0}^{\lfloor n/c \rfloor} \frac{b^{n-ck} \bmod p(k)}{p(k)} \bmod 1 + \sum_{k=\lfloor n/c \rfloor + 1}^{\infty} \frac{b^{n-ck}}{p(k)} \bmod 1$$

For each term of the first summation, the binary exponentiation scheme is used to evaluate the numerator. Then floating-point arithmetic is used to perform the division and add the result to the sum mod 1. The second summation, where the exponent of b is negative, may be evaluated as written using floating-point arithmetic. It is only necessary to compute a few terms of this second summation, just enough to insure that the remaining terms sum to less than the "epsilon" of the floating-point arithmetic being used. The final result, a fraction between 0 and 1, is then converted to the desired base b.

Since floating-point arithmetic is used here in divisions and in addition modulo 1, the result is of course subject to round-off error. If the floating-point arithmetic system being used has the property that the result of each individual floating-point operation is in error by at most one bit (as in systems implementing the IEEE arithmetic standard), then no more than $\log_2(2n)$ bits of the final result will be corrupted. This is actually a generous estimate, since it does not assume any cancelation of errors, which would yield a lower estimate. In any event, it is clear that ordinary IEEE 64-bit arithmetic is sufficient to obtain a numerically significant result for even a large computation, and "quad precision" (i.e. 128-bit) arithmetic, if available, can insure that the final result is accurate to several digits beyond the one desired. One can check the significance of a computed result beginning at position n by also performing a computation at position $n+1$ or $n-1$ and comparing the trailing digits produced.

The most basic interesting constant whose digits can be computed using this scheme is

$$\log(2) = \sum_{k=1}^{\infty} \frac{1}{k2^k}$$

in base 2. Using this scheme to compute hexademical digits of π from identity (1.2) is only marginally more complicated, since one can rewrite formula (1.2) using four sums of the required form. Details are given in the next section. In both cases, in order to compute the n-th binary digit (or a fixed number of binary digits at the n-th place) we must sum $O(n)$ terms of the series. Each term requires $O(\log(n))$ arithmetic operations and the required precision is $O(\log(n))$ digits. This gives a total bit complexity of $O(n\log(n)M(\log(n)))$ where $M(j)$ is the complexity of multiplying j bit integers. So even with ordinary multiplication the bit complexity is $O(n\log^3(n))$.

This algorithm is, by a factor of $\log(\log(\log(n)))$, asymptotically slower than the fastest known algorithms for generating the n-th digit by generating all of the first n digits of $\log(2)$ or π [7]. The asymptotically fastest algorithms for all the first n digits known requires a Strassen-Schönhage multiplication [19]; the algorithms actually employed use an FFT based multiplication and are marginally slower than our algorithm, from a complexity point of view, for computing just the n-th digit. Of course this complexity analysis is totally misleading: the strength of our algorithm rests mostly on its easy implementation in standard precision without requiring FFT methods to accelerate the computation.

It is clear that the above methods can easily be extended to evaluate digits of contstants defined by a formula of the form

$$S = \sum_{k=0}^{\infty} \frac{p(k)}{b^{ck}q(k)},$$

where p and q are polynomials with integer coefficients and c is a positive integer. Similarly if p and q are slowly growing analytic functions of various types the method extends.

4. Computations.

We report here computations of π, $\log(2)$, $\log^2(2)$, π^2 and $\log(9/10)$, based on the formulas (1.1), (2.2), (2.5), (2.6) and the identity $\log(9/10) = -L_1(1/10)$, respectively.

Each of our computations employed quad precision floating-point arithmetic for division and sum mod 1 operations. Quad precision is supported from Fortran on the IBM RS6000/590 and the SGI Power Challenge (R8000), which were employed by the authors in these computations. We were able to avoid the usage of explicit quad precision in the exponentiation scheme by exploiting a hardware feature common to these two systems, namely the 106-bit internal registers in the multiply-add operation. This saved considerable time, because quad precision operations are significantly more expensive than 64-bit operations.

Computation of π^2 and $\log^2(2)$ presented a special challenge, because one must perform the exponentiation algorithm modulo k^2 instead of k. When n is larger than only 2^{13}, some terms of the series (2.5) and (2.6) must be computed with a modulus k^2 that is greater than 2^{26}. Squares that appear in the exponentiation algorithm will then exceed 2^{52}, which is the nearly the maximum precision of IEEE 64-bit floating-point numbers. When n is larger than 2^{26}, then squares in the exponentiation algorithm will exceed 2^{104}, which is nearly the limit of quad precision.

This difficulty can be remedied using a method which has been employed for example in searches for Wieferich primes [10]. Represent the running value r in the exponentiation algorithm by the ordered pair (r_1, r_2), where $r = r_1 + kr_2$, and where r_1 and r_2 are positive integers less than k. Then one can write

$$r^2 = (r_1 + kr_2)^2 = r_1^2 + 2r_1r_2k + r_2^2k^2$$

When this is reduced mod k^2, the last term disappears. The remaining expression is of the required ordered pair form, provided that r_1^2 is first reduced mod k, the carry from this reduction is added to $2r_1r_2$, and this sum is also reduced mod k. Note that this scheme can be implemented with integers of size not exceeding $2k^2$. Since the computation of r^2 mod k^2 is the key operation of the binary exponentiation algorithm, this means that ordinary IEEE 64-bit floating-point arithmetic can be used to compute the n-th hexadecimal digit of π^2 or $\log^2(2)$ for n up to about 2^{24}. For larger n, we still used this basic scheme, but we employed the multiply-add "trick" mentioned above to avoid the need for explicit quad precision in this section of code.

Our results are given below. The first entry, for example, gives the 10^6-th through 10^6+13-th hexadecimal digits of π after the "decimal" point. In all cases we did the calculations twice — the second calculation was similar to the first, except shifted back one position. Since this changes all the arithmetic performed, it is a highly rigorous validity check. Thus we believe that all the digits shown below are correct.

Constant:	Base:	Position:	Digits from Position:
π	16	10^6	26C65E52CB4593
		10^7	17AF5863EFED8D
		10^8	ECB840E21926EC
		10^9	85895585A0428B
		10^{10}	921C73C6838FB2
$\log(2)$	16	10^6	418489A9406EC9
		10^7	815F479E2B9102
		10^8	E648F40940E13E
		10^9	B1EEF1252297EC
π^2	16	10^6	685554E1228505
		10^7	9862837AD8AABF
		10^8	4861AAF8F861BE
		10^9	437A2BA4A13591
$\log^2(2)$	16	10^6	2EC7EDB82B2DF7
		10^7	33374B47882B32
		10^8	3F55150F1AB3DC
		10^9	8BA7C885CEFCE8
$\log(9/10)$	10	10^6	80174212190900
		10^7	21093001236414
		10^8	01309302330968
		10^9	44066397959215
		10^{10}	82528693381274

These computations were done at NASA Ames Research Center, using workstation cycles that otherwise would have been idle.

5. Logs in base 2.

It is easy to compute, in base 2, the d-th binary digit of

$$\log(1-2^{-n}) = L_1(1/2^n)\,. \tag{5.1}$$

So it is easy to compute $\log(m)$ for any integer m that can be written as

$$m := \frac{(2^{a_1}-1)(2^{a_2}-1)\cdots(2^{a_h}-1)}{(2^{b_1}-1)(2^{b_2}-1)\cdots(2^{b_j}-1)}. \tag{5.2}$$

In particular the n-th cyclotomic polynomial evaluated at 2 is so computable. A check shows that all primes less than 19 are of this form. The beginning of this list is:

$$\{2,3,5,7,11,13,17,31,43,57,73,127,151,205,257\}\,.$$

Since

$$2^{18} - 1 = 7 \cdot 9 \cdot 19 \cdot 73,$$

and since 7, $\sqrt{9}$ and 73 are all on the above list we can compute $\log(19)$ in SC* from

$$\log(19) = \log(2^{18} - 1) - \log(7) - \log(9) - \log(73).$$

Note that $2^{11} - 1 = 23 \cdot 89$ so either both $\log(23)$ and $\log(89)$ are in SC* or neither is.

We would like to thank Carl Pomerance for showing that an identity of type (5.2) does not exist for 23. This is a consequence of the fact that each cyclotomic polynomial evaluated at two has a new distinct prime factor. We would also like to thank Robert Harley for pointing out that 29 and 37 are in SC^* in base 2 via consideration of the Aurefeuillian factors $2^{2n-1} + 2^n + 1$ and $2^{2n-1} - 2^n + 1$.

6. Relation Bounds.

One of the first questions that arises in the wake of the above study is whether there exists a scheme of this type to compute decimal digits of π. At present we know of no identity like (1.2) in base 10. The chances that there is such an identity are dimmed by some numerical results that we have obtained using the PSLQ integer relation algorithm [3, 12]. These computations establish (with the usual provisos of computer "proofs") that there are no identities (except for the case $n = 16$) of the form

$$\pi = \frac{a_1}{a_0} + \frac{1}{a_0} \sum_{k=0}^{\infty} \frac{1}{n^k} \left[\frac{a_2}{mk+1} + \frac{a_3}{mk+2} + \cdots + \frac{a_{m+1}}{mk+m} \right],$$

where n ranges from 2 to 128, where m ranges from 1 to $\min(n, 32)$, and where the Euclidean norm of the integer vector $(a_0, a_1, \cdots, a_{m+1})$ is 10^{12} or less. These results of course do not have any bearing on the possibility that there is a formula not of this form which permits computation of π in some non-binary base.

In fact, J. P. Buhler has reported a proof that any identity for π of the above form must have $n = 2^K$ or $n = \sqrt{2}^K$. This also does not exclude more complicated formulae for the computation of π base 10.

7. Questions.

As mentioned in the previous section, we cannot at present compute decimal digits of π by our methods because we know of no identity like (1.2) in base 10. But it seems unlikely that it is fundamentally impossible to do so. This raises the following obvious problem:

1] Find an algorithm for the n-th decimal digit of π in SC*. It is not even clear that π is in SC in base 10 but it ought to be possible to show this.

2] Show that π is in SC in all bases.

3] Are e and $\sqrt{2}$ in SC (SC*) in any base?

Similarly the treatment of log is incomplete:

4] Is log(2) in SC* in base 10?

5] Is log(23) in SC* in base 2?

8. Acknowledgments.

The authors wish to acknowledge the following for their helpful comments: V. Adamchik, J. Borwein, J. Buhler, R. Crandall, H. Ferguson, T. Hales, R. Harley, D. Knuth, C. Pomerance and S. Wagon.

References

1. M. Abramowitz & I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, NY, 1965.

2. V. Adamchik & S. Wagon, *Pi: A 2000-year search changes direction (preprint).*

3. A. V. Aho, J.E. Hopcroft, & J. D. Ullman, *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, MA, 1974.

4. D. H. Bailey, J. Borwein and R. Girgensohn, *Experimental evaluation of Euler sums*, Experimental Mathematics **3** (1994), 17–30.

5. J. Borwein, & P Borwein, *Pi and the AGM – A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, NY, 1987.

6. J. Borwein & P. Borwein, *On the complexity of familiar functions and numbers*, SIAM Review **30** (1988), 589–601.

7. J. Borwein, P. Borwein & D. H. Bailey, *Ramanujan, modular equations and approximations to pi*, Amer. Math. Monthly **96** (1989), 201–219.

8. R. Brent, *The parallel evaluation of general arithmetic expressions*, J. Assoc. Comput. Mach. **21** (1974), 201–206.

9. S. Cook, *A taxonomy of problems with fast parallel algorithms*, Information and Control **64** (1985), 2–22.

10. R. Crandall, K. Dilcher, and C. Pomerance, *A search for Wieferich and Wilson primes (preprint).*

11. R. Crandall and J. Buhler, *On the evaluation of Euler sums*, Experimental Mathematics **3**, (1995), 275–285.

12. H. R. P. Ferguson & D. H. Bailey, *Analysis of PSLQ, an integer relation algorithm (preprint).*

13. E. R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, NJ, 1975.

14. D. E. Knuth, *The Art of Computer Programming. Vol. 2: Seminumerical Algorithms*, Addison-Wesley, Reading, MA, 1981.

15. L. Lewin, *Polylogarithms and Associated Functions*, North Holland, New York, 1981.

16. L. Lewin, *Structural Properties of Polylogarithms*, Amer. Math. Soc., RI., 1991.

17. N. Nielsen, *Der Eulersche Dilogarithmus*, Halle, Leipzig, 1909.

18. S. D. Rabinowitz and S. Wagon, *A spigot algorithm for pi*, Amer. Math. Monthly **103** (1995), 195-203.

19. A. Schönhage, *Asymptotically fast algorithms for the numerical multiplication and division of polynomials with complex coefficients*, in: EUROCAM (1982) Marseille, Springer Lecture Notes in Computer Science, vol. 144, 1982, pp. 3–15.

20. J. Todd, *A problem on arc tangent relations*, MAA Monthly **56** (1940), 517–528.

21. H. S. Wilf, *Algorithms and Complexity*, Prentice Hall, Englewood Cliffs, NJ, 1986.

BAILEY: NASA AMES RESEARCH CENTER, MAIL STOP T27A-1, MOFFETT FIELD, CA, USA 94035-1000 dbailey@nas.nasa.gov

BORWEIN: DEPARTMENT OF MATHEMATICS AND STATISTICS, SIMON FRASER UNIVERSITY, BURNABY, B.C., CANADA V5A 1S6 pborwein@cecm.sfu.ca

PLOUFFE: DEPARTMENT OF MATHEMATICS AND STATISTICS, SIMON FRASER UNIVERSITY, BURNABY, B.C., CANADA V5A 1S6 plouffe@cecm.sfu.ca

Appendix I: On the Early History of Pi

Our earliest source on π is a problem from the *Rhind* mathematical papyrus, one of the few major sources we have for the mathematics of ancient Egypt. Although the material on the papyrus in its present form comes from about 1550 B.C. during the Middle Kingdom, scholars believe that the mathematics of the document originated in the Old Kingdom, which would date it perhaps to 1900 B.C. The Egyptian source does not state an explicit value for π, but tells, instead, how to compute the area of a circle as the square of eight-ninths of the diameter. (One is reminded of the 'classical' problem of squaring the circle.)

Quite different is the passage in I Kings 7, 23 (repeated in II Chronicles 4, 21), which clearly implies that one should multiply the diameter of a circle by 3 to find its circumference. It is worth noting, however, that the Egyptian and the Biblical texts talk about what are, *a priori*, different concepts: the diameter being regarded as known in both cases, the first text gives a factor used to produce the area of the circle and the other gives (by implication) a factor used to produce the circumference.[1]

Also from the early second millenium are Babylonian texts from Susa. In the mathematical cuneiform texts the standard factor for multiplying the diameter to produce the circumference is 3, but, according to some interpretations, the Susa texts give a value of $3\frac{1}{8}$.[2]

Another source of ancient values of π is the class of Indian works known as the *Śulba Sūtras*, of which the four most important, and oldest are Baudhāyana, Āpastamba, Kātyāyana and Mānava (resp. BSS, ASS, KSS and MSS), composed during the period from 800 to 200 B.C. The Śulba Sūtras are concerned with mathematical rules for complying with the requirements of of Hindu ritual, specifically with measuring and constructing sacrificial pits and altars. The requirements of religious ritual led to the problems of squaring the circle and, as it has been called, 'circling the square'.

For this latter problem all the Śulba Sūtras state varying forms of a rule that produces, for a circle whose area is equal to that of a square of side a, the radius

$$r = \frac{a(2+\sqrt{2})}{6} \quad (*),$$

which gives for π in the context of areas a value nearly 3.088.[3] For π as a

[1] The importance of making this distinction in historical investigations is pointed out in A. J. E. M. Smeur, "On the Value Equivalent to π in Ancient Mathematical Texts. A New Interpretation," *Archive for History of Exact Sciences* **6** (1970), 249 – 270.

[2] O. Neugebauer, "Exact Sciences in Antiquity," (2nd Edition) Dover, New York (1969), p. 47.

[3] This value results from circumscribing a circle around the given square and then increasing one-half the side of the square by $\frac{1}{3}$ of the difference between the radius of the circumscribed circle and half the side of the square. The resulting sum is alleged to be the radius of the circle equal to the given square.

factor for obtaining the circumference the MSS gives $3\frac{1}{5}$ – and advises the reader that 'not even a hair's length is extra'[4].

For squaring the circle BSS, KSS and ASS all state rules that give $(\frac{13d}{15})^2$ as the area of a square equal to a circle, which they all admit is only a coarse approximation and which, indeed, gives $\pi = 3.00\bar{4}$. BSS also gives for the side, a, of a square equal in area to a circle of diameter d

$$a = d - \frac{d}{8} + \frac{d}{8 \cdot 29} - \frac{d}{8 \cdot 29 \cdot 6} + \frac{d}{8 \cdot 29 \cdot 6 \cdot 8},$$

which was perhaps obtained by solving (*) above for a and using the Śulba's value of $\sqrt{2} = \frac{577}{408}$.[5]

Given that the problem of expressing the area of a circle as that of a certain square appeared in so many ancient cultures it is hardly surprising that it appeared in ancient Greece as well. There is a jesting reference to it in Aristophanes's play *The Birds* (413 B.C.), and geometrical attacks on the problem from around that time, in the works of Hippocrates of Chios, Bryson and Antiphon, are known.[6] In fact, Hippocrates was aware of the contents of what we know as Euclid's *Elements* XII, 2:

> 'Circles are to one another as the squares on their diameters,'

a proposition which implies that the circle and the square on its diameter (and hence on its radius) bear a constant ratio to one another, but there is no evidence that any of this early work resulted in a numerical value for π. It was Archimedes (277 – 212 B.C.) who first established rigorously a connection between the area of a circle and its circumference in Prop. 1 of his treatise *Measurement of the Circle*, which we include in this volume. It is in this treatise as well where his famous upper and lower bounds for π occur, in the form of Proposition 3: the circumference of any circle is greater than $3\frac{10}{71}$ times the diameter and less than $3\frac{10}{70}$ times the diameter.

Heron of Alexandria (fl. ca. 60 A.D.) writes, in his *Metrica* (i, 26), that Archimedes in a (now lost) work *Plinthides and Cylinders* made a better approximation to π. The figures are garbled, but it seems the denominators were on the orders of tens of thousands and the numerators on the orders of hundreds of thousands. The French historian, P. Tannery, has made two possible emendations of the text, either of which gives as the mean of the two limits, the approximation $\pi = 3.141596$. In his *Almagest* (vi, 7) Ptolemy (150 A.D.) gives for the circumference of a circle the value $3 + \frac{8}{60} + \frac{30}{60^2}$, i.e. $3.141\bar{6}$, times its diameter. Since this exactly corresponds to the perimeter of a regular 360-gon inscribed in a circle, calculated according to his table of chords in a

[4] For other values from the MSS and references to the literature on the Śulba Sūtras, see R. C. Gupta, "New Indian Values of pi from the Mānava Śulba Sūtra." *Centaurus* **31** (1988), 114 – 125.

[5] Datta B., "Science of the Sulba," University of Calcutta, Calcutta (1932), 143–4.

[6] See pp. 220–235 in T.L. Heath, *A History of Greek Mathematics*, vol. 1, Clarendon Press, Oxford, 1921, for details on this and the other Greek material cited below.

circle, there is no doubt that Ptolemy used his chord table (Almagest i,11) as the basis for calculating π.

In the period from the late fifth to the early seventh century of our era in India two astronomers stated rules for measuring circles that implied values of π. The earlier of these, Āryabhaṭa (b. 476), in his work known as the *Āryabhaṭiya* gives correctly the area of a circle as that of the rectangle whose sides are half the circumference and half the diameter, and then gives the volume of a sphere as the area of the [great] circle multiplied by its square root. [7] Then in Rule 10 we are told

> 'Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000.'[8]

This value of π, $\frac{62832}{20000}$, yields the decimal equivalent 3.1416.[9]

In his notes on Rule 10. Kaye states that the second of the two writers we referred to above, the astronomer Brahmagupta (b. 598), gives for π the values 3 and $\sqrt{10}$, and that Brahmagupta finds fault with Ārhybhata for using in one place $\frac{3393}{1080}$ and in another $\frac{3393}{1050}$.[10]

Both Greek and Indian science came to the court of the 'Abbasid califs in the latter part of the eighth and early ninth centuries. One of the most important writers of this latter period was Muḥmmad ibn Mūsā al-Khwārizmī, whose *Algebra* was composed during the first third of the ninth century and dedicated to the Calif al-Ma'mūn. Between its opening section on algebra and its third (and final) section on the calculation of inheritances, al-Khwārizmī's work contains, a section on mensuration,[11] with the following rules: The 'practical man' takes $3\frac{1}{7}$ as the value which, multiplied by the diameter, produces the circumference, 'though it is not quite exact.' Geometers, on the other hand, take the circumference either as $\sqrt{10 \cdot d \cdot d}$ or, if they are astronomers, as $\frac{62832d}{20000}$. He next gives the same rule as that found in Āryabhaṭa and the Banū Mūsā (see below), i.e. that any circle is equal to the rectangle whose one side is half the circumference and whose perpendicular side is the radius. Finally, al-Khwārizmī states that

> 'If you multiply the diameter of any circle by itself, and subtract from the product one-seventh and half of one-seventh of the same, then the remainder is equal to the area of the circle,'

[7] "The Āryabhaṭiya of Āryabhaṭa," (W.E. Clark, tr.), Chicago, U. of Chicago Press (1930). See Rule 7, p. 27.

[8] The use of large values for the diameter was one way of avoiding fractions.

[9] An earlier work than the Āryabhaṭiya is the Suryasiddhanta, although its present form incorporates later material. The value of π in an early part of the Suryasiddhanta is $\sqrt{10}(= 3.1623)$, but in the section on the Sine table the value used is $10,8000 : 3438 (= 3.14136)$.

[10] Kaye, G.R., "Notes on Indian Mathematics, No. 2," *J. Asiatic Society of Bengal* **4**, No. 3 (1908), 122.

[11] pp. 71-72, in: Al-Khwārizmī, Muḥammad ibn Musa, "The Algebra," (ed. and trans. F. Rosen) London: Oriental Translation Fund, 1831. Reprinted in Hildesheim, Zurich & New York: Georg Olms Verlag, (1986).

the implicit value for π in the rule $A = (1 - \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{7})d^2$ being $\frac{22}{7}$. He describes this as coming 'very nearly to the same result' as the previous rule, for this rule effectively replaces half the circumference by $3\frac{1}{7}r$.

Only slightly later than the *Algebra* is the material included in our volume from the Banū Mūsā, three brothers from a well–to–do family who worked in Baghdad and who aided the growth of mathematical sciences in medieval Islam by their patronage of scientists and translators as well as by their own scientific compositions. The material we have selected comes from a translation of a Latin version of their work *On the Measurement of Plane and Solid Figures* and states explicitly as a theorem that π is a constant. It also points to the use of sexagesimal (base-60) fractions for recording the results of employing Archimedes' method for calculating, decimal fractions not having been invented yet.

It is patent in the case of the Banū Mūsā, and can hardly be doubted in that of al-Khwārizmī, that their material on the circle depended wholly on earlier sources (Hellenistic and Hindu). In the centuries immediately following these writers Islamic authors obtained new and better values for π, but in the context of trigonometric investigations aimed at improved tables of the Sine. Thus the 10th century Persian mathematician Abu'l-Wafā' al-Būzjānī calculated the circumference of a circle of diameter 120 as 376 and a fraction which is greater than $\frac{59}{60} + \frac{10}{60^2} + \frac{59}{60^3}$ and less than $\frac{59}{60} + \frac{23}{60^2} + \frac{54}{60^3} + \frac{12}{60^4}$.

Al-Kāshī, who reports this value, says this means an uncertainty in the circumference of the Earth on the order of 1,000 cubits. He also notes that this introduced into the Chord tables of Abu'l-Wafā' an error of about $\frac{1}{60^4}$ too low for the chord of an arc of $1/2^\circ$ (in a circle of radius 60°). Of course, al-Kāshi who, as was said of Euler, could calculate as eagles can fly, gives the correct value of this chord, to six sexagesimal places.

The Persian polymath, al-Bīrūnī, a younger contemporary of Abu'l-Wafā', calculated the perimeters of inscribed and circumscribed 180-gons for a circle of diameter 120 and took the arithmetic mean of the two values to obtain the estimate of the circumference of the circle which, if applied to the Earth, would result in an error of 1 farsang. The approximation was, however, sufficiently accurate to ensure that the value al-Bīrūnī used in the Sine table of his astronomical work, the *Masūdī Canon*, was accurate to the third sexagesimal place. Incidentally, this Persian writer, al-Bīrūnī (fl. ca.1020), also made a deep study of Indian civilization and science during a stay there. He wrote in his *India* (I, 168) that the astronomical work, the *Paulisasiddhanta*, gives for π the value $3\frac{177}{1250}$, one equivalent to that of Āryabhaṭa.[12]

Al-Kāshī passes over in silence the roughly 350 years intervening between al-Bīrūnī's death and his own period of activity, which we take to mean that he knew of no important advances in the calculation of π in that interval. Of the

[12]Finding this value in a version of what was originally a Greek astronomical work (by a writer named Paul) that was translated into Sanskrit in the third or fourth century of our era is, as Kaye suggests in his notes, cited earlier, evidence that the value did not originate with Ārhybhaṭa.

Indian activity in this period we report only that of Bhaskara II (fl. ca. 1115) who takes the Earth's diameter to be 1581 yojnas and calls its circumference 4967 yojnas, which implies a value of of π of $\frac{4967}{1581}$, i.e. 3.1417. His famous mathematical work, the *Lilavati*, gives the two traditional values of $\frac{22}{7}$ and $\frac{3927}{1250}$.

After such a long hiatus, the 15th century witnessed two brilliant achievements, one due to the florescence of Islamic civilization taking place in the early years of that century at the court of Ulugh Beg in Samarqand, and the other taking place in South India, in what is now the state of Kerala. The first-mentioned was that of al-Kāshī who recorded it in his *Treatise on the Circumference of the Circle*, beginning with the words,

> 'Praise to Allah who knows the ratio of the diameter to the circumference ... and peace to Muḥammad, the Chosen, the center of the circle of prophets.'

Al-Kāshī was Director of the Observatory and one of the ornaments of the court of the grandson of Tamurlane, Ulugh Beg, who reigned in Samarqand during the years early in the 15th century. Ulugh Beg was himself a highly competent astronomer who seemed to delight in posing problems from the mathematical side of astronomy to confound his entourage of scientists. Al-Kāshī, who suffered from no false modesty, wrote more than one letter to his father explaining how often only he, of all the scholars there assembled, could immediately solve a problem posed by Ulugh Beg. Al-Kāshī's goal in his *Treatise on the Circumference*, written in 1424, is to determine the ratio of the circumference of a circle to its diameter so accurately that, when one uses that value to compute not just the circumference of the Earth but that of the whole cosmos,[13] one would find a value that varies from the true value by less that the width of a hair that is $\frac{1}{60}$ the width of a barley-corn. [14]

To do so he computes the circumferences of inscribed and circumscribed regular polygons having $3 \cdot 2^{28}$ (= 805,306,368) sides to obtain for the value of 2π the base-sixty approximation $6; 16, 59, 28, 01, 34, 51, 46, 14, 50$, where each pair of digits after the semicolon records the numerator of successive powers of $\frac{1}{60}$. For those of his readers who were not astronomers (and therefore might not find the base-60 system familiar) he also converts the value to decimal fractions. The treatise is remarkable in the history of mathematics up to al-Kāshī's time not only for the value of π it records but also, as Paul Luckey remarks in his preface, for the format of the sexagesimal calculations, his control of those calculations and his error estimates.

[13] Al-Kāshī took his value to be 600,000 times that of the Earth's circumference.

[14] Al-Kāshī, Jamshid ibn Mas'ud. *Al-Risala al-Muhitiya*, "Treatise on the Circumference," (P. Luckey, tr.), Abhandlungen der deutschen Akademie der Wissenschaften zu Berlin. Klasse für Math. und Allgemeine Naturwiss. Jahrgang 1950, Nr. 6. Berlin (1953).

This treatise has not been translated into English. There are, however, excellent Russian and German translations. the latter, by Paul Luckey, being the one we have used.

Only very shortly after al-Kāshī's accomplishment the Indian astronomer Mādhava gave a method for the (implicit) calculation of π which was quite different from that which had been known up to that time, and that was the discovery of calculating π by means of an infinite series, in fact the one known today as 'Gregory's series. What Mādhava showed was that the circumference of a circle of diameter d may be approximated by $C(n)$, where

$$C(n) = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \cdots + (-1)^{(n-1)}\frac{4d}{2n-1} + (-1)^n \cdot 4dF(n),$$

and $F(n)$ is an error term we shall discuss below. According to Gold and Pingree,

> 'Mādhava . . . lived and worked in large family compounds called *illams* within a small area on the Western coast of south India, in the modern state of Kerala. He himself was from an *illam* .. . a few miles north of Cochin. His family was Brāhmaṇa . . . He was both a mathematician and an astronomer, and his dated works were composed in 1403 and 1418.'[15]

This makes him a exact contemporary of al-Kāshī. The selection we reproduce in this book is translated from an account of Mādhava's discovery by Nīlakaṇṭha, who wrote it in 1501. By the time of Saṇkara (ca. 1530) there were three different forms for the term $F(n)$, namely

$$F_1(n) = \frac{1}{4n} \qquad F_2(n) = \frac{n}{4n^2+1} \qquad F_3(n) = \frac{n^2+1}{4n^3+5n}.$$

Hayashi, Kusuba and Yano point out in their paper[16] that $C(19)$, with the error term $F_3(19)$, yields 9 correct decimal places of π. (Their paper, apart from its intrinsic interest, also provides a useful bibliography for one wishing to further explore Keralese work on infinite series.)

[15] Gold, David and Pingree, David. "A Hitherto Unknown Sanskrit Work concerning Mādhava's Derivation of the Power Series for Sine and Cosine," *Historia Scientiarum*, No. 42 (1991), 49– 65.

[16] Hayashi, T., Kusuba, T. and Yano, M. "The Correction of the Mādhava Series for the Circumference of a Circle," *Centaurus* **33** (1990), 149 –174.

Appendix II: A Computational Chronology for Pi

The next two tables provide a reasonably complete accounting of computations from 2000 BCE to 1996 AD. The tables are taken from: David H. Bailey, Jonathan M. Borwein, Peter B. Borwein, and Simon Plouffe, "The Quest for Pi," *The Mathematical Intelligencer* **19** (1997), 50–57.

There is a considerable history of computations uncovering subtle (and not so subtle) computer problems in both hardware and software. There is also a long history of errors in calculation. Witness the error Shanks made in his 1853 computation that went unnoticed for almost a century.

As a result recent records tend only to be considered as such after some validation, often by a second computation using a different algorithm. Of course, provided the claimed record includes some of the digits it is easily checked by the next person aspiring to hold the record.

As this book goes to print the tables are already out of date. The current record for decimal digits (due to Yasumasa Kanada) is in excess of 17.1 billion (2^{34}) digits. This computation was performed on a Hitachi SR2201 Massively Parellel Processor with 2^{10} processors. This took under 7 hours to perform twice, once with the Brent–Salamin algorithm and once with the quartic algorithm of Borwein and Borwein both of which are given in Appendix III.

Also Fabrice Bellard has recently computed the 400 billionth binary digit of π by the methods given in [70]. He determines that the hexadecimal digits starting at the 100 billionth are $9C381872D27596F81D0E48B95A6C46$.

Babylonians	2000? BCE	1	3.125 ($3\frac{1}{8}$)
Egyptians	2000? BCE	1	3.16045 ($4(\frac{8}{9})^2$)
China	1200? BCE	1	3
Bible (1 Kings 7:23)	550? BCE	1	3
Archimedes	250? BCE	3	3.1418 (ave.)
Hon Han Shu	130 AD	1	3.1622 ($=\sqrt{10}$?)
Ptolemy	150	3	3.14166
Chung Hing	250?	1	3.16227 ($\sqrt{10}$)
Wang Fau	250?	1	3.15555 ($\frac{142}{45}$)
Liu Hui	263	5	3.14159
Siddhanta	380	3	3.1416
Tsu Ch'ung Chi	480?	7	3.1415926
Aryabhata	499	4	3.14156
Brahmagupta	640?	1	3.162277 ($=\sqrt{10}$)
Al-Khowarizmi	800	4	3.1416
Fibonacci	1220	3	3.141818
Al-Kashi	1429	16	
Otho	1573	6	3.1415929
Viete	1593	9	3.1415926536 (ave.)
Romanus	1593	15	
Van Ceulen	1596	20	
Van Ceulen	1615	35	
Newton	1665	16	
Sharp	1699	71	
Seki	1700?	10	
Kamata	1730?	25	
Machin	1706	100	
De Lagny	1719	127	(112 correct)
Takebe	1723	41	
Matsunaga	1739	50	
Vega	1794	140	
Rutherford	1824	208	(152 correct)
Strassnitzky and Dase	1844	200	
Clausen	1847	248	
Lehmann	1853	261	
Rutherford	1853	440	
Shanks	1874	707	(527 correct)

Table 1: History of π Calculations (Pre 20th Century)

Ferguson	1946	620
Ferguson	Jan. 1947	710
Ferguson and Wrench	Sep. 1947	808
Smith and Wrench	1949	1,120
Reitwiesner et al. (ENIAC)	1949	2,037
Nicholson and Jeenel	1954	3,092
Felton	1957	7,480
Genuys	Jan. 1958	10,000
Felton	May 1958	10,021
Guilloud	1959	16,167
Shanks and Wrench	1961	100,265
Guilloud and Filliatre	1966	250,000
Guilloud and Dichampt	1967	500,000
Guilloud and Bouyer	1973	1,001,250
Miyoshi and Kanada	1981	2,000,036
Guilloud	1982	2,000,050
Tamura	1982	2,097,144
Tamura and Kanada	1982	8,388,576
Kanada, Yoshino and Tamura	1982	16,777,206
Ushiro and Kanada	Oct. 1983	10,013,395
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada, Tamura, Kubo, et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Chudnovskys	Jun. 1989	525,229,270
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1989	1,011,196,691
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Takahashi and Kanada	Jun. 1995	3,221,225,466
Kanada	Aug. 1995	4,294,967,286
Kanada	Oct. 1995	6,442,450,938

Table 2: History of π Calculations (20th Century)

Appendix III - Selected Formulae for Pi

Fifty decimal digits of π. One hundred thousand digits may be found in [38].

$$\pi = 3.14159265358979323846264338327950288419716939937 51\ldots$$

Archimedes ([3],[4])

Let $a_0 := 2\sqrt{3}$, $b_0 := 3$ and

$$a_{n+1} := \frac{2a_n b_n}{a_n + b_n} \quad \text{and} \quad b_{n+1} := \sqrt{a_{n+1}b_n}.$$

Then a_n and b_n converge linearly to π (with an error $O(4^{-n})$).

(ca 250 BC)

Francois Viète [9]

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

(ca 1579)

John Wallis ([10],[11])

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}$$

(ca 1655)

William Brouncker [37]

$$\pi = \cfrac{4}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cdots}}}}$$

(ca 1658)

Mādhava, James Gregory, Gottfried Wilhelm Leibnitz ([7],[13],[14])

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(1450–1671)

Isaac Newton [16]

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \cdots\right)$$

(ca 1666)

Machin Type Formulae ([31],[32],[37])

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$$

$$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{3})$$

$$\frac{\pi}{4} = 2\arctan(\frac{1}{2}) - \arctan(\frac{1}{7})$$

$$\frac{\pi}{4} = 2\arctan(\frac{1}{3}) + \arctan(\frac{1}{7})$$

(1706–1776)

Leonard Euler [17]

$$\begin{aligned}
\frac{\pi^2}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \\
\frac{\pi^4}{90} &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots \\
\frac{\pi^2}{6} &= 3 \sum_{m=1}^{\infty} \frac{1}{m^2 \binom{2m}{m}}
\end{aligned}$$

(ca 1748)

Srinivasa Ramanujan ([29],[64])

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left(\begin{array}{c} 2n \\ n \end{array} \right)^3 \frac{42n+5}{2^{12n+4}}$$

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}$$

Each additional term of the latter series adds roughly 8 digits.

(1914)

Louis Comtet [49]

$$\frac{\pi^4}{90} = \frac{36}{17} \sum_{m=1}^{\infty} \frac{1}{m^4 \binom{2m}{m}}$$

(1974)

Eugene Salamin [46], **Richard Brent** [47]

Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. For $k = 1, 2, 3, \cdots$ compute

$$\begin{aligned} a_k &= \frac{a_{k-1} + b_{k-1}}{2} \\ b_k &= \sqrt{a_{k-1} b_{k-1}} \\ c_k &= a_k^2 - b_k^2 \\ s_k &= s_{k-1} - 2^k c_k \\ p_k &= \frac{2a_k^2}{s_k} \end{aligned}$$

Then p_k converges *quadratically* to π.

(1976)

Jonathan Borwein and Peter Borwein [64]

Set $a_0 = 1/3$ and $s_0 = (\sqrt{3} - 1)/2$. Iterate

$$\begin{aligned} r_{k+1} &= \frac{3}{1 + 2(1 - s_k^3)^{1/3}} \\ s_{k+1} &= \frac{r_{k+1} - 1}{2} \\ a_{k+1} &= r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1) \end{aligned}$$

Then $1/a_k$ converges *cubically* to π.

(1991)

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$\begin{aligned} y_{k+1} &= \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \\ a_{k+1} &= a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2) \end{aligned}$$

Then a_k converges *quartically* to $1/\pi$.

(1985)

David Chudnovsky and Gregory Chudnovsky [63]

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \frac{13591409 + n545140134}{(640320^3)^{n+1/2}}.$$

Each additional term of the series adds roughly 15 digits.

(1989)

Jonathan Borwein and Peter Borwein [62]

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{(A + nB)}{C^{n+1/2}}$$

where

$$\begin{aligned} A &:= 212175710912\sqrt{61} + 1657145277365 \\ B &:= 13773980892672\sqrt{61} + 107578229802750 \\ C &:= [5280(236674 + 30303\sqrt{61})]^3. \end{aligned}$$

Each additional term of the series adds roughly 31 digits.

(1989)

The following is not an identity but is correct to over 42 billion digits

$$\left(\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{10^{10}}} \right)^2 \doteq \pi.$$

(1985)

Roy North [65]
Gregory's series for π, truncated at 500,000 terms gives to forty places

$$4 \sum_{k=1}^{500,000} \frac{(-1)^{k-1}}{2k-1}$$

$$= 3.14159\underline{0}65358979324\underline{0}4626433832\underline{6}9502884197.$$

$$= 3.14159\underline{0}6535897932\underline{40}46264338326950...$$

Then only the underlined digits are incorrect.

(1989)

David Bailey, Peter Borwein and Simon Plouffe [70]

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$

(1996)

Appendix IV - Translations of Viète and Huygens

Translation of Article 9 (Excerpt 1): Viète. Various Responses on Mathematical Matters: Book VIII (1593)

CHAPTER XVIII.

Ratio of the inscribed polygons in order in a circle to the circle.

Archimedes squared the parabola through continuously inscribing a triangle existing *"in a rational ratio"*. In fact since he inscribed, over a very large triangle inscribed in the parabola triangles in a continuous ratio with this largest one, constantly sub-quadruple, to the infinite: for this reason he concluded that the parabola is the four thirds of this very large triangle. Thus Antiphon could not square the circle since triangles continuously inscribed exist *"in an irrational ratio"*. Could not a circle then be squared? If in fact the figure composed of triangles established to the infinite in the sub-quadruple ratio with a given larger triangle is made four-thirds to the same one there is a certain knowledge of the infinite. And a plane figure will equally be able to be composed of inscribed triangles *"in a ratio"* continuously on the circle up to infinity, albeit *"irrational"* and vague. And this figure will be in a certain ratio with the largest inscribed triangle. On the other hand as the Euclidians assure it they pretend that an angle larger than acute and smaller than an obtuse is not a right angle. I so propose about it so that it is allowed to philosophise more freely about this uncertain and inconsequent matter of a polygon which you want regularly inscribed up to a polygon with an infinite number of sides or if it is so a circle; I propose so.

Proposition 1.

If two ordered polygons are inscribed in the same circle and if on the other hand the number of sides of angles of the first one is sub-double of the number of sides or angles of the second: the first polygon will be to the second one as the apotome of the side of the first one is to the diameter.

I call the apotome of a side the subtended of the periphery that is left to the half circle by the one on which the side is subtended.

Thus in a circle with center A let be inscribed a scalene ordered polygon with side BD. And let the circumference BD cut into two at E be subtended by BE. And thus let another polygon with side BE be inscribed. Then the number of sides or angles of the first polygon will be subdoubled of the number of sides or angles of the second one. On the other hand let D and C be joined. I say that the first polygon with side BD is to the polygon with side BE.

END OF PAGE 398

---

or ED as DC is to BC. This is why. Let D and A and E and D be joined. The first polygon is made of as many triangles as the sides or angles of the first polygon. On the other hand the first polygon is made altogether of as many trapezoids $BEDA$. Then the first polygon is to the second polygon what the triangle BAD is to the trapezoid $BEDA$. Of course this trapezoid $BEDA$ is divided into two triangles BAD and BED with BD as a common base. Then the triangles with the same base are to each other what the heights are to each other. And thus let the half diameter AE be drawn intersecting BD at F. Then since the circumference was cut in half at E, AE intersects BD at a right angle. And thus AF is the height of triangle BDA and FE the height of triangle BED. This is why triangle BAD is to triangle BED as AF is to EF and, by construction of triangle BAD, is to triangles BAD and BED put together which is a trapezoid $BEDA$ as AF is to AE. And then the first polygon will be to the second in the same ratio. But AF is to AE or AB as DC is to BC. But angle BCD is like BFA and for this reason AF and DC are parallel. Then the first polygon with side BD is to the second polygon with side BE or ED as DC is to BC. Which was to be proved.

Proposition II.

If infinitely ordered polygons are inscribed in the same circle and if the number of the side of the first is sub-doubled of the number of sides of the second and sub-quadruple of the number of sides of the third sub-octuple of the fourth the sixth to the fifth and with no interruption with a continuous sub ratio.

The first polygon will be to the third as the plane made by the apotome of the first and second polygon is to the square of the diameter.

In addition to the fourth as the solid made by the apotome of the first second and third polygons is to the cube of the diameter.

To the fifth as the plano-plane made by the apotome of the sides of the first, second, third, and fourth polygons is to the quadrato-square of the diameter.

To the sixth as the plano-solid made by the apotome of the sides of the first, second, third, fourth, and fifth polygons is to the quadrato-cube of the diameter.

And will be to the seventh as the solido-solid made by the apotome of the sides of the first, second, third, fourth, fifth, and sixth polygons to the cubo-cube of the diameter. And this continually progressing into infinite.

Let B be the apotome of the side of the first polygon C of the second D, of the third F, of the fourth G, of the fifth H, of the sixth. Let Z be the diameter of the circle. From the previous proposition the first polygon will be to the second polygon as B to Z. And thus what is done to B in the second polygon will be equal to what is done to Z in the first polygon. And the second polygon will be to the third as C is to Z. And subsequently the area made by the second polygon and B is the area made by the first, and Z to the area made by the third polygon and B as C is to Z. This is why the area made by the first polygon and Z squared is equal to the area made by the third polygon as the plane B times C. Thus the first polygon is to the third polygon as the plane B times C is to Z squared. And the area made by the third and the plane B times C will be equal to the area made by the first one and Z squared. Again according to the same above proposition the third polygon is to the fourth polygon as D is to Z. And subsequently the area made by the third plane and B times C the area made by the first and Z squared, to the area made by the fourth plane and B times C, is as D to Z. This is why the area made by the first and Z cubed will be equal to the area made by the fourth and the solid B times C times D. Then the first polygon is to the fourth as B times C times D to Z cubed. And by the same method of demonstration it will be of the fifth one as B times C times D times F is to Z quadrato-squared. To the sixth, as B times C times D times F times G is to Z quadrato-cubed. To the seventh, as B times C times D times D times F times G times H is to Z cubo-cubed. And this is through a constant progression towards infinity.

END OF PAGE 399

Corollary.

And thus the square inscribed in the circle will be as the side of this square to the highest power of the diameter divided by the area continuously made by the side of the apotome of the octagon of the sixteen sided, of the thirty-two sided, of the sixty-four sided, of the one hundred and twenty-eight sided, of the two hundred and fifty-six sided polygons and in this sub-double ratio of all the remaining angles or sides.

Let the square be the first polygon inscribed in the circle, the octagon will be the second, the sixteenth sided polygon the third, the thirty-two sided polygon the fourth, and this continuously in order. And thus the square inscribed in the circle will be to the extreme polygon (with an infinite number of sides) as the area made by the apotomes of the sides of the square of the octagon of the sixteen sided polygon, and of all the remaining in this sub-double ratio to the infinite is to the highest power of the diameter. And through a common division as the apotome of the square is to the highest power of the diameter divided by the area made by the apotomes of the sides of the octagon of the sixteen sided polygon and of all the remaining in this sub-double ratio to the infinite. On the other hand the apotome of the square inscribed in the circle is equal to the side itself and the polygon with an infinite number of sides is equal to the circle itself.

Let 2 be the diameter of the circle. The side of the inscribed square in that circle will be $\sqrt{2}$ *the square itself 2. The apotome of the side of the octagon* $\sqrt{2+\sqrt{2}}$. *The apotome of the side of the sixteen sided polygon* $\sqrt{2+\sqrt{2+\sqrt{2}}}$. *The apotome of the side of the thirty-two side polygon* $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$. *The apotome of the side of the sixty-four sided polygon* $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$. *And this in a continuous pattern.*

On the other hand let 1 be the diameter and N the circle; $\frac{1}{2}$ *will be to N as* $\sqrt{\frac{1}{2}}$ *to the unit divided by* $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}$ *times* $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}}$ *times* $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}}}$ *times* $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}}}}$.

Let x be the diameter; A the plane of the circle. Half of the square of x will be to the plane A as the root of $\frac{1}{2}x$ squared to x to the highest power divided by the binomial root of $\frac{1}{2}x$ squared + the root of $\frac{1}{2}x$ quadrato-squared times the binomial root of $\frac{1}{2}x$ squared + the binomial root of $\frac{1}{2}x$ quadrato-squared + the root of x quadrato-quadrato-quadrato-squared times the binomial root of $\frac{1}{2}x$ squared + the root of binomial $\frac{1}{2}x$ quadrato-squared + the root of binomial of $\frac{1}{2}x$ quadrato-quadrato-quadrato-squared + the root of $\frac{1}{2}x$ quadrato-quadrato-quadrato-quadrato-quadrato-quadrato-quadrato-squared times the root etc $\cdots$ this uniform method being kept up to the infinite.

Translation of Article 9 (Excerpt 2): François Viète Defense for the New Cyclometry or "Anti-Axe"

Those who tried to set the circle equal to thirty-six figures of hexagon segments, unluckily failed in this attempt. Indeed, what certain result can be reached by using completely uncertain quantities to solve them? If they add or subtract equals to equals, if they divide or multiply by equals, if they invert, permute and at last increase or decrease with certain ratios of proportions, they won't go forward one inch in their research, but they will make the mistake that Logicians call *"requiring what is demanded"*, Diophantus's *"inequality"* or they will get mistaken through a wrong calculation, whereas they would have foreseen it, if any light from the true analytic doctrine had enlightened them. On the other hand, those who hate these double bladed weapons, *"one mouthed"* are fainthearted and resent now Archimedes being attacked. But Archimedes lives. And indeed *"the false proofs about what is true, the false arithmetic"*, the Unproved, magnificent words, shock him.

I make known

"those who carry round shields decorated with clouds,
and weapons unreached by actions",

and even *"double bladed axes"*, of which they are first fortified, to inspire *"wars"*, in case maybe the enemies' boldness would be strong enough.

Proposition I.

The ratio of the circumference of the dodecagon inscribed in the circle to the diameter is smaller than the triple and one eighth.

I draw from center A, with a scalene interval AB, the circle BCD, on which is taken the circumference of the hexagon, which is cut into two parts at D, and let DB be subtended. Then DB is the side of the dodecagon which, multiplied by twelve up to E, DE, will be equal to the circumference of the dodecagon inscribed in the circle BDC. On the other hand, let the diameter DF be drawn. I say that the ratio of DE to DF is smaller than the ratio of the triple to one eighth.

In fact, let BC and BA be joined and let DF intersect BC even in G. It will divide it into halves and at right angles. On the other hand, let triangle DEH be drawn similar to the triangle DBG.

Since line BC is subtended on the hexagon circumference, for this reason BA or DA is equal to BC itself. This is why AC or BC being made of eight parts, BG is made of four same parts. And the square of AG is 48, and for this reason AG is larger than $6\frac{12}{13}$. On the other hand, DG itself is smaller than $1\frac{1}{13}$. And since DE was made of twelve times DB itself, EH will also be twelve times BG itself, and

---

END OF PAGE 437

DH is twelve times DG itself. This is why EH will be made of 48 same parts. And DH will be smaller than 13, much smaller than $12\frac{12}{13}$. On the other hand, the square of the side 48 is 2,304; surely the square of 13 is 169. And these two squares make 2473, not quite 2500, square of the side 50. This is why line DE, the square of which is equal to the squares of EH, DH, is smaller than 50. And the ratio of 50 to 16 is the triple and one eighth exactly. Then the ratio of DE to DF is smaller than the triple and one eighth. Which was to be proved.

Science is absolutely as much arithmetic as geometric. Rational magnitudes are precisely designated by rational numbers, the irrationals by irrationals. And if someone measures quantities with numbers, and if he perceives them through his calculation different from what they really are, it is not the art's mistake but the worker's mistake.

Even more truly, as Proclus says, "Arithmetic is more exact than Geometry". If carefully calculated, the diameter being made of one part, the circumference of the dodecagon is made of the side of the binomial $72 - \sqrt{3,888}$. *Whoever says otherwise is mistaken, either the geometer in his measurements, or the calculator in his numbers.*

On the other hand, the circumference of the circle is greater than the triple plus one eighth, as the school of mathematicians did not doubt up to now that it is smaller than one seventh. Because Archimedes truly demonstrated that. Then, "a manifest absurdity" it was not to be induced according to a wrong calculation. The straight line is larger than the circular contained by the same endpoints, Archimedes taking the contrary "from the common nations" and also Eutocius demonstrating the same, and generally exposing "of all lines having the same extremities, the straight line is the shortest".

Proposition II.

The part of the half diameter of the circle divided by the quadratix, from the center up to the quadratix, is larger than the mean proportional between the half diameter and two fifths of the half diameter.

Given the quarter of circle ABC, BD the quadratix; let AE be taken equal to two fifths of the half diameter AB or AC; and let the mean proportional of AE, AC be AF. I say that AD is larger than AF.

Indeed from what was demonstrated by Pappus about the quadratix, the half diameter AB or AC is the mean proportional of the circumference BC and AD. Let AB be made of 7 parts. The circumference BC, which is one quarter of the perimeter, will be less than 11. Indeed, the existing diameter 14 is less than the perimeter 44. On the other hand let AB be 35. The circumference [BC] will be less than 55. And

---

END OF PAGE 438

AD, the area contained by [the arc] BC, is equal to the square of AB. This is why AD will be larger than $22\frac{3}{11}$. On the other hand, AB, which is AC, is 35, and AE 14 and AF is smaller then $22\frac{3}{22}$. Then AD will be larger than AF. Which was to be proved.

This is why if from the diameter AB is subtracted the line AG equal to AF itself and if the parallelogram GHDA is completed, it will be "another length", not a square. And when the square BC is completed, the diagonal BK will not go through H, but through a certain point I further from point D. And it was worth noting that this wrong representation was to be avoided.

Proposition III.

The square of the circumference of the circle is less than the decuple of the square of the diameter.

Let the diameter be 7. The square of the diameter will be 49. And the decuple of the same 490. But the circumference of the circle will be less than 22, and consequently its square less than 484.

On the other hand, this opinion from the Arabs about the circle to be squared has been rejected for a long time: the square of the circumference of the circle is the decuple of the square of the diameter. *Surely what would be proposed opposed to, "contradictory to" against Archimedes' demonstration, will not be related.*

Proposition IV.

The ratio of the circle to the inscribed hexagon is greater than the ratio of 6 to 5.

Let hexagon BCDEFG be inscribed in the circle with center A. I say that the ratio of the circle with center A to the hexagon BCDEFG is greater than the ratio of 6 to 5.

Indeed, AB, AC, BC, being joined, let the perpendicular AZ be on BC.

Then since in triangle ABC, the legs AB, AC are equal, the base was cut in two at Z and BZ, ZC are equal. On the other hand, triangle ABC is equilateral. Indeed, the two legs are a half diameter. But since the base is the side of the hexagon, it is equal to the half diameter. Then half diameter BA or AC being made 30, BZ or ZC is 15 and AZ is less than 26, the square of which is 676, since indeed the difference of the squares of AB, BZ is exactly 675. Then the rectangle contained by BZ, AZ is equal to triangle BAC. And thus, let 15 be multiplied by 26, which gives 390. Then as the square of AB will be 900, as triangle ABC will be less than 390, or (all being divided by 30) the square of AB being 30, triangle ABC will be less than 13. Let AD, AE, AF, AG be joined. Then hexagon BCDEFG is made of six triangles equal to BAC itself. This is why if the square of AB is 30, the hexagon will be less than 78. Or if the square AB is five, the hexagon will be less than thirteen parts.

END OF PAGE 439

But surely as is the ratio of the perimeter of the circle to the diameter, such is the ratio of the area made by the perimeter of the circle and the quarter of the diameter to the area made by the diameter and the quarter of the diameter. But the area made by the perimeter of the circle and the quarter of the diameter is equal to the circle. On the other hand the area made by the diameter and the quarter of the diameter is the square itself of the half diameter. Then as is the ratio of the perimeter to the diameter, such is the ratio of the circle to the square of the half diameter. On the other hand, if diameter is 1, the perimeter is larger than $3\frac{10}{71}$ and even more obviously larger than $3\frac{10}{80}$ or $3\frac{1}{8}$. Then if the square of the half diameter AB is five, like before, the circle will be larger than $15\frac{5}{8}$. On the other hand, the hexagon was in the same parts less than 13. This is why the ratio of the circle to the inscribed hexagon will be larger than the ratio of $15\frac{5}{8}$ to 13, which is, of 125 to 104, or 6 to $4\frac{124}{125}$, and, more obviously, greater than the ratio of six to five. Which was to be proved.

Then those who made it equal to the hexagon and the fifth part of the hexagon, do not square the circle "according to the thing", since it is made larger according to the limits set by Archimedes "from the same principles". On the other hand, *they belong to our Platonic School, oh naïve professors. This is why do not fight against our geometric principles. And surely so that they will not amputate the circle "using axes", that in spite of the received damage, they might be shortened toward a sharper end of their sparrow tail.*

Proposition V.

Thirty-six segments of the hexagon are larger than the circle.

Indeed since the ratio of the circle to its inscribed hexagon is larger than the ratio of six to five, or of one to five sixths, for this reason, the difference between the circle and the hexagon is six segments of the hexagon. The six segments of the hexagon are more than one sixth of the circle and so, thirty-six segments will be larger than the unit of the circle. Which was to be proved.

Proposition VI.

Any segment of the circle is larger than the sixth of the similar segment similarly drawn in a circle whose radius is equal to the base of the proposed segment.

In circle BDC drawn under center A, let the circumference BD that you want, be subtended; on the other hand, let line BE touch the circle and from center B with the interval BC let another circle be drawn.

So the circumference ED will be similar to half of the circumference BD. And thus let circumference DF be taken double of DE itself and let BF, AD be joined. Then the sectors BAD, FBD will be similar. I say that the segment of circle BDC contained by line BD and the circumference by which it is subtended, is larger than one sixth of sector FBD.

END OF PAGE 440

Indeed, let the spiral line be drawn, with origin B, passing through D, BD increasing the beginning of rotation BEZ as large as angle BED is the part of four right angles. Then, since the third part of sector EBD is the area contained between line BD and the spiral. Because Pappus demonstrated that according to Archimedes, proposition XXII of book IV of Mathematical Collections, surely, the area of sector FBD will consequently be the same as the sextuple part [of the area contained between the line BD and the spiral]; because sector FBD was made double of sector EBD. And surely the spiral will not intersect with the circumference, because it would be absurd, and the spiral in its progression does not come outside the circle before reaching point D; for however that angle EBD is cut by line BGH, intercepting the spiral at G, and the circumference at H. The ratio of line BG to line BD will be the one of angle EBD to angle EBG, which is the one of circumference BD to circumference BH, according to the condition on spirals. But the ratio of circumference BD to circumference BH is larger than the ratio of "the sublending BD to BH. Indeed the ratio of larger circumference to smaller ones is larger than the ratio of the lines with the lines subtended by the same circumferences. This is why line BH will exceed line BG. And the same will happen in any line we want, intersecting angle EBD. And thus the spiral will go under circumference BD and it will leave a certain space between itself and the circumference. And surely through this space the segment of circle limited by line BD and the circumference, will exceed the space between the same line and the spiral which was proved equal to a sixth of sector FBD then this segment will be larger than the sixth of sector FBD. Which was to be proved.

Corollary.

And from there, it is equally obvious that thirty-six segments of the hexagon are greater than the circle.

Indeed, when BD will reach to be the segment of the hexagon, the sectors FBD, BAD, will be equal since their half diameters BD, AD will be equal. Then six segments of the hexagon will be larger than sector BAD, in such a way that thirty-six segments will be larger than six sectors, which is, than the whole circle.

A theorem no less general could be proposed to be proved by the geometric mean, through parabolas, or rather by the way through which parabolas are squared. Any circle segment is larger than one and one-third the segment itself fixed at the base of the inscribed isosceles triangle. *As it will soon appear that the ratio of thirty-six hexagon segments to the circle, is larger than the one of 48 to 47. Even more through more cautious calculation, only the thirty-four segments and the space a little larger than two thirds of the segment, about less than three quarters, are found to fill up the circle. On the other hand it is possible to show this way to the eyes what in the segment exceeds one third of the twelfth.*

Proposition VII.

In a given circle, it is required to cut off from the hexagon segment, the thirty-sixth part of the circle itself.

Given a circle of center A, diameter BC, the hexagon segment BD. The thirty-sixth part of the circle itself needs to be taken apart from the given circle BDC, the hexagon segment BD limited by line BD and the circumference which is subtended.

Let line BE touch the circle and let a spiral line starting at B, going through D be drawn, BD being a part of the conversion as much as part of an angle EBD is four rights, then the third part of sector EBD is the space limited by line BD and the spiral. On the other hand, BD is equal to half diameter BA, because BD is through hypothesis the hexagon side and angle EBD is the third of one right, since the circumference BD is by size two thirds of a right. Then sector EBD is the twelfth of the circle, and consequently the spiral space BD is the third of on twelfth, which is the thirty-sixth part of the circle. On the other hand, the spiral will intersect the segment, it will not be concurrent with the circle, or it will cut off a circular space in the advancement from B, before

---

END OF PAGE 441

---

it reaches D, like it was proved.

Then in the given circle BDC, the thirty-sixth part of the circle itself was cut off from the hexagon segment BD. Which was to be done.

And let the blunt hatchet blade be weakened enough by this shield made of seven soft leathers.

If some wanted a rough draft of this *"fight with the axe"* itself, let them not miss it, let them watch it in these short pages.

Analysis of the Circle.

According to the "*axe cuts*".

I.

The circle consists of six hexagon cuttings.

II.

The hexagon cutting consists of the hexagon segment and the hexagon triangle, or the larger.

III.

The hexagon triangle, be it larger, consists of the hexagon segment and one hatchet.

IV.

The hatchet consists of two hexagon segments and the hatchet complement.

V.

The complement of the hatchet consists of the hexagon segment and the remaining segment.

VI.

Again the complement of the hatchet consists of the smaller triangle and the remaining of the smaller triangle. On the other hand, the smaller triangle is the fifth part of hexagon triangle, or of the larger.

END OF PAGE 442

Two True Lemmas.

The first.

Ten smaller triangles are equal to six hexagon segments and to two complements of the hatchet.

Indeed the hexagon triangle consists of the segment and the hatchet, and the hatchet consists of two segments and the complement; for this reason, two hexagon triangles consist in six segments and two complements. Which was to be proven.

The second.

Forty smaller triangles equal the circle and two complements of the hatchet.

Indeed since the circle is equal to six cuttings of the hexagonal sectors, since on the other hand the sectors are equal to six hexagon triangles and six segments, since then six hexagon triangles make thirty smaller triangles: for this reason, the circle equals thirty smaller triangles and six segments. On both sides, let two complements of the hatchet be added. Then the circle along with two complemants of the hatchet will be equal to thirty smaller triangles, according to the above lemma: then forty smaller triangles equal six hexagon segments and two complements of the hatchet. Which was to be proven.

"Fallacies".

I say that a smaller triangle equals its remaining.

"As proof".

Indeed since the circle with the complements of the hatchet (which indeed make two smaller triangles and two remainings of the smaller triangle) equals thirty-six smaller triangles and again four. On the other hand, let two smaller triangles be taken way. There, when they will be taken out of two complements, they will leave two remainings of the triangle. Here, when they will be taken out of four triangles, they will leave two triangles. So the two remaining are equal to two triangles.

"Illogical Refutation".

Equal parts can be taken out of equal whole parts, not out of parts of equals, so that the ones left are equal. To take away a part of equals is to take the remaining of the whole equal to the remaining, as here the circle equals thirty-six smaller triangles. Surely, this is formally denied and is very wrong. To accept that it is demonstrated by itself, is to want to seem to be mistaken demonstratively.

Two Other True Lemmas for Another "*Fallacy*".

The first.

Twenty-four parts of the hexagon triangle and six segments equal twenty-four segments and six complements of the hatchet.

Indeed, since the hexagon triangle consists of three segments and one complement of the hatchet, on the other hand, since the circle is made of six triangles and six segments with six complements equal the circle. And because four fourths make the unit, twenty-four quarters of the hexagon circle, with at the same time six segments, will equally equal the circle. On the other hand the ones equal to the unit are equal between themselves. This is why twenty-four quarters of the hexagon triangle and six segments equal twenty-four segments and six complements of the hatchet. Which was to be proven.

END OF PAGE 443

The Second.

Given three unequal quantities, if the medium taken twenty-four times and added to the same one taken six times equals the smallest one taken twenty-four times and added to the smallest on taken six time equals the smallest one taken twenty-four times and added to the largest taken six times: the difference between the quadruple of the middle one and the triple of the smallest equals the largest one.

Indeed let B be the smallest, D the middle one, A the largest. Then according to the hypothesis 6B plus 24D will equal 6A plus 24B. On both sides, let 24B be subtracted. Then 24D minus 18B will equal 6A. And all being divided by 6D, 4D minus 3B will equal A. Which had to be stated.

"Unproved Theorem".

There are three unequal figures in the plane and commensurable between themselves, the smallest is a hexagon segment; the middle one is the quarter of the hexagon triangle; the largest is the complement of the hatchet-

The inequality and the degree of this inequality could be proved but nobody will ever prove either the symmetry or the asymmetry, except if he first compares the hexagon triangle to the circle or another line; surely this comparison is not known up to here, *"and if precisely there is an equal success, it is lying in the Gods' knees"*.

"Pseudo-Porism".

And thus as the quarter of the hexagon triangle is made of five parts, the segment must be four.

"As proof".

Let B be the hexagon segment, D the quarter of the hexagon triangle, Z the complement of the hatchet. Then the three quantities are unequal, and the smallest of these, D the medium, Z the largest, are related to each other as a number to another. Let D be made of five parts, such as B will be made of three or four, and no more. On the other part, but if this can be done, let it be made of three parts; according to the first and second lemma, Z will be made of eleven. And thus the complement will consist of two segments and three-quarters. On the other hand, this meaning inspires repugnance. This is why B is made of four.

"Illogical Refutation".

Quantity D is given made of five parts, B can be shown larger than three parts. Surely will B made of four for this reason? This being accepted, is it denied that B relates to D as a number to a number? The conclusion is not at all syllogistic. Indeed, if B is made of four parts and some small rational fraction, why is not four and a half related to five as a number to a number, which is like 9 to 10, other than *"illogic or non geometric"*. Reasonably D is being made of 11 parts, B is made a little larger than 9, and Z a little smaller than 17, according to Archimedes limits. On the other hand, out of these two *"false fights"* were spread the remainings of the *"sectors with the hatchet according to the area of the circle and the wrong surfaces of the spheres".*

"End of comparison to a hatchet's sector".

END OF PAGE 444

Second Example of "Fight with an Axe".

"An outline, From the Addendum".

In the circle with center A, let the hexagon circumference BCD be taken and let AB, AD, BD be joined. On the other hand, from AB, let a line be drawn, having the ratio of its square to the square of AB the same as the ratio of one to five. And that, through E, the parallel to AD itself be drawn intersecting BD at F.

And thus let triangle BEF made of equal angles, and the fifth of it.

Lemma I. True.

Thirty-seven triangles BEF are larger than circle BCD.

Indeed, in what is written for the Mathematical Table, it was proved that the circle relates to the square of the half diameter very close to 31, 415, 926, 536 to 10, 000, 000, 000. On the other hand, side AB, which is the half diameter, being made of 100, 000 small parts, the height of the equilateral triangle ABD is 86, $602^{5}4, 036\backslash 100, 000$. And thus:

The triangle ABD is made of	4, 330, 127, 019
The triangle BEF,	866, 025, 404
Thirty-seven triangles BE	32, 042, 939, 948
Exceeding the circle	31, 415, 926, 536
By the small parts	627, 013, 412

Lemma II. True.

The circle BCD is not larger than thirty-six segments BCDF.

Even more, circle BCD is much smaller than thirty-six segments BCDF. Indeed sector BAD is one sixth of the whole circle.

And thus:

If the circle is	31, 415, 926, 536
The sector BAD is	5, 235, 987, 756
Let triangles ABD be subtracted from it	4, 330, 127, 019
It remains the hexagon sector	
or the mixtiline space BCDF	905, 860, 737
On the other hand three dozens of such segments are	32, 610, 986, 532
Exceeding the circle by the small parts	1, 195, 059, 996

END OF PAGE 445

"Pseudo-Porism".

Then thirty-seven triangles BEF are larger than thirty segments BCDF.

"Illogical Refutation".

In the grammar, to give to the ship the south winds, or to give the ships to the south winds, have the same meaning.

But in geometry, to say that circle BCD is not larger than thirty-six segments BCDF is different from saying that thirty-six segments BCDF are not larger than circle BCD. The first is true, the second is false.

Then when I argue this way:

Thirty-seven triangles are larger than the circle.

But thirty-six segments are not larger than the circle.

Then thirty-seven triangles are larger than thirty-six segments.

I conclude syllogistically, but wrongly, because I take the wrong. On the other hand, I sin in the logic laws when I establish the syllogism in this formula:

The circle is smaller than thirty-seven triangles.

The circle is not larger than thirty-six segments.

Then thirty-seven triangles are large than thirty-six segments.

On the other hand, there is "*loss of the eyes*" not "*of the intelligence*". Indeed since truly, at the beginning, it was proposed that the measure of the circumference of the circle is not larger than thirty-six hexagon segments, it is read, according to what was done above, that it is neither smaller, and, from there, was extracted its wrong corollary.

END.

Translation of Article 12 : Christiani Hugenii Determining the Magnitude of the Circle

Problem IV. — Proposition XX.

To find the ratio between the circumference and the diameter; and through given chords in a given circle, to find the length of the arcs which they subtend.

Let a circle of center D, with CD as a diameter, and let AB be the arc one sextant of the circumference, for which we draw the chord AB and the sine AM. If we suppose then that the half diameter DB is 100,000 parts, the chord BA will contain the same number. But AM will be made of 86,603 parts and not one less (which means that if we would take away one part or one unit of the 86,603 we would have less than what it should be), since it is half of the side of the equilateral triangle inscribed in the circle.

From there, the excess of AB over AM becomes 13397, less than the true value. One third of it is $4465\frac{2}{3}$, which, added to 100,000 of AB, gives $104{,}465\frac{2}{3}$, which is less than arc AB. And this is a first lower limit; in the following, we will find another one, closer to the real value. But first we must also find an upper limit, according to the same theorem.

Then a fourth proportional is to be found for three numbers. The first equals the double parts of AB and the triple of AM. It will then be 459807, less than the real value, (as we also have to make sure that this number here is less; and in the same way with the other as we will specify) the second is equal to the quadruple of AB and to AM, which is 486,603, more than the real value. And the third is one third of the excess of AB over AM, 4466, more than the real value, which, added to AB or 100,000 gives 104 727, larger than the number of parts which are AB sextant of the periphery (according to the above). Then we already found the length of arc AB within an upper and lower limit, of which two the last is far closer to the real value because the number 104, 719 is closer to the real value.

But through these two, we will obtain another lower limit more exact than the first one, using the following precept, which results from a more precise examination of the center of gravity.

END OF PAGE 384

Add the four thirds of the difference of the above limits to the double of the chord and the triple of the sine, and the same ratio as between the line made this way and three and one third, or $\frac{10}{3}$ times the sum of the sine and the chord, also exists between the excess of the chord over the sine and another line ; this last one added to the sine will be a line smaller than the arc.

The lower limit was $104{,}465\frac{2}{3}$; the upper one is 10,427; their difference is $261\frac{1}{3}$. Again we need to find a fourth proportional to three numbers. The first one is the double of the parts AB increased by the triple of AM and by the four thirds of the difference of the limits; we find 460, 158, larger than the real value. The second is the $\frac{10}{3}$ of AB and AM taken together, 622,008, smaller than the real value. Last the third is the excess of AB over AM, 13, 397, smaller than the real value. The fourth proportional to these numbers is 18,109, smaller than the real value.

Then if we add this to the number of parts of AM, $86{,}602\frac{1}{2}$, less than the real value, we get 104, $711\frac{1}{2}$ less than arc AB. Thus the sextuple of these parts, 628,269, will be less than the whole circumference. But because 104727 of these parts were found larger than Arc AB, their sextuple will be larger than the circumference. Thus the ratio of the circumference to the diameter is smaller than the one of 628,362 to 200,000 and larger than the one of 628,268 to 200,000, or smaller than the one of between 314,181 and 100,000 and larger than the one of 314,135 to 100,000. From that, the ratio is certainly smaller than $3\frac{1}{7}$ and larger than $3\frac{10}{71}$. From there also is refuted Longomontanus' mistake, who wrote that the periphery is larger than 314,182 parts, when the radius contains 100,000.

Let us suppose that arc AB is $\frac{1}{8}$ of the circumference; than AM, half of the side of the square inscribed in the circle, will measure 7,071,068 parts, and the radius DB measures 1,000,000, and not one less. On the other hand AB, side of the octagon, measures 7,653,668 parts and not one more. Through this data, we will find, in the same manner as above, as first lower limit of the length of arc AB 7,847,868. Then as upper limit 7,854,066. And from these two, again, a more precise lower limit 7,853,885. This results in the ratio of the periphery to the diameter being less than $31{,}416\frac{1}{4}$ and more than 31,415 to 10,000.

And as the difference between the upper limit 7,854,066 and the real length of the arc

---

END OF PAGE 385

---

is less than 85 parts (arc AB, according to what we proved above is larger than 7,853,981) and as 85 parts are less than two seconds, which is $\frac{21}{1{,}096{,}000}$ of the circumference, because the whole circumference has more than 60,000,000 parts, it is obvious that, if we try to find the angles of a right triangle using the given sides, the same way as we did for the upper limit above, the error can never be more than two seconds; even if the sides of the right angle are equal, as they were here in triangle DAM.

But if the ratio of side DM to MA is such that the angle ADM does not exceed a quarter of a right angle, the error will not be more than a third. Because, taking arc AB equal to $\frac{1}{10}$ of the circumference, AM will be half of the side of the equilateral octagon inscribed in the circle, and equal to 382,683,433 parts and not more, but AB will be the side of the sixteen sides polygon and then will contain 390,180,644 parts, and not one more, with the radius DB containing 1,000,000,000 parts. This way is found a first lower limit of the length of arc AB of 392,679,714 parts. And the upper limit is 392,699,148. And from there again a lower limit of 392,699,010. But, what was proved above results in arc AB, $\frac{1}{10}$ of the circumference, being larger than 392,699,081 parts, which the upper limit exceeds by 67 parts. But these are less than a third, which is $\frac{1}{77,700,000}$ of the whole circumference , since the circumference is larger than 6,000,000,000.

Then, out of these new limits just found, the ratio of the circumference to the diameter will come smaller than $314593\frac{1}{6}$ to 1,000,000 but larger than 314,592 to 1,000,000.

And if we take an arc AB equal to $\frac{1}{60}$ of the circumference, which is six parts of the total 360, AM will be half of the side of the (inscribed) polygon with 30 angles, made of 10,452,846,326,766 parts with a radius of 100,000,000,000,000, and not one less. And AB is the side of the (inscribed)8 polygon with 60 angles, 10,467,191,248,588 and not one more. Through this data is found arc AB according to the first lower limit 10,471,972,889,195 then the upper limit 10,471,975,512,584. And from there the other lower limit 10,471,975,511,302. This results in the ratio of the periphery to the diameter being less than

---

END OF PAGE 386

31,415,926,538 to 10,000,000,000, but larger than 31,415,926,533 to 10,000,000,000.

If we had to find these limits through adding the sides of the inscribed and circumscribed polygons, we would have to go up to 400,000 sides . Because with the 60 angles inscribed and circumscribed polygons, we only prove that the ratio of the periphery to the diameter is less than 3,145 to 1,000, and larger than 3,140. Thus, the number of exact digits through this calculation seems to be three times higher, and even more. But if someone tries it, he will see that the same always happens with the following polygons; we know why but it would take a long explanation .

On the other hand, I believe it is clear enough how, for other inscribed scalene polygons, it is possible to find, through the above methods, the length of the arcs subtended. Because, if they are larger than the side of the inscribed square, we will have to find the length of the remaining arc on the half circumference, the chord of which is then also given. But we must also know how to find the chords of the half-arcs, when the chord of the full arc is given. And this way, if we want to use bisections, we will be able to find without any difficulty for any chord the length of its arc, as close as we want. This is useful for examining tables of sines, and even for their composition; because, knowing the chord of a given arc, we can determine with enough accuracy the length of the arc which is slightly larger or smaller.

Bibliography

Papers marked with '*' appear in the Source Book.

*W. W. Adams, Asymptotic Diophantine Approximations to E, *Proceedings of the National Academy of Sciences U.S.A.* **55** (1966), 28-31.

S. Akiyama, Lehmer Numbers and an Asymptotic Formula for π, *Journal of Number Theory* **36** (1990), 328-331.

K. Alladi and M. L. Robinson, *On Certain Irrational Values of the Logarithm,* in: Lecture Notes in Mathematics 751, Springer-Verlag, Berlin, 1979, pp. 1-9.

T. S. Ang and F. J. Swetz, A Chinese Mathematical Classic of the Third Century: *The Sea Island Mathematical Manual* of Liu Hui, *Historia Mathematica* **13** (1986), 99-117.

*T. M. Apostol, A Proof that Euler Missed: Evaluating $\zeta(2)$ the Easy Way, *The Mathematical Intelligencer* **5** (1983), 59-60.

*D. H. Bailey, The Computation of π to 29,360,000 Decimal Digits Using Borweins' Quartically Convergent Algorithm, *Mathematics of Computation* **50** (1988), 283-296.

D. H. Bailey, J. M. Borwein, P. B. Borwein and S. Plouffe, The Quest for Pi, *The Mathematical Intelligencer* **19** (1997), 50-57.

*D. H. Bailey, P. B. Borwein and S. Plouffe, On the Rapid Computation of Various Polylogarithmic Constants, *Mathematics of Computation* **66** (1997), 903-913.

*A. Baker, Approximations to the Logarithms of Certain Rational Numbers, *Acta Arithmetica* **10** (1964), 315-323.

*J. P. Ballantine, The Best (?) Formula for Computing π to a Thousand Places, *The American Mathematical Monthly* **46** (1939), 499-501.

A. K. Bag, Indian Literature on Mathematics During 1400-1800 AD, *Indian J. Hist. Sci.* **15** (1980), 79-93.

P. Beckmann, *A History of Pi*, 4th ed., Golem Press, Boulder, CO, 1977.

M. Beeler, R. W. Gosper, and R. Schroeppel, "Hakmem", MIT Artificial Intelligence Lab., MIT. Cambridge, MA, 1972.

*F. Beukers, A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$, *Bulletin of the London Mathematical Society* **11** (1979), 268-272.

*F. Beukers, J. P. Bézivin and P. Robba, An Alternative Proof of the Lindemann-Weierstrass Theorem, *The American Mathematical Monthly* **97** (1990), 193-197.

*R. H. Birch, An Algorithm for the Construction of Arctangent Relations, *Journal of the London Mathematical Society* **21** (1946), 173-174.

G. Birkhoff, *A Source Book in Classical Analysis,* Harvard Univ. Press, Cambridge, MA, 1973.

C. W. Borchardt, *Gesammelte Werke,* Berlin, 1888.

*J. M. Borwein and P. B. Borwein, The Arithmetic-Geometric Mean and Fast Computation of Elementary Functions, *SIAM Review* **26** (1984), 351-366.

J. M. Borwein and P. B. Borwein, *Pi and the AGM,* John Wiley & Sons, Inc., New York, 1987.

*J. M. Borwein and P. B. Borwein, Ramanujan and Pi, *Scientific American* **256** (1988), 112-117.

*J. M. Borwein, P. B. Borwein and D. H. Bailey, Ramanujan, Modular Equations, and Approximations to Pi, or How to Compute One Billion Digits of Pi, *The American Mathematical Monthly* **96** (1989), 201-219.

*J. M. Borwein, P. B. Borwein and K. Dilcher, Pi, Euler Numbers, and Asymptotic Expansions, *The American Mathematical Monthly* **96** (1989), 681-687.

C. B. Boyer, *A History of Mathematics,* John Wiley & Sons, Inc., NY, 1968.

R. P. Brent, Multiple-Precision Zero-Finding Methods and the Complexity of Elementary Function Evaluation, in: *Analytic Computational Complexity* (J. F. Traub, Ed.), Academic Press, New York, 1976, pp. 151-176.

*R. P. Brent, Fast Multiple-Precision Evaluation of Elementary Functions, *Journal of the ACM* **23** (1976), 242-251.

R. P. Brent, The Complexity of Multiple-Precision Arithmetic, in: *Complexity of Computational Problem Solving* (R. S Anderssen and R. P. Brent, Eds.), Univ. of Queensland Press, Brisbane, 1976.

*R. P. Brent and E. M. McMillan, Some New Algorithms for High-Precision Computation of Euler's Constant, *Mathematics of Computation* **34** (1980), 305-312.

*A. B. Chace, *The Rhind Mathematical Papyrus,* The National Council of Teachers of Mathematics, Reston, Virginia, 1979, pp. 92-93 & Plate 72 (Problem 50).

J. Chan, As Easy As Pi, *Math Horizons* **Winter** (1993), 18-19.

*D. V. Chudnovsky and G. V. Chudnovsky, Approximations and Complex Multiplication According to Ramanujan, in: *Ramanujan Revisited* (G. E. Andrews, R. A. Askey, B. C. Berndt, K. G. Ramanathan & R. A. Rankin, Eds.), Academic Press Inc., Boston, MA, 1988, pp. 375-396 & 468-472.

*M. Clagett, *Archimedes in the Middle Ages, Volume 1: The Arabo-Latin Tradition,* University of Wisconsin Press, Madison, 1964, pp. 261-269.

W. E. Clark (trans.), *The* Āryabhaṭīya *of Arybhaṭa,* Univ. of Chicago Press, Chicago, Ill., 1930.

*D. A. Cox, The Arithmetic-Geometric Mean of Gauss, *L'Enseignement Mathématique* **30** (1984), 275-330.

P. J. Davis, *The Lore of Large Numbers,* Random House, New York, 1961.

D. Desbrow, On the Irrationality of π^2, *The American Mathematical Monthly* **97** (1990), 903-906.

G. A. Dickinson, Wallis's Product for $\pi/2$, *The Mathematical Gazette* **21** (1937), 135-139.

H. -D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukrich, A. Prestel & R. Remmert, *Numbers,* Springer-Verlag, New York, 1991.

*Umberto Eco, *Foucault's Pendulum,* Ballantine, 1988, p. 3.

*W. E. Edington, House Bill No. 246, Indiana State Legislature, 1897, *Proceedings of the Indiana Academy of Science* **45** (1935), 206-210.

C. H. Edwards, Jr., *The Historical Development of the Calculus,* Springer-Verlag, New York, 1979.

*H. Engels, Quadrature of the Circle in Ancient Egypt, *Historia Mathematica* **4** (1977), 137-140.

*L. Euler, *Introduction to Analysis of the Infinite. Book 1,* Springer-Verlag, New York, 1988-1990, pp. 137-153. (Translated from the Latin by J. D. Blanton)

*H. W. Eves, *In Mathematical Circles; A Selection of Mathematical Stories and Anecdotes,* Prindle Weber & Schmidt, Inc., Boston, MA, 1969, pp. 81-82.

*H. W. Eves, *Mathematical Circles Revisited; A Second Collection of Mathematical Stories and Anecdotes,* Prindle, Weber & Schmidt, Inc., Boston, MA, 1971, pp. 25-34.

J. A. Ewell, A New Series Representation for $\zeta(3)$, *The American Mathematical Monthly* **97** (1990), 219-220.

J. Fauvel and J. Gray, *The History of Mathematics: A Reader,* Macmillan Education Ltd in association with the Open University, Basingstoke, 1987.

C. F. Gauss, *Werke,* vol. 3, Göttingen, 1866-1933, pp. 361-403.

*E. J. Goodwin, Quadrature of the Circle, *The American Mathematical Monthly* **1** (1894), 246-247.

C. C. Gillespie (ed.), *Dictionary of Scientific Biography,* Scribner, New York, 1970-1980.

R. C. Gupta, New Indian Values of π from the Mānava Sulba Sūtra, *Centaurus* **31** (1988), 114-125.

G. H. Hardy, *Ramanujan,* Cambridge Univ. Press, London, 1940.

M. Hata, Improvement in the Irrationality Measures of π and π^2, *Proceedings Japan Acad. Ser. A Math. Sci.* **68** (1992), 283-286.

M. Hata, Rational Approximations to π and Some Other Numbers, *Acta Arithmetica* **63** (1993), 335-349.

M. Hata, A Lower Bound for Rational Approximations to π, *Journal of Number Theory* **43** (1993), 51-67.

T. Hayashi, T. Kusuba and T. Yano, The Correction of the Mādhava Series for the Circumference of a Circle, *Centaurus* **33** (1990), 149-174.

*T. L Heath (trans.), *The Works of Archimedes with the Method of Archimedes,* Dover Publications Inc., NY, 1953, pp. 91-98.

*C. Hermite, *Oeuvres,* vol. 3, Paris, 1912, pp. 150-181.

C. Hermite, *Oeuvres,* vol. 4, Paris, 1917, pp. 357-377.

*D. Hilbert, Ueber die Transcendez der Zahlen e und π, *Mathematische Annalen* **43** (1893), 216-219.

E. W. Hobson, H. P. Hudson, A. N. Singh and A. B. Kempe, *Squaring the Circle and Other Monographs,* Chelsea Publishing Co., New York, 1953.

*W. Hope-Jones, Correspondence. Ludolph (or Ludolff or Lucius) van Ceulen, *The Mathematical Gazette* **22** (1938), 281-282.

*C. Huygens, *Christiani Hugenii Opera Varia, I,* Leiden, 1724, pp. 384-388.

C. Jami, Une Histoire Chinoise du "nombre pi," *Archive for History of Exact Sciences* **38** (1988), 39-50.

*W. Jones, The First Use of Pi for the Circle Ratio, in: *A Source Book in Mathematics*, Volume II (D. E. Smith, Ed.), Dover Publications Inc., NY, 1959, pp. 346-347.

*Y. Kanada, Vectorization of Multiple-Precision Arithmetic Program and 201,326,000 Decimal Digits of π Calculation, *Supercomputing 88: Volume II, Science and Applications,* 117-128.

Y. Kanada, Y. Tamura, S. Yoshino & Y. Ushiro, Calculation of π to 10,013,395 Decimal Places Based on the Gauss-Legendre Algorithm and Gauss Arctangent Relation, Computer Centre, University of Tokyo, Technical Report-84-01.

G. R. Kaye, Notes on Indian Mathematics. No. 2.Āryabhata, *Journal of the Asiatic Society of Bengal* **IV** (1908), 111-141.

*M. Keith, Circle Digits. A Self-Referential Story, *The Mathematical Intelligencer* **8** (1986), 56-57.

*M. Keith, Pi Mnemonics and the Art of Constrained Writing (1996), unpublished.

P. Kiss and F. Mátyás, An Asymptotic Formula for π, *Journal of Number Theory* **31** (1989), 255-259.

F. Klein, *Famous Problems of Elementary Geometry,* transl. by W. W. Beman & D. E. Smith, Dover Publication Inc., New York, 1956.

D. E. Knuth, *The Art of Computer Programming.* vol. 2 *Seminumerical Algorithms,* Addison-Wesley Publishing Co., Reading, MA 1981.

*L. Y. Lam and T. S. Ang, Circle Measurements in Ancient China, *Historia Mathematica* **13** (1986), 325-340.

*A. M. Lambert, Mémoire sur Quelques Propriétés Remarquables des Quantitiés Transcendentes Circulaires et Logarithmiques, *Histoire de l'Académie Royale des Sciences et des Belles-Lettres der Berlin* **17** (1761), 265-276.

*A. M. Lambert, Irrationality of π, in: *A Source Book in Mathematics*, 1200-1800 (D. J. Struik, Ed.), Harvard University Press, Cambridge, MA, 1969, pp. 369-374.

A. M. Legendre, *Exercises de Calcul Intégral,* vols. 1-3, Paris, 1811-1819.

D. H. Lehmer, On Arctangent Relations for π, *The American Mathematical Monthly* **45** (1938), 657-664.

Le Petit Archimède, no. 64, "Numéro Spécial π," (1980).

F. Lindemann, Über die LUDOLPH'sche Zahl, *Berichte der Berliner Akademie* **2** (1882), 679-682.

*F. Lindemann, Ueber die Zahl π, *Mathematische Annalen* **20** (1882), 213-225.

J. Liouville, Sur des Classes Trés Étendues de Quantités dont la Valeur n'est ni Rationnelle ni même Réductible à des Irrationnelles Algébriques, *Comptes Rendus* **18** (1844), 883-885, 910-911.

K. Mahler, Zur Approximation der Exponentialfunktion und des Logarithmus, I, II, *J. Reine Ang. Math.* **166** (1931), 118-136, 137-150.

*K. Mahler, On the Approximation of π, *Indagationes Mathematicae* **15** (1953), 30-42.

*K. Mahler, Applications of Some Formulae by Hermite to the Approximation of Exponentials and Logarithms, *Mathematische Annalen* **168** (1967), 200-227.

E. Maor, *e: The Story of A Number,* Princeton University Press, Princeton, NJ, 1994.

K. M. Marur and C. T. Rajagopal, On the Hindu Quadrature of the Circle, *Journal of the Bombay Branch of the Royal Asiatic Society* **20** (1944), 65-82.

G. Miel, Of Calculations Past and Present: The Archimedian Algorithm, *The American Mathematical Monthly* **90** (1983), 17-35.

M. Mignotte, Approximations Rationnelles de π et Quelques Autres Nombres, *Bull. Soc. Math. France* Mem. **37** (1974), 121-132.

G. Németh, Relative Rational Approximation of the Function e^x, *Mathematical Notes* **21** (1977), 325-328.

*D. J. Newman, A Simplified Version of the Fast Algorithms of Brent and Salamin, *Mathematics of Computation* **44** (1985), 207-210.

*M. Newman and D. Shanks, On a Sequence Arising in Series for π, *Mathematics of Computation* **42** (1984), 199-217.

*I. Newton, *The Mathematical Works of Isaac Newton,* vol. 1, Johnson Reprint Corporation, New York, 1964, pp. 128-131.

*I. Niven, A Simple Proof that π is Irrational, *Bulletin of the American Mathematical Society* **53** (1947), 509.

*L. O'Shaugnessy, Putting God Back in Math, *The Mathematical Intelligencer* **5** (1983), 76-77.

*G. M. Phillips, Archimedes the Numerical Analyst, *The American Mathematical Monthly* **88** (1981), 165-169.

R. Preston, Profiles: The Mountains of Pi, *The New Yorker* **March 2** (1992), 36-67.

S. Rabinowitz and S. Wagon, A Spigot Algorithm for the Digits of π, *The American Mathematical Monthly* **102** (1995), 195-203.

C. T. Rajagopal and T. V. Vedamurthi Aiyar, On the Hindu Proof of Gregory's Series, *Scripta Mathematica* **17** (1951), 65-74.

*S. Ramanujan, Squaring the Circle, in: *S. Ramanujan; Collected Papers* (G. H. Hardy, P. V. Seshuaigar & B. M. Wilson, Eds.), Chelsea, New York, 1962, p. 22.

*S. Ramanujan, Modular Equations and Approximations to π, in: *S. Ramanujan: Collected Papers* (G. H. Hardy, P. V. Seshuaigar & B. M. Wilson, Eds.), Chelsea, New York, 1962, pp. 23-39.

*G. W. Reitwiesner, An ENIAC determination of π and e to more than 2000 Decimal Places *Mathematical Tables and other Aids to Computation* **4** (1950), 11-15.

*R. Roy, The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha *Mathematics Magazine* **63** (1990), 291-306.

F. Rudio, *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung,* Leipzig, 1892.

*E. Salamin, Computation of π Using Arithmetic-Geometric Mean, *Mathematics of Computation* **30** (1976), 565-570.

*H. C. Schepler, The Chronology of π, *Mathematics Magazine* **23** (1950), 165-170, 216-228 & 279-283.

D. Shanks, Dihedral Quartic Approximations and Series for π, *Journal of Number Theory* **14** (1982), 397-423.

*D. Shanks and J. W. Wrench, Jr., Calculation of π to 100,000 Decimals, *Mathematics of Computation* **16** (1962), 76-99.

*W. Shanks, *Contributions to Mathematics Comprising Chiefly the Rectification of the Circle to 607 Places of Decimals,* G. Bell, London, 1853, pp. i-xvii & 10.

*D. Singmaster, The Legal Values of Pi, *The Mathematical Intelligencer* **7** (1985), 69-72.

A. J. E. M. Smeur, On the Value Equivalent to π in Ancient Mathematical Texts. A New Interpretation, *Archive for History of Exact Sciences* **6** (1970), 249-270.

D. E. Smith and Y. Mikami, *A History of Japanese Mathematics,* Open Court Publishing Co., Chicago, 1914.

*M. D. Stern, A Remarkable Approximation to π, *The Mathematical Gazette* **69** (1985), 218-219.

J. Stillwell, *Mathematics and Its History,* Springer-Verlag, New York, 1989.

H. Sugiyama, Bayesian Point Estimation of π by Buffon's Needle and the Inverse Problem of Measuring Related Length, *Journal of Mathematical Analysis and Applications* **119** (1986), 72-81.

*D. W. Sweeney, On the Computation of Euler's Constant, *Mathematics of Computation* **17** (1963), 170-178.

F. Swetz, The Evolution of Mathematics in Ancient China, *Mathematics Magazine* **52** (1979), 10-19.

Y. Tamura and Y. Kanada, Calculation of π to 4,194,293, Decimals Based on Gauss-Legendre Algorithm, Computer Centre, University of Tokyo, Technical Report-83-01.

*J. Todd, The Lemniscate Constants, *Communications of the ACM* **18** (1975), 14-19.

H. W. Turnbull (ed.), *James Gregory Tercentenary Memorial Volume,* Bell, London, 1939, pp. 168-171.

*A. J. van der Poorten, A Proof that Euler Missed... , Apéry's Proof of the Irrationality of $\zeta(3)$, *The Mathematical Intelligencer* **1** (1979), 195-203.

* F. Viète, *Opera Mathematica,* (reprinted) Georg Olms Verlag, Hildesheim, New York, 1970, pp. 398-400 & 436-446.

*S. Wagon, Is π Normal?, *The Mathematical Intelligencer* **7** (1985), 65-67.

*J. Wallis, *Arithmetica Infinitorum, sive Nova Methodus Inquirendi in Curvilineorum Quadratum, aliaque difficiliora Matheseos Problemata,* Oxford, 1655/6, reprinted in vol. 1 (pp. 357-478) of *Opera Mathematica,* 3 vols., Oxford, vol. 1 1693; vol. 2, 1695; vol. 3, 1699.

*J. Wallis, *Computation of π by Successive Interpolations,* in: A Source Book in Mathematics, 1200-1800 (D. J. Struik, Ed.), Harvard University Press, Cambridge, MA, 1969, pp. 244-253.

*G. N. Watson, The Marquis and the Land Agent; A Tale of the Eighteenth Century, *The Mathematical Gazette* **17** (1933), 5-17.

*R. Webster, *The Tail of π,* Fourteenth Annual IMO Lecture at the Royal Society (Sept. 17, 1991).

*K. W. Weierstrass, Zu Lindemanns Abhandlung: "Über die LUDOLPH'sche Zahl", *Berichte der Berliner Akademie* **5** (1885), 1067-1085.

T. I. Williams (ed.), *A Biographical Dictionary of Scientists,* Wiley, New York, 1969.

* J. W. Wrench, Jr., The Evolution of Extended Decimal Approximations to π, *The Mathematics Teacher* **53** (1960), 644-650.

Credits

We would like to extend our thanks to various publishers for granting permission to reprint the following articles:

[1] A.B. Chace, *The Rhind Mathematical Papyrus*, The National Council of Teachers of Mathematics, Reston, Virginia, 1979, pp. 92-93 and Plate 72 (Problem 50). Reprinted with permission from the *The Rhind Mathematical Papyrus* by Arnold Buffum Chace. 1927-29. Reprint, with a note on the text by Phillip S. Jones and Bruce E. Meserve, Reston, VA: National Council of Teachers of Mathematics, 1979. Second Printing, 1986.

[2] H. Engels, Quadrature of the Circle in Ancient Egypt, *Historia Mathematica* **4** (1977), 137-140. Reprinted by permission of Academic Press.

[3] T.L. Heath (trans)., *The Works of Archimedes with the Method of Archimedes*, Dover Publications Inc., NY, 1953, pp. 91-98.

[4] G.M. Phillips, Archimedes the Numerical Analyst, *The American Mathematical Monthly* **88** (1981), 165-169. Reprinted by permission of the author.

[5] Lam Lay-Yong and Ang Tian-Se, Circle Measurements in Ancient China, *Historia Mathematica* **13** (1986), 325-340. Reprinted by permission of Academic Press.

[8] W. Hope-Jones, Correspondence. Ludoph (or Ludolff or Lucius) van Ceulen, *The Mathematical Gazette* **22** (1938), 281-282. Reprinted by permission of the Mathematical Association.

[10] J. Wallis, Computation of Pi by Successive Interpolations, in *A Source Book in Mathematics*, 1200–1800. Copyright © 1969 by Harvard University Press. Reprinted by permission of Harvard University Press.

[14] R. Roy, The Discovery of the Series Formula for Pi by Leibniz, Gregory and Nilakantha, *Mathematics Magazine* **63** (1990), 291-306. Reprinted by permission of the author.

[15] W. Jones, The First Use of Pi for the Circle Ratio, in *A Source Book in Mathematics*, Volume II (D.E. Smith, Ed.), Dover Publications Inc., NY, 1959, pp. 346-347.

[16] I. Newton, Of the Method of Fluxions and Infinite Series, *The Mathematical Works of Isaac Newton, Volume I*, Johnson Reprint Corporation, New York, 1964, pp. 128-131.

[17] L. Euler, *Introduction to Analysis of the Infinite. Book I*, Springer-Verlag, New York, 1988-1990, pp. 137-153. (Translated from the Latin by J.D. Blanton). Reprinted by permission of Springer-Verlag, New York, Inc.

[19] Lambert, A.M. Irrationality of Pi, in *A Source Book in Mathematics*, 1200–1800. Copyright 1969 by Harvard University Press. Reprinted by permission of Harvard University Press.

[22] F. Lindemann, Ueber die Zahl Pi, *Mathematische Annalen* **20** (1882), 213-225. Reprinted by permission of Springer-Verlag, New York, Inc.

[24] D. Hilbert, Ueber die Transcendenz der Zahlen e und pi. *Mathematische Annalen* **43** (1893), 216–219. Reprinted by permission of Springer-Verlag, New York, Inc.

[25] E.J. Goodwin, Quadrature of the Circle, *The American Mathematical Monthly* **1** (1894), 246–247. Reprinted by permission of the Mathematical Association of America.

[26] W.E. Eddington, House Bill No. 246, Indiana State Legislature, 1897, *Proceedings of the Indiana Academy of Science* **45** (1935), 206–210. Reprinted by permission of the Indiana Academy of Science.

[27] D. Singmaster, The Legal Values of Pi, *The Mathematical Intelligencer* **7** (1985), 69–72. Reprinted by permission of Springer-Verlag, New York, Inc.

[28] and [29] S. Ramanujan, Squaring the Circle and Modular Equations and Approximations to Pi in: *S. Ramanujan: Collected Papers* (G.H. Hardy, P.V. Seshuaigar, and B.M. Wilson, Eds.), Chelsea, New York, 1962, pp. 22 and 23–29. Reprinted by permission of Academic Press.

[30] G.N. Watson, The Marquis and the Land Agent; A Tale of the Eighteenth Century, *The Mathematical Gazette* **17** (1933), 5–17. Reprinted by permission of the Mathematical Association.

[31] J.P. Ballantine, The Best (?) Formula for Computing Pi to a Thousand Places, *The American Mathematical Monthly* **46** (1939), 499–501. Reprinted by permission of The Mathematical Association of America.

[32] R.H. Birch, An Algorithm for the Construction of Arctangent Relations, *Journal of the London Mathematical Society* **21** (1946), 173–174. Reprinted by permission of the Canadian Mathematical Society.

[33] Reprinted from "A Simple Proof that Pi is Irrational," by Ivan Niven, Bulletin of the American Mathematical Society, Volume 53, Number 7, July (1947), 509, by permission of the American Mathematical Society.

[34] G.W. Reitwiesner, An ENIAC determination of pi and e to more than 2000 Decimal Places, *Mathematical Tables and Other Aids to Computation* **4** (1950), 11–15. Reprinted by permission of the AMS.

[35] H.C. Schepler, The Chronology of Pi, *Mathematics Magazine* **23** (1950), 165–170, 216–228, and 279–283. Reprinted by permission of *Mathematics Magazine*.

[36] K. Mahler, On the Approximation of Pi, *Indagationes Mathematicae* **15** (1953), 30–42. Reprinted by permission of *Indagationes Mathematicae*.

[37] J.W. Wrench, Jr., The Evolution of Extended Decimal Approximations to Pi, *The Mathematics Teacher* **53** (1960), 644–650. Reprinted with permission from the *Mathematics Teacher*, copyright © 1960 by the National Council of Teachers of Mathematics.

[38] D. Shanks and J.W. Wrench, Jr., Calculation of Pi to 100,000 Decimals, *Mathematics of Computation* **16** (1962), 76–99. Reprinted by permission of *Mathematics of Computation*.

[39] D.W. Sweeney, On the Computation of Euler's Constant, *Mathematics of Computation* **17** (1963), 170–178. Reprinted by permission of *Mathematics of Computation*.

[40] A. Baker, Approximations to the Logarithms of Certain Rational Numbers, *Acta Arithmetica* **10** (1964), 315–323. Reprinted by permission of the Institute of Mathematics, Polish Academy of Sciences.

[42] K. Mahler, Applications of Some Formulae by Hermite to the Approximation of Exponentials and Logarithms, *Mathematische Annalen* **168** (1967), 200–277. Reprinted by permission of Springer-Verlag, New York, Inc.

[43] and [44] Eves, H.W. *In Mathematical Circles: A Selection of Mathematical Stories and Anecdotes* and *Mathematical Circles Revisited: A Second Collection of Mathematical Stories and Anecdotes*. PWS Publishing Company, Boston, 1969 and 1971.

[45] J. Todd, The Lemniscate Constants, *Communications of the ACM* **18** (1975), 14–19. Copyright © 1975 Association for Computing Machinery, Inc. (ACM). Reprinted by permission.

[46] E. Salamin, Computation of Pi Using Arithmetic-Geometric Mean, *Mathematics of Computation* **30** (1976), 565–570. Reprinted by permission.

[47] R.P. Brent, Fast Multiple-Precision Evaluation of Elementary Function, *Journal of the ACM* **23** (1976), 242–251. Copyright © 1976 Association for Computing Machinery, Inc. Reprinted by permission.

[48] F. Beukers, A Note on the Irrationality of Zeta(2) and Zeta(3), *Bulletin of the London Mathematical Society* **11** (1979), 268–272.

[49] A.J. van der Poorten, A Proof that Euler Missed . . . Apery's Proof of the Irrationality of Zeta(3) *The Mathematical Intelligencer,* **1** (1979), 195–203. Reprinted by permission of Springer-Verlag, New York, Inc.

[50] R.P. Brent and E.M. McMillan, Some New Algorithms for High-Precision Computation of Euler's Constant, *Mathematics of Computation* **34** (1980), 305–312. Reprinted by permission of *Mathematics of Computation.*

[51] T.M. Apostol, A Proof that Euler Missed: Evaluating Zeta(2) the Easy Way, *The Mathematical Intelligencer* **5** (1983), 59–60. Reprinted by permission of Springer-Verlag, New York, Inc.

[52] L. O'Shaughnessy, Putting God Back in Math, *The Mathematical Intelligencer* **5** (1983), 76–77. Reprinted by permission of Springer-Verlag, New York, Inc.

[53] M.D. Stern, A Remarkable Approximation to Pi, *The Mathematical Gazette* **69** (1985), 218–219. Reprinted by permission of the Mathematical Association of America.

[54] M. Newman and D. Shanks. On a Sequence Arising in Series for Pi, *Mathematics of Computation* **42** (1984), 199–217. Reprinted by permission of *Mathematics of Computation.*

[55] D.A. Cox, The Arithmetic–Geometric Mean of Gauss, *L'Ensignement Mathematique* **30** (1984), 275–330. Reprinted by permission of *L'Enseignement Mathematique.*

[56] J.M. Borwein and P.B. Borwein, The Arithmetic-Geometric Mean and Fast Computation of Elementary Functions. Reprinted with permission from *SIAM Review*, pp. 351–365, Vol. 26, 1984. Copyright © 1984 by the Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania. All rights reserved.

[57] D.J. Newman, A Simplified Version of the Fast Algorithms of Brent and Salamin, *Mathematics of Computation* **44** (1985), 207–210. Reprinted by permission of *Mathematics of Computation.*

[58] S. Wagon, Is Pi Normal? *The Mathematical Intelligencer* **7** (1985), 65–67. Reprinted by permission of Springer-Verlag, New York, Inc.

[59] M. Keith, Circle Digits: A Self-Referential Story, *The Mathematical Intelligencer* **8** (1986), 56–57. Reprinted by permission of Springer-Verlag, New York, Inc.

[60] D.H. Bailey, The Computation of Pi to 29,360,000 Decimal Digits Using Borweins' Quartically Convergent Algorithm, *Mathematics of Computation* **50** (1988), 283–296. Reprinted by permission of *Mathematics of Computation.*

[61] Y. Kanada, Vectorization of Multiple-Precision Arithmetic Program and 201,326,000 Decimal Digits of Pi Calculation. Copyright © 1988 IEEE. Reprinted with permission from *Supercomputing 88: Volume II, Science and Applications*, 117–128.

[62] J.M. Borwein and P.B. Borwein, Ramanujan and Pi, *Scientific American* 256 (1988), 112–117. Reprinted with permission. Copyright © 1988 by *Scientific American,* Inc. All rights reserved.

[63] D.V. Chudnovsky and G.V. Chudnovsky, Approximations and Complex Multiplication According to Ramanujan, in *Ramanujan Revisited* (G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, and R.A. Rankin, Eds.), Academic Press, Inc., Boston, MA, 1988, pp. 375–396 & 468–472. Reprinted by permission of Academic Press.

[64] J.M. Borwein, P.B. Borwein, and D.H. Bailey, Ramanujan, Modular Equations, and Approximations to Pi, or How to Compute One Billion Digits of Pi, *The American Mathematical Monthly* **96** (1989), 201–219. Reprinted by permission of authors.

[65] J.M. Borwein, P.B. Borwein, and K. Dilcher, Pi, Euler Numbers, and Asymptotic Expansion, *The American Mathematical Monthly* **96** (1989), 681–687. Reprinted by permission of the authors.

[66] F. Beukers, J.P. Bezivin, and P. Robba, An Alternative Proof of the Lindemann–Weierstrass Theorem, *The American Mathematical Monthly* **97** (1990), 193–197. Reprinted by permission of the authors.

[67] Webster, *The Tail of Pi.* Fourteenth Annual IMO Lecture at the Royal Society (17th September 1991). Reprinted by permission of the author.

[70] D.H. Bailey, P.B. Borwein and S. Plouffe, On the Rapid Computation of Various Polylogarithmic Constants, *Mathematics of Computation* **66** (1997), 903–913. Reprinted by permission of the AMS.

A Pamphlet on Pi
serving as a
Supplement for the Third Edition
of
Pi: A Source Book[1]

[1]Centre for Experimental and Constructive Mathematics (CECM) and Department of Mathematics, Simon Fraser University, Burnaby BC Canada V5A 1S6.
Research supported by NSERC, the Canada Foundation for Innovation and the Canada Research Chair Program.
E-mail: berggren@sfu.ca, jborwein@cecm.sfu.ca, pborwein@cecm.sfu.ca.

Contents

1

Pi and Its Friends

This chapter and the next are paraphrased from the book *Mathematics by Experiment* [11].

1.1 A Recent History of Pi

The first truly electronic computation of π was performed in 1949 on the original ENIAC. This calculation was suggested by John von Neumann, who wished to study the digits of π and e. Computing 2037 decimal places of π on the ENIAC required 70 h. A similar calculation today could be performed in a fraction of a second on a personal computer.

Later computer calculations were further accelerated by the discovery of advanced algorithms for performing the required high-precision arithmetic operations. For example, in 1965 it was found that the newly discovered fast Fourier transform (FFT) could be used to perform high-precision multiplications much more rapidly than conventional schemes. These methods dramatically lowered the computer time required for computing π and other mathematical constants to high precision. See also Refs. 3 and 19.

In spite of these advances, until the 1970s all computer evaluations of π still employed classical formulas, usually one of the Machin-type formulas. Some new infinite-series formulas were discovered by Ramanujan around 1910, but these were not well known until quite recently, when his writings were widely published. Ramanujan's related mathematics may be followed in Refs. 10, 18, and 23. One of these series is the remarkable formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}. \tag{1.1.1}$$

Each term of this series produces an additional *eight* correct digits in the result. Bill Gosper used this formula to compute 17 million digits of π in 1985. Gosper also computed the first 17 million terms of the continued fraction expansion of π. At about the same time, David and Gregory Chudnovsky found the following variation of Ramanujan's formula:

$$\frac{1}{\pi} = 12\sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!(13591409+545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \tag{1.1.2}$$

Each term of this series produces an additional 14 correct digits. The Chudnovskys implemented this formula using a clever scheme that enabled them to utilize the results of an initial level of precision to extend the calculation to even higher precision. They used this method in several large calculations of π, culminating with a computation to over 4 billion decimal digits in 1994.

Along this line, it is interesting to note that the Ramanujan-type series [see Ref. 17 (p. 188)]

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\left(\frac{\binom{2n}{n}}{16^n}\right)^3 \frac{42\,n+5}{16} \tag{1.1.3}$$

permits one to compute the billionth binary digit of $1/\pi$ without computing the first half of the series.

Although the Ramanujan and Chudnovsky series are considerably more efficient than the classical formulas, they share with them the property that the number of terms one must compute increases linearly with the number of digits desired in the result. In other words, if you want to compute π to twice as many digits, you have to evaluate twice as many terms of the series.

In 1976, Eugene Salamin and Richard Brent independently discovered an algorithm for π based on the arithmetic–geometric mean (AGM) and some ideas originally due to Gauss in the 1800s (although, for some reason, Gauss never saw the connection to computing π). The Salamin–Brent algorithm may be stated as follows. Set $a_0 = 1, b_0 = 1/\sqrt{2}$, and $s_0 = 1/2$. Calculate

$$\begin{aligned} a_k &= \frac{a_{k-1}+b_{k-1}}{2}, \\ b_k &= \sqrt{a_{k-1}b_{k-1}}, \\ c_k &= a_k^2 - b_k^2, \\ s_k &= s_{k-1} - 2^k c_k, \\ p_k &= \frac{2a_k^2}{s_k}. \end{aligned} \tag{1.1.4}$$

Then, p_k converges *quadratically* to π: Each iteration of this algorithm approximately *doubles* the number of correct digits—successive iterations produce 1, 4, 9, 20, 42, 85, 173, 347, and 697 correct decimal digits of π. Twenty-five iterations are sufficient to compute π to over 45-million-decimal-digit accuracy. However, each of these iterations must be performed using a level of numeric precision that is at least as high as that desired for the final result.

Beginning in 1985, one of the present authors (Jonathan Borwein) and his brother Peter Borwein discovered some additional algorithms of this type [13]. One is as follows. Set $a_0 = 1/3$ and $s_0 = (\sqrt{3}-1)/2$. Iterate

$$\begin{aligned} r_{k+1} &= \frac{3}{1+2(1-s_k^3)^{1/3}}, \\ s_{k+1} &= \frac{r_{k+1}-1}{2}, \\ a_{k+1} &= r_{k+1}^2 a_k - 3^k(r_{k+1}^2 - 1). \end{aligned} \tag{1.1.5}$$

Then, $1/a_k$ converges *cubically* to π—each iteration approximately triples the number of correct digits. Another algorithm is as follows: Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$\begin{aligned} y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}}, \\ a_{k+1} &= a_k(1+y_{k+1})^4 - 2^{2k+3}y_{k+1}(1+y_{k+1}+y_{k+1}^2). \end{aligned} \tag{1.1.6}$$

Then, a_k converges *quartically* to $1/\pi$. This particular algorithm, together with the Salamin–Brent scheme, has been employed by Yasumasa Kanada of the University of Tokyo in several computations of π over the past 15 years or so, including his 1999 computation of π to over 206 billion decimal digits.

Shanks, who in 1961 computed π to over 100,000 digits, once declared that a billion-digit computation would be "forever impossible." However, both Kanada and the Chudnovskys computed over 1 billion digits in 1989. Similarly, the intuitionist mathematicians Brouwer and Heyting once asserted the "impossibility" of ever knowing whether the sequence "0123456789" appears in the decimal expansion of π [17]. This sequence was found in 1997 by Kanada, beginning at position 17,387,594,880. Even as late as 1989, British mathematical physicist Roger Penrose, ventured in the first edition of his book *The Emperor's New Mind* that we are not likely to know whether a string of "ten consecutive sevens" occurs in the decimal expansion of π [22, p. 115]. By the time his book was published, Kanada had already found a string of 10 consecutive sixes in his 480-million-digit computation of π. When one of the present authors mentioned this to Penrose in 1990, he replied that he was "startled to learn how far the combination of human mathematical ingenuity with computer technology has enabled the calculation of the decimal expansion of π to be carried out." Accordingly, he changed his text to "twenty consecutive sevens," which appeared in subsequent printings of the book. This was just in time, as a string of 10 consecutive sevens was found by Kanada in 1997, beginning at position 22,869,046,249.

In December 2002, Kanada, with a team consisting of Y. Ushiro of Hitachi, H. Kuroda and M. Kudoh of the University of Tokyo, and the assistance of nine others from Hitachi, completed computation of π to over 1.24 *trillion* decimal digits. Kanada and his team first computed π in hexadecimal (base 16) to 1,030,700,000,000 places, using the following two arctangent relations for π:

$$\pi = 48\tan^{-1}\left(\frac{1}{49}\right) + 128\tan^{-1}\left(\frac{1}{57}\right) - 20\tan^{-1}\left(\frac{1}{239}\right) + 48\tan^{-1}\left(\frac{1}{110443}\right),$$
$$\pi = 176\tan^{-1}\left(\frac{1}{57}\right) + 28\tan^{-1}\left(\frac{1}{239}\right) - 48\tan^{-1}\left(\frac{1}{682}\right) + 96\tan^{-1}\left(\frac{1}{12943}\right). \tag{1.1.7}$$

The first formula was found in 1982 by K. Takano, a high school teacher and song writer. The second formula was found by F. C. W. Störmer in 1896.

Kanada and his team evaluated these formulas using a scheme analogous to that employed by Gosper and the Chudnovskys, in that they were able to avoid explicitly storing the multiprecision numbers involved. This resulted in a scheme that is roughly competitive in efficiency compared to the Salamin–Brent and Borwein quartic algorithms they had previously used, yet with a significantly lower total memory requirement. In particular, they were able to perform their latest computation on a system with 1 Tbyte (10^{12} bytes) main memory, the same as with their previous computation, yet obtain six times as many digits.

After Kanada and his team verified that the hexadecimal-digit strings produced by these two computations were in agreement, they performed an additional check by directly computing 20 hexadecimal digits beginning at position 1,000,000,000,001. This calculation employed an algorithm that we will describe in Section 1.2 and required 21 h run time, much less than the time required for the first step. The result of this calculation, `B4466E8D21 5388C4E014`, perfectly agreed with the corresponding digits produced by the two arctan formulae. At this point, they converted their hexadecimal value of π to decimal and converted back to hexadecimal as a check. These conversions employed a numerical approach similar to that used in the main and verification calculations. The entire computation, including hexadecimal and decimal evaluations and checks, required roughly 600 h run time on their 64-node Hitachi parallel supercomputer. The main segment of the computation ran at nearly 1 Tflop/s (i.e., 1 trillion floating-point operations per second), although this performance rate was slightly lower than the rate of their previous calculation of 206 billion digits. Full details will appear in an upcoming paper [18].

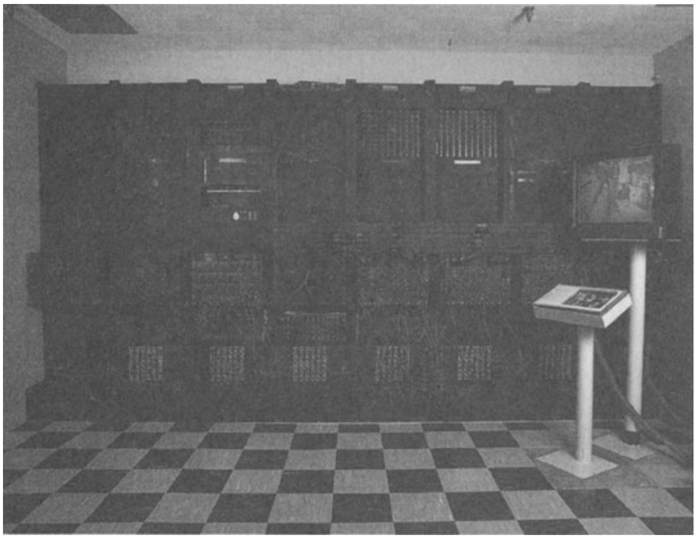


Figure 1.1: **The ENIAC "Integrator and Calculator"**

According to Kanada, the 10 decimal digits ending in position 1 trillion are 6680122702, and the 10 hexadecimal digits ending in position 1 trillion are `3F89341CD5`]. Some data on the frequencies of digits in π, based on Kanada's computations, are given in Section 2.1. Additional information of this sort is available from Kanada's Web site: `http://www.super-computing.org`. Additional historical background on record-breaking computations of π is available at `http://www.cecm.sfu.ca/personal/jborwein/pi_cover.html`.

A listing of some milestones in the recent history of the computation of π is given in Table 1.1.

In retrospect, one might wonder why in antiquity π was not measured to an accuracy in excess of 22/7. One conjecture is that it reflects not an inability to do so but, instead, a very different mind-set to a modern (Baconian) experimental one.

For those who are familiar with *The Hitchhiker's Guide to the Galaxy* it is amusing that 042 occurs at the digits ending at the 50-billionth decimal place in each of π and $1/\pi$—thereby providing an excellent answer to the ultimate question "What is forty-two?"

Much lovely additional material, "both sensible and silly" can be found in *Pi Unleashed* [2] and in *the Joy of Pi* [10] (`www.joyofpi.com/`).

Table 1.1: **Digital era π calculations**

Ferguson	1946	620
Ferguson	1947	710
Ferguson and Wrench	1947	808
Smith and Wrench	1949	1,120
Reitwiesner et al. (ENIAC)	1949	2,037
Nicholson and Jeenel	1954	3,092
Felton	1957	7,480
Genuys	1958	10,000
Felton	1958	10,021
Guilloud	1959	16,167
Shanks and Wrench	1961	100,265
Guilloud and Filliatre	1966	250,000
Guilloud and Dichampt	1967	500,000
Guilloud and Bouyer	1973	1,001,250
Miyoshi and Kanada	1981	2,000,036
Guilloud	1982	2,000,050
Tamura	1982	2,097,144
Tamura and Kanada	1982	4,194,288
Tamura and Kanada	1982	8,388,576
Kanada, Yoshino and Tamura	1982	16,777,206
Ushiro and Kanada	Oct. 1983	10,013,395
Gosper	Oct. 1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada, Tamura, Kubo, et al.	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Chudnovskys	Jun. 1989	525,229,270
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1989	1,011,196,691
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Takahashi and Kanada	Jun. 1995	3,221,225,466
Kanada	Aug. 1995	4,294,967,286
Kanada	Oct. 1995	6,442,450,938
Kanada and Takahashi	Jun. 1997	51,539,600,000
Kanada and Takahashi	Sep. 1999	206,158,430,000
Kanada, Ushiro, Kuroda, Kudoh	Dec. 2002	1,241,100,000,000

The ENIAC Integrator and Calculator

ENIAC, built in 1946 at the University of Pennsylvania, had 18,000 vacuum tubes, 6000 switches, 10,000 capacitors, 70,000 resistors, and 1500 relays, was 10 ft tall, occupied 1800 ft^2 and weighed 30 tons. ENIAC could perform 5000 arithmetic operations per second—1000 times faster than any earlier machine, but a far cry from today's leading-edge microprocessors, which can perform more than 4 billion operations per second.

The first stored-memory computer, ENIAC could store 200 digits, which, again, is a far cry from the hundreds of megabytes in a modern personal computer system. Data flowed from one accumulator to the next, and after each accumulator finished a calculation, it communicated its results to the next in line. The accumulators were connected to each other manually. A photo is shown in Figure 1.1. We observe that the photo—obtained digitally—requires orders of magnitudes more data than ENIAC could store.

1.2 Computing Individual Digits of Pi

An outsider might be forgiven for thinking that essentially everything of interest with regard to π has been discovered. For example, this sentiment is suggested in the closing chapters of Beckmann's 1971 book on the history of π [6, p. 172]. Ironically, the Salamin–Brent quadratically convergent iteration was discovered only 5 years later, and the higher-order convergent algorithms followed in the 1980s. In 1990, Rabinowitz and Wagon discovered a "spigot" algorithm for π, which permits successive digits of π (in any desired base) to be computed with a relatively simple recursive algorithm based on the previously generated digits (see Ref. 31).

However, even insiders are sometimes surprised by a new discovery. Prior to 1996, almost all mathematicians believed that if you want to determine the dth digit of π, you have to generate the entire sequence of the first d digits. (For all of their sophistication and efficiency, the above-described schemes all have this property.) However, it turns out that this is not true, at least for hexadecimal (base 16) or binary (base 2) digits of π. In 1996, Peter Borwein, Simon Plouffe, and one of the present authors (Bailey) found an algorithm for computing individual hexadecimal or binary digits of π [3]. To be precise, this algorithm has the following properties:

1. directly produces a modest-length string of digits in the hexadecimal or binary expansion of π, beginning at an arbitrary position, without needing to compute any of the previous digits

2. can be implemented easily on any modern computer

3. does not require multiple precision arithmetic software

4. requires very little memory

5. has a computational cost that grows only slightly faster than the digit position.

Using this algorithm, for example, the 1 millionth hexadecimal digit (or the 4 millionth binary digit) of π can be computed in less than a minute on a 2001-era computer. The new algorithm is not fundamentally faster than best known schemes for computing all digits of π up to some position, but its elegance and simplicity are nonetheless of considerable interest. This scheme is based on the following remarkable new formula for π:

Theorem 1.2.1:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right). \tag{1.2.8}$$

Proof. First note that for any $k < 8$,

$$\begin{aligned} \int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\, dx &= \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\, dx \\ &= \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}. \end{aligned} \tag{1.2.9}$$

Thus, one can write

$$\begin{aligned} &\sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \\ &\qquad = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}\, dx, \end{aligned} \tag{1.2.10}$$

which, on substituting $y = \sqrt{2}x$, becomes

$$\begin{aligned} \int_0^1 \frac{16\,y - 16}{y^4 - 2\,y^3 + 4\,y - 4}\, dy &= \int_0^1 \frac{4y}{y^2-2}\, dy - \int_0^1 \frac{4y-8}{y^2-2y+2}\, dy \\ &= \pi. \end{aligned} \tag{1.2.11}$$

$\square$

However, in presenting this formal derivation, we are disguising the actual route taken to the discovery of this formula. This route is a superb example of experimental mathematics in action.

It all began in 1995, when Peter Borwein and Simon Plouffe of Simon Fraser University observed that the following well-known formula for $\log 2$ permits one to calculate isolated digits in the binary expansion of $\log 2$:

$$\log 2 = \sum_{k=0}^{\infty} \frac{1}{k2^k}. \tag{1.2.12}$$

This scheme is as follows. Suppose we wish to compute a few binary digits beginning at position $d + 1$ for some integer $d > 0$. This is equivalent to calculating $\{2^d \log 2\}$, where $\{\cdot\}$ denotes fractional part. Thus, we can write

$$\begin{aligned} \{2^d \log 2\} &= \left\{\left\{\sum_{k=0}^{d} \frac{2^{d-k}}{k}\right\} + \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\} \\ &= \left\{\left\{\sum_{k=0}^{d} \frac{2^{d-k} \bmod k}{k}\right\} + \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\}. \end{aligned} \tag{1.2.13}$$

We are justified in inserting "mod k" in the numerator of the first summation because we are only interested in the fractional part of the quotient when divided by k.

Now, the key observation is this: The numerator of the first sum in Eqn. (1.2.13), namely $2^{d-k} \bmod k$, can be calculated very rapidly by means of the binary algorithm for exponentiation, performed modulo k. The binary algorithm for exponentiation is merely the formal name for the observation that exponentiation can be economically performed by means of a factorization based on the binary expansion of the exponent. For example, we can write $3^{17} = ((((3^2)^2)^2)^2) \cdot 3$, thus producing the result in only 5 multiplications, instead of the usual 16. According to Knuth, this technique dates back at least to 200 BCE [19]. In our application, we need to obtain the exponentiation result modulo a positive integer k. This can be done very efficiently by reducing modulo k the intermediate multiplication result at each step of the binary algorithm for exponentiation. A formal statement of this scheme is as follows:

Algorithm 1.2.2: *Binary algorithm for exponentiation modulo k.*

To compute $r = b^n \bmod k$, where r, b, n, and k are positive integers: First, set t to be the largest power of two such that $t \leq n$, and set $r = 1$. Then

$$\begin{array}{lllll} \text{A: if } n & \geq t & \text{then } r \leftarrow br \bmod k; & n \leftarrow n - t; & \text{endif} \\ \quad t & \leftarrow t/2 & & & \\ \text{if } t & \geq 1 & \text{then } r \leftarrow r^2 \bmod k & \text{go to A;} & \text{endif} \end{array}$$

□

Note that the above algorithm is performed entirely with positive integers that do not exceed k^2 in size. Thus, ordinary 64-bit floating-point or integer arithmetic, available on almost all modern computers, suffices for even rather large calculations. The 128-bit floating-point arithmetic (double-double or quad precision), available at least in software on many systems, suffices for the largest computations currently feasible.

We can now present the algorithm for computing individual binary digits of $\log 2$.

Algorithm 1.2.3: *Individual digit algorithm for* $\log 2$.

To compute the $(d+1)$st binary digit of $\log 2$: Given an integer $d > 0$, (1) calculate each numerator of the first sum in Eqn. (1.2.13), using Algorithm 1.2.2, implemented using ordinary 64-bit integer or floating-point arithmetic; (2) divide each numerator by the respective value of k, again using ordinary floating-point arithmetic; (3) sum the terms of the first summation, discarding any integer parts; (4) evaluate the second summation as written using floating-point arithmetic—only a few terms are necessary because it rapidly converges; and (5) add the result of the first and second summations, discarding any integer part. The resulting fraction, when expressed in binary, gives the first few digits of the binary expansion of $\log 2$ beginning at position $d+1$. □

As soon as Borwein and Plouffe found this algorithm, they began seeking other mathematical constants that shared this property. It was clear that any constant α of the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)2^k}, \tag{1.2.14}$$

where $p(k)$ and $q(k)$ are integer polynomials, with $\deg p < \deg q$ and q having no zeros at positive integer arguments, is in this class. Further, any rational linear combination of such constants also shares this property. Checks of various mathematical references eventually uncovered about 25 constants that possessed series expansions of the form given by Eqn. (1.2.14).

As you might suppose, the question of whether π also shares this property did not escape these researchers. Unfortunately, exhaustive searches of the mathematical literature did not uncover any formula for π of the requisite form. However, given the fact that any rational linear combination of constants with this property also shares this property, Borwein and Plouffe performed integer relation searches to see if a formula of this type existed for π. This was done, using computer programs written by one of the present authors (Bailey), which implement the "PSLQ" integer relation algorithm in high-precision, floating-point arithmetic [4,14,15,21].

In particular, these three researchers sought an integer relation for the real vector $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_1 = \pi$ and $(\alpha_i,\ 2 \le i \le n)$ is the collection of constants of the requisite form gleaned from the literature, each computed to several-hundred-decimal-digit precision. To be precise, they sought an n-long vector of integers (a_i) such that $\sum_i a_i \alpha_i = 0$, to within a very small "epsilon." After a month or two of computation, with numerous restarts using new α vectors (when additional formulas were found in the literature), the identity (1.2.8) was finally uncovered. The actual formula found by the computation was

$$\pi = 4F(1/4, 5/4; 1; -1/4) + 2\tan^{-1}(1/2) - \log 5, \tag{1.2.15}$$

where $F(1/4, 5/4; 1; -1/4) = 0.955933837\ldots$ is a hypergeometric function evaluation. Reducing this expression to summation form yields the new π formula:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right). \tag{1.2.16}$$

To return briefly to the derivation of formula (1.2.16), let us point out that it was discovered not by formal reasoning or even by computer-based symbolic processing, but, instead, by numerical computations using a high-precision implementation of the PSLQ integer relation algorithm. It is most likely the first instance in history of the discovery of a new formula for π by a computer. We might mention that, in retrospect, formula (1.2.16) could be found much more quickly, by seeking integer relations in the vector $(\pi, S_1, S_2, \cdots, S_8)$, where

$$S_j = \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}. \tag{1.2.17}$$

Such a calculation could be done in a few seconds on a computer, even if one did not know in advance to use 16 in the denominator and 9 terms in the search, but instead had to stumble on these parameters by trial and error. However, this observation is, as they say, 20–20 hindsight. The process of real mathematical discovery is often far more tortuous and less elegant than the polished version typically presented in textbooks and research journals.

It should be clear at this point that the scheme for computing individual hexadecimal digits of π is very similar to Algorithm 1.2.3. For completeness, we state it as follows:

Algorithm 1.2.4: *Individual digit algorithm for π.*

To compute the $(d+1)$st hexadecimal digit of π: Given an integer $d > 0$, we can write

$$\{16^d\pi\} = \{4\{16^d S_1\} - 2\{16^d S_4\} - \{16^d S_5\} - \{16^d S_6\}\}, \qquad (1.2.18)$$

using the S_j notation of Eqn. (1.2.17). Now, apply Algorithm 1.2.3, with

$$\begin{aligned}\{16^d S_j\} &= \left\{\left\{\sum_{k=0}^{d} \frac{16^{d-k}}{8k+j}\right\} + \sum_{k=d+1}^{\infty} \frac{16^{d-k}}{8k+j}\right\} \\ &= \left\{\left\{\sum_{k=0}^{d} \frac{16^{d-k} \bmod 8k+j}{8k+j}\right\} + \sum_{k=d+1}^{\infty} \frac{16^{d-k}}{8k+j}\right\} \qquad (1.2.19)\end{aligned}$$

instead of Eqn. (1.2.13), to compute $\{16^d S_j\}$ for $j = 1, 4, 5, 6$. Combine these four results, discarding integer parts, as shown in Eqn.(1.2.18). The resulting fraction, when expressed in hexadecimal notation, gives the hex digit of π in position $d+1$, plus a few more correct digits. □

As with Algorithm 1.2.3, multiple-precision arithmetic software is not required—ordinary 64-bit or 128-bit floating-point arithmetic suffices even for some rather large computations. We have omitted here some numerical details for large computations—see Ref.[3]. Sample implementations in both C and Fortran-90 are available from the Web site `http://www.nersc.gov/~dhbailey`.

One mystery that remains unanswered is why formula (1.2.8) was not discovered long ago. As you can see from the above proof, there is nothing very sophisticated about its derivation. There is no fundamental reason why Euler, for example, or Gauss or Ramanujan could not have discovered it. Perhaps the answer is that its discovery was a case of "reverse mathematical engineering." Lacking a motivation to find such a formula, mathematicians of previous eras had no reason to derive one. However, this still does not answer the question of why the algorithm for computing individual digits of $\log 2$ had not been discovered before—it is based on a formula, namely Eqn. (1.2.12), that has been known for centuries.

Needless to say, Algorithm 1.2.4 has been implemented by numerous researchers. In 1997, Fabrice Bellard of INRIA computed 152 binary digits of

Table 1.2: **Computed Hexadecimal Digits of π**

Position	Hex Digits Beginning at This Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B
2.5×10^{14}	E6216B069CB6C1

π starting at the trillionth position. The computation took 12 days on 20 workstations working in parallel over the Internet. His scheme is actually based on the following variant of 1.2.8:

$$\begin{aligned}\pi &= 4\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(2k+1)}\\ &\quad -\frac{1}{64}\sum_{k=0}^{\infty}\frac{(-1)^k}{1024^k}\left(\frac{32}{4k+1}+\frac{8}{4k+2}+\frac{1}{4k+3}\right). \qquad (1.2.20)\end{aligned}$$

This formula permits individual hex or binary digits of π to be calculated roughly 43% faster than Eqn.(1.2.8).

A year later, Colin Percival, then a 17-year-old student at Simon Fraser University, utilized a network of 25 machines to calculate binary digits in the neighborhood of position 5 trillion, and then in the neighborhood of 40 trillion. In September 2000, he found that the quadrillionth binary digit is "0," based on a computation that required 250 CPU-years of run time, carried out using 1734 machines in 56 countries. Table 1.2 gives some results known as of this writing.

One question that immediately arises in the wake of this discovery is whether or not there is a formula of this type and an associated computational scheme to compute individual *decimal* digits of π. Searches conducted by numerous researchers have been unfruitful. Now it appears that there is no nonbinary formula of this type—this is ruled out by a new result coauthored by one of the present authors (see Section 1.3) [12]. However, none of this removes the possibility that there exists some completely different approach that permits rapid computation of individual decimal digits of π. Also, there do exist formulas for certain other constants that admit individual digit calculation schemes in various nonbinary bases (including base 10).

1.3 Does Pi Have a Nonbinary BBP Formula?

As mentioned earlier, from the day that the BBP formula (Bailey, Borwein, and Plouffe) for π was discovered, researchers have wondered whether there exist BBP-type formulas that would permit computation of individual digits in bases other than powers of 2 (such as base 10). This is not such a far-fetched possibility, because both base-2 and base-3 formulas are known for π^2, as well as for $\log 2$. However, extensive computations failed to find any nonbinary formulas for π.

Recently, it has been shown that there are no nonbinary Machin-type arctangent formulas for π. We believe that if there is no nonbinary Machin-type arctangent formula for π, then there is no nonbinary BBP-type formula of any form for π. We summarize this result here. Full details and other related results can be found in Ref. [16].

We say that the integer $b > 1$ is not a proper power if it cannot be written as c^m for any integers c and $m > 1$. We will use the notation $\mathrm{ord}_p(z)$ to denote the p-adic order of the rational $z \in Q$. In particular, $\mathrm{ord}_p(p) = 1$ for prime p, and $\mathrm{ord}_p(q) = 0$ for primes $q \neq p$ and $\mathrm{ord}_p(wz) = \mathrm{ord}_p(w) + \mathrm{ord}_p(z)$. The notation $\nu_b(p)$ will mean the order of the integer b in the multiplicative group of the integers modulo p. We will say that p is a primitive prime factor of $b^m - 1$ if m is the least integer such that $p|(b^m - 1)$. Thus, p is a primitive prime factor of $b^m - 1$ provided $\nu_b(p) = m$. Given the Gaussian integer $z \in Q[i]$ and the rational prime $p \equiv 1 \pmod 4$, let $\theta_p(z)$ denote $\mathrm{ord}_{\mathfrak{p}}(z) - \mathrm{ord}_{\bar{\mathfrak{p}}}(z)$, where $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are the two conjugate Gaussian primes dividing p and where we require $0 < \Im(\mathfrak{p}) < \mathrm{R}(\mathfrak{p})$ to make the definition of θ_p unambiguous. Note that

$$\theta_p(wz) \quad = \quad \theta_p(w) + \theta_p(z). \tag{1.3.21}$$

Given $\kappa \in R$, with $2 \leq b \in Z$ and b not a proper power, we say that κ has a Z-linear or Q-linear *Machin-type BBP formula* to the base b if and only if κ can be written as a Z-linear or Q-linear combination (respectively) of generators of the form

$$\arctan\left(\frac{1}{b^m}\right) \quad = \quad \Im \log\left(1 + \frac{i}{b^m}\right) \quad = \quad b^m \sum_{k=0}^{\infty} \frac{(-1)^k}{b^{2mk}(2k+1)}. \tag{1.3.22}$$

We will also use the following theorem, first proved by Bang in 1886:

Theorem 1.3.1: *The only cases where $b^m - 1$ has no primitive prime factor(s) are when $b = 2$, $m = 6$, and $b^m - 1 = 3^2 \cdot 7$, and when $b = 2^N - 1$, $N \in Z$, $m = 2$, and $b^m - 1 = 2^{N+1}(2^{N-1} - 1)$.*

We can now state the main result of this section:

Theorem 1.3.2: *Given $b > 2$ and not a proper power, then there is no Q-linear Machin-type BBP arctangent formula for π.*

Proof: It follows immediately from the definition of a Q-linear Machin-type BBP arctangent formula that any such formula has the form

$$\pi = \frac{1}{n}\sum_{m=1}^{M} n_m \Im \log(b^m - i) \tag{1.3.23}$$

where $n > 0 \in Z$, $n_m \in Z$, and $M \geq 1$, $n_M \neq 0$. This implies that

$$\prod_{m=1}^{M}(b^m - i)^{n_m} \in e^{ni\pi}Q^\times = Q^\times \tag{1.3.24}$$

For any $b > 2$ and not a proper power, we have $M_b \leq 2$, so it follows from Bang's theorem that $b^{4M} - 1$ has a primitive prime factor, say p. Furthermore, p must be odd, because $p = 2$ can only be a *primitive* prime factor of $b^m - 1$ when b is odd and $m = 1$. Since p is a primitive prime factor, it does not divide $b^{2M} - 1$, and so p must divide $b^{2M} + 1 = (b^M + i)(b^M - i)$. We cannot have both $p | b^M + i$ and $p | b^M - i$, as this would give the contradiction that $p | (b^M + i) - (b^M - i) = 2i$. It follows that $p \equiv 1 \pmod 4$ and that p factors as $p = \mathfrak{p}\bar{\mathfrak{p}}$ over $Z[i]$, with exactly one of $\mathfrak{p}$, and $\bar{\mathfrak{p}}$ dividing $b^M - i$. Referring to the definition of θ, we see that we must have $\theta_p(b^M - i) \neq 0$. Furthermore, for any $m < M$, neither $\mathfrak{p}$ nor $\bar{\mathfrak{p}}$ can divide $b^m - i$ because this would imply $p \mid b^{4m} - 1$, $4m < 4M$, contradicting the fact that p is a primitive prime factor of $b^{4M} - 1$. Thus, for $m < M$, we have $\theta_p(b^m - i) = 0$. Referring to Eqn. (1.3.23), using Eqn. (1.3.21) and the fact that $n_M \neq 0$, we get the contradiction

$$0 \neq n_M\theta_p(b^M - i) = \sum_{m=1}^{M} n_m\theta_p(b^m - i) = \theta_p(Q^\times) = 0. \tag{1.3.25}$$

Thus, our assumption that there was a b-ary Machin-type BBP arctangent formula for π must be false. □

2

Normality of Numbers

2.1 Normality: A Stubborn Question

Given a real number α and an integer $b > 2$, we say that α is *b-normal* or *normal base b* if *every* sequence of k consecutive digits in the base-b expansion of α appears with limiting frequency b^{-k}. In other words, if a constant is 10-normal, then the limiting frequency of "3" (or any other single digit) in its decimal expansion is $1/10$, the limiting frequency of "58" (or any other two-digit pair) is $1/100$, and so forth. We say that a real number α is *absolutely normal* if it is b-normal for all integers $b > 1$ simultaneously.

In spite of these strong conditions, it is well known from measure theory that the set of absolutely normal real numbers in the unit interval has measure one or, in other words, that almost all real numbers are absolutely normal [20]. Further, from numerous analyses of computed digits, it appears that all of the fundamental constants of mathematics are normal to commonly used number bases. By "fundamental constants," we include $\pi, e, \sqrt{2}$, the golden mean $\tau = (1 + \sqrt{5})/2$, as well as $\log n$ and the Riemann zeta function $\zeta(n)$ for positive integers $n > 1$, and many others. For example, it is a reasonable conjecture that *every* irrational algebraic number is absolutely normal, because there is no known example of an irrational algebraic number whose decimal expansion (or expansion in any other base) appears to have skewed digit-string frequencies.

Decimal values are given for a variety of well-known mathematical constants in Table 2.1[14, 16]. In addition to the widely recognized constants such as π and e, we have listed Catalan's constant (G), Euler's constant (γ), an evaluation of the elliptic integral of the first kind $K(1/\sqrt{2})$, an evaluation of an elliptic integral of the second kind $E(1/\sqrt{2})$, Feigenbaum's α and δ constants, Khintchine's constant $\mathcal{K}$ and Madelung's constant $\mathcal{M}_3$. Binary values for some of these constants, as well as Chaitin's Ω constant (from the field of computational complexity) [14], are given in Table 2.2. As you can see, none of the expansions in either table

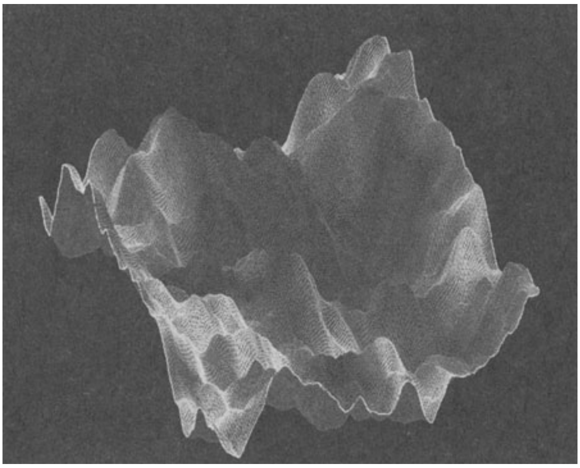


Figure 2.1: A random walk based on a million digits of π.

exhibits any evident "pattern."

The digits of π have been studied more than any other single constant, in part because of the widespread fascination with π. Along this line, Yasumasa Kanada of the University of Tokyo has tabulated the number of occurrences of the 10 decimal digits "0" through "9" in the first one trillion decimal digits of π. These counts are shown in Table 2.3. For reasons given in Section 1.2, binary (or hexadecimal) digits of π are also of considerable interest. To that end, Kanada has also tabulated the number of occurrences of the 16 hexadecimal digits "0" through "`F`" as they appear in the first one trillion hexadecimal digits. These counts are shown in Table 2.4. As you can see, both the decimal and hexadecimal single-digit counts are entirely reasonable.

Some readers may be amused by the PiSearch utility, which is available at `http://pi.nersc.gov`. This online tool permits one to enter one's name (or any other modest-length alphabetic string, or any modest-length hexadecimal string) and see if it appears encoded in the first 4 billion binary digits of π (i.e., the first 1 billion hexadecimal digits of π). Along this line, a graphic based on a random walk of the first million decimal digits of π, courtesy of David and Gregory Chudnovsky, is shown in Figure 2.1. It maps the digit stream to a surface in ways similar to those used by Mandelbrot and others.

The question of whether π, in particular, or, say, $\sqrt{2}$ is normal or not has intrigued mathematicians for centuries. However, in spite of centuries of effort, not a single one of the fundamental constants of mathematics has ever been proven to be b-normal for any integer b, much less for all integer bases simultaneously. This is not for lack of trying—some very good mathematicians have seriously

Table 2.1: **Decimal Values of Various Mathematical Constants**

Constant	Value
$\sqrt{2}$	1.4142135623730950488...
$\sqrt{3}$	1.7320508075688772935...
$\sqrt{5}$	2.2360679774997896964...
$\phi = \frac{\sqrt{5}-1}{2}$	0.61803398874989484820...
π	3.1415926535897932385...
$1/\pi$	0.31830988618379067153...
e	2.7182818284590452354...
$1/e$	0.36787944117144232160...
e^{π}	23.140692632779269007...
$\log 2$	0.69314718055994530942...
$\log 10$	2.3025850929940456840...
$\log_2 10$	3.3219280948873623478...
$\log_{10} 2$	0.30102999566398119522...
$\log_2 3$	1.5849625007211561815...
$\zeta(2)$	1.6449340668482264365...
$\zeta(3)$	1.2020569031595942854...
$\zeta(5)$	1.0369277551433699263...
G	0.91596559417721901505...
γ	0.57721566490153286061...
$\Gamma(1/2) = \sqrt{\pi}$	1.7724538509055160273...
$\Gamma(1/3)$	2.6789385347077476337...
$\Gamma(1/4)$	3.6256099082219083121...
$K(1/\sqrt{2})$	1.8540746773013719184...
$E(1/\sqrt{2})$	1.3506438810476755025...
α_f	4.669201609102990...
δ_f	2.502907875095892...
$\mathcal{K}$	2.6854520010653064453...
$\mathcal{M}_3$	1.7475645946331821903...

Table 2.2: **Binary Values of Various Mathematical Constants**

Constant	Value
π	11.00100100001111110110101010001000100001011010001100001000 1...
e	10.101101111110000101010001011000101000101011101101001010100...
$\sqrt{2}$	1.011010100000100111100110011001111111001110111100110010010 0...
$\sqrt{3}$	1.101110110110011110101110100001011000010011001010101001110 0...
$\log 2$	0.101100010111001000010111111101111110100011100111101111001 10...
$\log 3$	1.000110010011111010100111101010101101000000110000101010010 1...
Ω	0.000000100000001000010000010000111011100110010011110001001 00...

Table 2.3: **Statistics for the First Trillion Decimal Digits of π**

Digit	Occurrences
0	99999485134
1	99999945664
2	100000480057
3	99999787805
4	100000357857
5	99999671008
6	99999807503
7	99999818723
8	100000791469
9	99999854780
Total	1000000000000

Table 2.4: **Statistics for the First Trillion Hexadecimal Digits of π**

Digit	Occurrences
0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
5	62500007205
6	62499925426
7	62499878794
8	62500216752
9	62500120671
A	62500266095
B	62499955595
C	62500188610
D	62499613666
E	62499875079
F	62499937801
Total	1000000000000

investigated this problem, but to no avail. Even much weaker results, such as "the digit '1' appears with nonzero limiting frequency in the binary expansion of π" and "the digit '5' appears infinitely often in the decimal expansion of $\sqrt{2}$" have heretofore remained beyond the reach of modern mathematics.

One result in this area is the following. Let $f(n) = \sum_{1\le j\le n} \lfloor \log_{10} j \rfloor$. Then, the Champernowne number

$$\sum_{N=0}^{\infty} \frac{n}{10^{n+f(n)}} = 0.12345678910111213141516171819202122232425\ldots,$$

(i.e., where the positive integers are concatenated) is known to be 10-normal, with a similar form and normality result for other bases (the authors are indebted to Richard Crandall for the formula above). However no one, to the authors' knowledge, has ever argued that this number is a "natural" or "fundamental" constant of mathematics.

Consequences of a proof in this area would definitely be interesting. For starters, such a proof would immediately provide an inexhaustible source of provably reliable pseudorandom numbers for numerical or scientific experimentation. We also would obtain the mind-boggling but uncontestable consequence that if π, for example, is shown to be 2-normal, then the entire text of the Bible, the Koran, and the works of William Shakespeare, as well as the full LaTeX source text for this book, must all be contained somewhere in the binary expansion of π, where consecutive blocks of eight bits (two hexadecimal digits) each represent one ASCII character. Unfortunately, this would not be much help to librarians or archivists, because every conceivable misprint of each of these books would also be contained in the binary digits of π.

2.2 BBP Constants and Normality

Until recently, the BBP formulas mentioned in Sections 1.2 and 1.3 were assigned by some to the realm of "recreational" mathematics—interesting but of no serious consequence. However, the history of mathematics has seen many instances where results once thought to be idle curiosities were later found to have significant consequences. This now appears to be the case with the theory of BBP-type constants.

What we shall establish here, in a nutshell, is that the 16-normality of π (which, of course, is equivalent to the 2-normality of π), as well as the normality of numerous other irrational constants that possess BBP-type formulas, can be reduced to a certain plausible conjecture in the theory of chaotic sequences. We do not know at this time what the full implications of this result are. It may be the first salvo in the resolution of this age-old mathematical question or it may be merely a case of reducing one very difficult mathematical problem to another. However at the least, this result appears to lay out a structure—a "road map" of sorts—for the analysis of this question. Thus, it seems worthy of investigation.

We will also establish that a certain well-defined class of real numbers, uncountably infinite in number, *is* indeed b-normal for certain bases b, a result not dependent on any unproven conjecture. We will also present some results on the digit densities of algebraic irrationals. All of these recent results are direct descendants of the theory of BBP-type constants that we have presented in Section 1.2 and Ref. [11].

The results for BBP-type constants derive from the following observation, which was given in a recent paper by one of the present authors and Richard Crandall [4]. Here, we define the norm $||\alpha||$ for $\alpha \in [0, 1)$ as $||\alpha|| = \min(\alpha, 1-\alpha)$. With this definition, $||\alpha - \beta||$ measures the shortest distance between α and β on the unit circumference circle in the natural way. Suppose α is given by a BBP-type formula, namely,

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)} \tag{2.2.1}$$

where p and q are polynomials with integer coefficients, with $0 \leq \deg p < \deg q$, and with q having no zeros at positive-integer arguments. Now, define the recursive sequence (x_n) as $x_0 = 0$, and

$$x_n = \left\{ b x_{n-1} + \frac{p(n)}{q(n)} \right\}, \tag{2.2.2}$$

where the notation $\{\cdot\}$ denotes fractional part, as earlier. Recall from Section 1.2 that we can write the base-b expansion of α beginning at position $n+1$, which we denote α_n, as

$$\begin{aligned} \alpha_n &= \{b^n \alpha\} \\ &= \left\{ \sum_{k=0}^{\infty} \frac{b^{n-k} p(k)}{q(k)} \right\} \\ &= \left\{ \left\{ \sum_{k=0}^{n} \frac{b^{n-k} p(k)}{q(k)} \right\} + \sum_{k=n+1}^{\infty} \frac{b^{n-k} p(k)}{q(k)} \right\}. \end{aligned} \tag{2.2.3}$$

Now, observe that the sequence (x_n) generates the first part of this expression. In particular, given $\epsilon > 0$, assume that n is sufficiently large such that $p(k)/q(k) < \epsilon$

for all $k \geq n$. Then, we can write, for all sufficiently large n,

$$
\begin{aligned}
||x_n - \alpha_n|| &= \left| \sum_{k=n+1}^{\infty} \frac{b^{n-k} p(k)}{q(k)} \right| \\
&\leq \epsilon \sum_{k=n+1}^{\infty} b^{n-k} \\
&= \frac{\epsilon}{b-1} < \epsilon.
\end{aligned}
\tag{2.2.4}
$$

With this argument, we have established the following, which we observe is also true if the expression $p(k)/q(k)$ is replaced by any more general sequence $r(k)$ that tends to zero for large k:

Theorem 2.2.1: *Let α be a BBP-type constant as defined earlier, with α_n the base-b expansion of α beginning at position $n+1$, and (x_n) the BBP sequence associated with α, as given in Eqn. (2.2.2). Then, $|x_n - \alpha_n| \to 0$ as $n \to \infty$.*

In other words, the *BBP sequence* associated with α (as given in formula (2.2.2)) is a close approximation to the sequence (α_n) of shifted digit expansions, so much so that we might expect that if one has a property such as equidistribution in the unit interval, then the other does also. We now state a hypothesis, which is believed to be true, based on experimental evidence, but which is not yet proven:

Hypothesis A (Bailey–Crandall). Let $p(x)$ and $q(x)$ be polynomials with integer coefficients, with $0 \leq \deg p < \deg q$, and with q having no zeros for positive-integer arguments. Let $b \geq 2$ be an integer and let $r_n = p(n)/q(n)$. Then, the sequence $x = (x_0, x_1, x_2, \ldots)$ determined by the iteration $x_0 = 0$ and

$$x_n = \{b x_{n-1} + r_n\} \tag{2.2.5}$$

either has a finite attractor or is equidistributed in $[0, 1)$.

The terms "equidistributed" and "finite attractor" are defined in Refs. [4, 11]. Here, we rely on intuition.

Theorem 2.2.2: *Assuming Hypothesis A, any constant α given by a formula of the type $\alpha = \sum_k p(k)/(b^k q(k))$, with $p(k)$ and $q(k)$ polynomials as given in Hypothesis A, is either rational or normal base b.*

We should note here that even if a particular instance of Hypothesis A could be established, it would have significant consequences. For example, if it could be established that the simple iteration given by $x_0 = 0$ and

$$x_n = \left\{ 2x_{n-1} + \frac{1}{n} \right\} \tag{2.2.6}$$

is equidistributed in $[0,1)$, then it would follow from Theorem 2.2.2 that $\log 2$ is 2-normal. Observe that this sequence is simply the BBP sequence for $\log 2$. In a similar vein, if it could be established that the iteration given by $x_0 = 0$ and

$$x_n = \left\{16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\} \tag{2.2.7}$$

is equidistributed in $[0,1)$, then it would follow that π is 16-normal (and thus 2-normal also). This is the BBP sequence for π. The fractional term here is obtained by combining the four fractions in the BBP formula for π, namely Eqn. (1.2.8), into one fraction and then shifting the index by 1.

Before continuing, we mention a curious phenomenon. Suppose we compute the binary sequence $y_n = \lfloor 2x_n \rfloor$, where (x_n) is the sequence associated with $\log 2$ as given in Eqn. (2.2.6). In other words, (y_n) is the binary sequence defined as $y_n = 0$ if $x_n < 1/2$ and $y_n = 1$ if $x_n \geq 1/2$. Theorem 2.2.1 tells us, in effect, that (y_n) eventually should agree very well with the true sequence of binary digits of $\log 2$. In explicit computations, we have found that the sequence (y_n) disagrees with 15 of the first 200 binary digits of $\log 2$, but in only one position over the range 5000 to 8000.

As noted earlier, the BBP sequence for π is $x_0 = 0$, and x_n as given in Eqn. (2.2.7). In a similar manner as with $\log 2$, we can compute the hexadecimal digit-sequence $y_n = \lfloor 16x_n \rfloor$. In other words, we can divide the unit interval into 16 equal subintervals, labeled $(0, 1, 2, 3, \ldots, 15)$ and set y_n to be the label of the subinterval in which x_n lies. When this is done, a remarkable phenomenon occurs: the sequence (y_n) appears to perfectly (not just approximately) produce the hexadecimal expansion of π. In explicit computations, the first one million hexadecimal digits generated by this sequence are identical with the first one million hexadecimal digits of $\pi - 3$. (This is a fairly difficult computation, as it must be performed to very high precision and is not easily performed on a parallel computer system.)

Conjecture. The sequence $(\lfloor 16x_n \rfloor)$, where (x_n) is the sequence of iterates defined in Eqn. (2.2.7), precisely generates the hexadecimal expansion of $\pi - 3$.

Evidently this phenomenon arises from the fact that in the sequence associated with π, the perturbation term $r_n = p(n)/q(n)$ is summable, whereas the corresponding expression for $\log 2$, namely $1/n$, is not summable. In particular, note that expression (2.2.4) for $\alpha = \pi$ is

$$\begin{aligned} ||\alpha_n - x_n|| &= \sum_{k=n+1}^{\infty} \frac{120k^2 - 89k + 16}{16^{j-n}(512k^4 - 1024k^3 + 712k^2 - 206k + 21)} \\ &\approx \frac{120(n+1)^2 - 89(n+1) + 16}{16(512(n+1)^4 - 1024(n+1)^3 + 712(n+1)^2 - 206(n+1) + 21)} \end{aligned} \tag{2.2.8}$$

so that

$$\sum_{n=1}^{\infty} ||\alpha_n - x_n|| \approx 0.01579\ldots. \tag{2.2.9}$$

For the sake of heuristic argument, let us assume for the moment that the α_n are independent, uniformly distributed random variables in $(0,1)$ and let $\delta_n = ||\alpha_n - x_n||$. Note that an error (i.e., an instance where x_n lies in a different subinterval of the unit interval than α_n) can only occur when α_n is within δ_n of one of the points $(0, 1/16, 2/16, \ldots, 15/16)$. Because $x_n < \alpha_n$ for all n (where $<$ is interpreted in the wrapped sense when x_n is slightly less than 1), this event has probability $16\delta_n$. Then, the fact that the sum (2.2.9) has a finite value implies, by the first Borel–Cantelli lemma, that there can only be finitely many errors [9, p. 153]. The comparable figure for $\log 2$ is infinite, which implies by the second Borel–Cantelli lemma that discrepancies can be expected to appear indefinitely, but with decreasing frequency. Further, the small value of the sum (2.2.9) suggests that it is unlikely that any errors will be observed. If instead of summing relation (2.2.9) from 1 to infinity, we instead sum from 1,000,001 to infinity (since we have computationally verified that there are no errors in the first one million elements), then we obtain 1.465×10^{-8}, which suggests that it is very unlikely that any errors will ever occur.

2.3 A Class of Provably Normal Constants

We now summarize an intriguing recent development in this arena, due to one of the present authors and Richard Crandall, which offers additional hope that the BBP approach may eventually yield the long-sought proof of normality for π, $\log 2$ and other BBP-type constants [5]. In the previous section, we noted that the 2-normality of

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n} \tag{2.3.10}$$

rests on the (unproven) conjecture that the iteration given by $x_0 = 0$ and $x_n = \{2x_{n-1} + 1/n\}$ is equidistributed in the unit interval. We now consider the class of constants where the summation defining $\log 2$, namely Eqn. (2.2.6), is taken over a certain subset of the positive integers:

$$\alpha_{b,c} = \sum_{n=c^k>1} \frac{1}{nb^n} = \sum_{k=1}^{\infty} \frac{1}{c^k b^{c^k}}, \tag{2.3.11}$$

where $b > 1$ and $c > 2$ are integers. The simplest instance of this class is

$$\begin{aligned} \alpha_{2,3} &= \sum_{n=3^k>1} \frac{1}{n2^n} = \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}} \qquad (2.3.12)\\ &= 0.041883680831502985071252898624571682426096758465485 7\ldots_{10} \\ &= 0.0AB8E38F684BDA12F684BF35BA781948B0FCD6E9E06522C3F35B\ldots_{16}\,. \end{aligned}$$

We first prove the following interesting fact:

Theorem 2.3.1: *Each of the constants $\alpha_{b,c}$, where $b > 1$ and $c > 2$ are integers, is transcendental.*

Proof. A famous theorem due to Roth, states [23] that if $|P/Q - \alpha| < 1/Q^{2+\epsilon}$ admits infinitely many rational solutions P/Q (i.e., if α is approximable to degree $2 + \epsilon$ for some $\epsilon > 0$), then α is transcendental. We show here that $\alpha_{b,c}$ is approximable to degree $c - \delta$. Fix a k and write

$$\alpha_{b,c} = P/Q + \sum_{n>k} \frac{1}{c^n b^{c^n}}, \tag{2.3.13}$$

where $\gcd(P, Q) = 1$ and $Q = c^k b^{c^k}$. The sum over n gives

$$|\alpha_{b,c} - P/Q| < \frac{2}{c^{k+1}(Q/c^k)^c} < \frac{c^{kc}}{Q^c}. \tag{2.3.14}$$

Now, $c^k \log b + k \log c = \log Q$, so that $c^k < \log Q/\log b$, and we can write

$$c^{kc} < (\log Q/\log b)^c = Q^{c(\log\log Q - \log\log b)/\log Q}. \tag{2.3.15}$$

Thus, for any fixed $\delta > 0$,

$$|\alpha_{b,c} - P/Q| < \frac{1}{Q^{c(1+\log\log b/\log Q - \log\log Q/\log Q)}} < \frac{1}{Q^{c-\delta}} \tag{2.3.16}$$

for all sufficiently large k. $\square$

Consider now the BBP sequence associated with $\alpha_{2,3}$, namely, the sequence defined by $x_0 = 0$ and

$$x_n = \{2x_{n-1} + r_n\} \tag{2.3.17}$$

where $r_n = 1/n$ if $n = 3^k$ and $r_n = 0$ otherwise. Successive iterates of this sequence are:

$$\begin{array}{l}
0,\ 0,\ 0, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3},\\
\frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9},\\
\frac{13}{27}, \frac{26}{27}, \frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{11}{27}, \frac{22}{27}, \frac{17}{27}, \frac{7}{27}, \frac{14}{27}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}, \frac{8}{27}, \frac{16}{27}, \frac{5}{27}, \frac{10}{27}, \frac{20}{27},\\
\frac{13}{27}, \frac{26}{27}, \frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{11}{27}, \frac{22}{27}, \frac{17}{27}, \frac{7}{27}, \frac{14}{27}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}, \frac{8}{27}, \frac{16}{27}, \frac{5}{27}, \frac{10}{27}, \frac{20}{27},\\
\frac{13}{27}, \frac{26}{27}, \frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{11}{27}, \frac{22}{27}, \frac{17}{27}, \frac{7}{27}, \frac{14}{27}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}, \frac{8}{27}, \frac{16}{27}, \frac{5}{27}, \frac{10}{27}, \frac{20}{27},
\end{array} \tag{2.3.18}$$

A pattern is clear: the sequence consists of a concatenation of triply-repeated segments, each consisting of fractions whose denominators are successively higher powers of 3 and whose numerators range over all integers less than the denominator that are coprime to the denominator. Indeed, the successive numerators in each subsequence are given by the simple linear congruential pseudorandom number generator $z_n = 2z_{n-1} \mod 3^j$ for a fixed j.

What we have observed is that the question of the equidistribution of the sequence (x_n) (and, hence, the question of the normality of $\alpha_{2,3}$) reduces to the behavior of a concatenation of normalized pseudorandom sequences of a type (namely, linear congruential) that have been studied in mathematical literature and which, in fact, are widely implemented for use by scientists and engineers. These observations lead to a rigorous proof of normality for many of these constants. In particular, we obtain the result that each of the constants

$$\alpha_{b,c} = \sum_{n=c^k>1} \frac{1}{nb^n} = \sum_{k=1}^{\infty} \frac{1}{c^k b^{c^k}}, \tag{2.3.19}$$

where $b > 1$, and c is odd and coprime to b, is b-normal. This result was first given in Ref. [5]. One may present significantly simpler proof, although it requires a modest excursion into measure theory and ergodic theory.

3

Historia Cyclometrica

3.1 1 Kings, 2 Chronicles, and Maimonides

We quote two versions of the famous and controversial biblical text suggesting that setting Pi equal to three sufficed for the Old Testament:

> "Then he [Solomon] made the molten sea: it was round, ten cubits from brim to brim, and five cubits high. A line of thirty cubits would encircle it completely." [21, 1 Kings 7:23]

and:

> "Then he made the molten sea: it was round, ten cubits from rim to rim, and five cubits high. A line of thirty cubits would encircle it completely." [21, 2 Chronicles 4:2]

Several millennia later, the great Rabbi Moses ben Maimon Maimonedes (1135–1204) is translated by Tzvi Langermann, in "The 'true perplexity' " [15, p. 165] as clearly asserting the irrationality of Pi.

> "You ought to know that the ratio of the diameter of the circle to its circumference is unknown, nor will it ever be possible to express it precisely. This is not due to any shortcoming of knowledge on our part, as the ignorant think. Rather, this matter is unknown due to its nature, and its discovery will never be attained."

3.2 François Viète. Book VIII, Chapter XVIII.

Ratio of Regular Polygons, Inscribed in a Circle, to the Circle

Archimedes squared the parabola by continuously inscribing triangles that are IN A RATIONAL RATIO.[1] For, having inscribed the greatest possible triangle in the parabola, he further inscribed triangles in continuous proportion to the greatest, namely in the constant ratio 1 to 4, infinitely. And so he concluded that the parabola is four thirds of that greatest triangle. But Antiphon could not square the circle in that way, since triangles inscribed continuously in a circle are in an IRRATIONAL RATIO and constantly changing. But will it not be possible to square the circle, then? For if a figure, composed of triangles that are constructed successively and infinitely in the ratio 1 to 4 to the given greatest triangle, is made four thirds of the same, then there is a certain knowledge of the infinitely many. And it is possible to compose a plane figure of triangles that are infinitely and continuously inscribed in a circle in A RATIO, albeit IRRATIONALS and constantly changing. And this composed figure will have a certain ratio to the greatest inscribed figure. The Euclideans, however, will maintain with authority that an angle greater than an acute and smaller than an obtuse is not a right angle. About that I propose the following so that it is possible to philosophize more freely about the uncertain and changing [ratio] of any regular polygon, inscribed in a circle, to a polygon with an infinite number of sides, or a circle if you will.

Proposition 1. *If two regular polygons be inscribed in the same circle, and if furthermore the number of sides or angles of the first one is half of the sides or angles of the second one, then the first polygon will be to the second as the apotome of the first side is to the diameter.* ("Apotome of a side" is my name for the cord which subtends the arc of the semicircle that supplements the arc subtended by the side.)

[**Proof of Proposition 1.**] Thus, in a circle with centre A, diameter BC, let any regular polygon be inscribed, whose side is BD. And let the arc BD, bisected at E, be subtended by BE [and ED]. That is to say, let another polygon be inscribed whose side is BE. So the number of sides or angles of the first polygon will be half of the number of sides or angles of the second. Let DC be joined. I say that the first polygon with side BD is to the second polygon

[1] Greek in original is rendered in small capitals.

END OF PAGE 398 [2]

with side BE or ED as DC to BC. For let DA, ED be joined. The first polygon consists of as many triangles BAD as there are sides or angles in the first polygon. And the second polygon consists of as many trapezia BEDA. Therefore the first polygon is to the second as the triangle BAD is to the trapezium BEDA. But the trapezium BEDA is divided into two triangles BAD, BED, whose common base is BD. And triangles whose base is the same are [to each other] as the heights. Therefore, let the half diameter AE be drawn intersecting BD in F. Thus, since the arc BD is bisected in E, AE intersects BD at right angles. Therefore AF is the height of the triangle BDA and FE is the height of the triangle BED. And so the triangle BAD is to the triangle BED as AF to EF, and *componendo* the triangle BAD to the triangles BAD, BED together, that is the trapezium BEDA, as AF to AE. And the first polygon will be to the second in that ratio, too. But, AF is to AE or AB as DC is to BC; for, the angle BDC is right, as is the angle BFA, and therefore AF and DC are parallel. Thus the first polygon, whose side is BD, is to the second polygon, whose side is BE or ED, as DC to BC. Which was to be shown.

Proposition 2. *If in one and the same circle infinitely many regular polygons are inscribed, and the number of sides of the first is 1/2 of the sides of the second, and 1/4 of the number of sides of the third, 1/8 of the fourth, 1/16 of the fifth, and so on in continuous halvings, then the first polygon will be to the third as the product of [lit. "the rectangle contained by"] the apotomes of the sides of the first and the second is to the square on the diameter. To the fourth it will be as the product of [lit. "the solid made of"] the apotomes of the sides of the first, the second, and the third is to the cube on the diameter. To the fifth it will be as the product of the four lengths [lit. "plano-planum"] of the apotomes of the sides of the first, the second, the third, and the fourth is to the fourth power of [lit "quadrato-quadratum on"] the diameter. To the sixth it will be as the product of the five lengths [lit. "plano-solidum"] of the apotomes of the sides of the first, the second, the third, the fourth, and the fifth is to the fifth power of [lit. "quadrato-cubum on"] the diameter. To the seventh it will be as the product of the six lengths [lit. "solido-solidum"] of the apotomes of the sides of the first, the second, the third, the fourth, the fifth, and the sixth is to the sixth power of [lit. "cubo-cubum on"] the diameter. And so on in continuous progression* ad infinitum. [Having noted Viète's terms for various kinds of products and powers

[2]These page numbers refer to those of the original printed Latin text of the work reprinted in this volume.

of geometrical magnitudes we henceforth translate them by modern terminology for products.]

[**Proof.**] Let B be the apotome of the side of the first polygon, C of the second, D of the third, F of the fourth, G of the fifth, H of the sixth. And let the diameter of the circle be Z. According to the first proposition the first polygon will be to the second as B to Z; therefore the product of B and the second polygon will be equal to the product of Z and the first polygon. But the second polygon will be to the third as C to Z; consequently the product of the second polygon and B, that is the product of the first polygon and Z, will be to the product of the third polygon and B as C to Z. Therefore the product of the first polygon and Z squared equals the product of the third polygon and the product of B,C. Therefore the first polygon is to the third as the product of B,C to Z squared. And the product of the third polygon and the product $B\times C$ is equal to the product of the first polygon and Z squared. Again, according to the same previous proposition, as the third polygon is to the fourth, so is D to Z. And consequently, the product of the third and the rectangle B times C, that is the product of the first and Z squared, is to the product of the fourth and the rectangle B times C, as D to Z. Therefore, the product of the first [polygon] and Z to the third power is equal to the product of the fourth [polygon] and B times C times D. Therefore the first polygon is to the fourth as B times C times D to Z cubed. By the same method of demonstration it [the first polygon] will be to the fifth as B times C times D times F to Z to the fourth power. To the sixth as B times C times D times F times G to Z to the fifth power. To the seventh as B times C times D times F times G times H to Z to the sixth power. And so forth in this constant progression ad infinitum.

END OF PAGE 399

Corollary. *Therefore the square inscribed in the circle will be as the side of this square to the highest power of the diameter divided by the continuous product of the apotomes of the sides of the octagon, the sixteen-gon, the polygon with 32 sides, with 64 sides, with 128 sides, with 256 sides, and of all the others in the half ratio of angles and sides.*

[**Proof.**] For, let the square be the first polygon inscribed in the circle; then the octagon will be the second, the sixteen-gon the third, the thirty-two-gon the fourth, and so on in continuous order. Thus the square inscribed in the circle will have the same ratio to the extreme polygon—with infinitely many sides—as the product made by the apotomes of the sides of the square, the octagon, the sixteen-gon, and all the others in the half ratio *ad infinitum* has to the highest power of the diameter. And by a common division [the square inscribed in the circle will have the same ratio to the extreme polygon] as the apotome of the side of the square has to the highest power of the diameter divided by the product of the apotomes of the sides of the octagon, sixteen-gon, and the others in double ratio *ad infinitum.* But the apotome of the side of the square inscribed in a circle is the side itself, and the polygon with infinitely many sides is the circle itself.

Let the circle's diameter be 2, and the side of the inscribed square be $\sqrt{2}$. The apotome of the 8-gon is $\sqrt{2+\sqrt{2}}$. The apotome of the 16-gon is $\sqrt{2+\sqrt{2+\sqrt{2}}}$. The apotome of the 32-gon is $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$. The apotome of the 64-gon is $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$. And so on in that progression.

But then, let the diameter be 1 and the [area of the] circle N. [Viète writes, in the manner of Diophantus "1N".] 1/2 [the area of the inscribed square] will be to N as $\sqrt{1/2}$ to the unit divided by

$\sqrt{1/2+\sqrt{1/2}}$, times $\sqrt{1/2+\sqrt{1/2+\sqrt{1/2}}}$,

times $\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1/2}}}}$,

times $\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1/2}}}}}$,

times $\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1+\sqrt{1/2}}}}}}$.

Let the diameter be X and the circle (equal to) the plane area A. The half square on X will be to the area A as the side of the half square on X to the greatest power of X divided by the product of the binomial square root [i.e., a square root of the sum of two terms] of ($1/2X^2$ the square root of $1/2X^4$) times the binomial square root of ($1/2X^2$ + the binomial square root of ($1/2X^4$ + the square root of $1/2X^8$))) times the binomial square root of ($1/2X^2$+ the binomial square root of ($1/2X^4$+ the binomial square root of ($1/2X^8$+ the square root of $1/2X^{16}$)))$\cdots$ times ... etc. *ad infinitum* while observing this uniform method.

Defense Against the New Cyclometry or ANTI-AXE

Those who have tried to set the circle equal to thirty-six segments of the hexagon by means of their figures which they call hatchets, unluckily are wasting their efforts. For how can determined results be obtained from splitting completely undetermined magnitudes? If they add or subtract equals, if they divide or multiply by equals, if they invert, permute and at last increase or decrease by arbitrary degrees of proportion, they will not advance one inch in their research, but will make the mistake that logicians call BEGGING THE QUESTION, and Diophantines [call] NON-QUANTITIES. Or they delude themselves by false calculations, though they could have foreseen it if any light from the true analytic doctrine had enlightened them. Others, however, who are terrified by these ONE-EDGED double bladed weapons and already are lamenting that Archimedes is wounded by them, are quite unfit for war. But Archimedes lives, nor do THE FALSE WRITINGS ABOUT THE TRUTH, THE FALSE RECKONINGS, the Non-proofs, the magnificent words, shock him. But in order that they may live more secure, I bring them "Shields decorated with clouds and weapons untouched by slaughter," but HARD TO AXE, wherewith they can fortify themselves to begin with, ready to reinforce them with means FOR WAR if the enemies' impudence be too fierce.

Proposition 1. *The circumference of a dodecagon inscribed in a circle has a ratio to the diameter (that is) less than triple plus one-eighth.*

With centre A and an arbitrary radius AB let the circle BCD be described, in which let BC be taken as arc of the hexagon [i.e., an arc equal to 1/6 of the circumference], which is bisected at D, and let DB be subtended. Thus DB is the side of the dodecagon; and if it be extended twelve times to E, DE will be equal to the circumference of the dodecagon inscribed in the circle BDC. Let the diameter DF be drawn. I say that DE to DF has a ratio less than triple plus one-eighth.

For, let BC and BA be joined, and let the diameter DF intersect BC at G. Therefore it will bisect it at right angles. And let the triangle DEH be constructed similar to the triangle DBG.

Since the line BC is subtended under an arc of the hexagon, BA or DA is equal to BC. Therefore, if AC or BC are composed of eight (equal) parts, BG is four of the same (parts). So the square on AG is $[64 - 16] = 48$, and so AG is greater than 6 12/13 [since $48 = 7(7 - 1/7)$, and the arithmetic mean of the two factors is $7 - 1/14$. Since the geometric mean of the two factors (which is the square root of their product) is less than the arithmetic mean $\sqrt{48} < 6\ 13/14$. A good guess therefore is that $\sqrt{48} > 6\ 12/13$, which is easily verified.]

Therefore DG is less than 1 1/13. And since DE is composed of twelve times DB, EH will also be twelve times

---

END OF PAGE 437

BG, and DH twelve times DG. Therefore EH is 48 of the same parts. And DH will be less than 13, namely less than 12 12/13. The square on the side 48 is 2304, and on 13 it is 169. Those two squares added make 2473, not quite 2500, the square on the side 50. Therefore the line DE, whose square is equal to the sum of the squares on EH and DH, is less than 50. And the ratio of 50 to 16 is triple and one-eighth exactly. Therefore the ratio of DE to DF is less than triple and one-eighth. Which was to be proved.

Arithmetic is absolutely as much science as geometry [is]. Rational magnitudes are conveniently designated by rational numbers, and irrational [magnitudes] by irrational [numbers]. If someone measures magnitudes with numbers and by his calculation get them different from what they really are, it is not the reckoning's fault but the reckoner's.

Rather, says Proclus, ARITHMETIC IS MORE EXACT THAN GEOMETRY. To an accurate calculator, if the diameter is set to one unit, the circumference of the inscribed dodecagon will be the side of the binomial [i.e., square root of the difference] $72 - \sqrt{3888}$. Whoever declares any other result, will be mistaken, either the geometer in his measurements or the calculator in his numbers.

That the ratio of the circumference of the circle to its diameter is greater than triple and one-eighth as well as less than triple and one-seventh has not been doubted up till now by the school of mathematicians, for Archimedes proved that convincingly. So one should not, by a false calculation, have induced a MANIFEST ABSURDITY, that a straight line is longer than the circular arc terminating at the same endpoints, since Archimedes assumes the contrary FROM THE COMMON NOTION and Eutocius demonstrates the same, defining generally that OF ALL LINES HAVING THE SAME EXTREMITIES, THE STRAIGHT LINE IS THE SHORTEST.

Proposition 2. *If the half diameter of the circle be divided by the quadratrix, the part from the centre to the quadratrix is greater than the mean proportional between the half diameter and two fifths of the half diameter.*

Let ABC be a quadrant of a circle, and BD the quadratrix; let AE be taken equal to two fifths of the half diameter AB or AC; and let AF be made the mean proportional of AE, AC. I say that AD is greater than AF. For, from what was proved by Pappus about the quadratrix, the half diameter AB or

---

END OF PAGE 438

AC is the mean proportional of the arc BC and AD. Let AB be 7 parts; then the arc BC, which is a quarter of the perimeter, will be less than 11, for the diameter is 14, and the perimeter is less than 44. But then, let AB be 35 parts; then the arc BC will be less than 55. But the [area] contained by AD and [the arc] BC is equal to the square on AB. Therefore AD is greater than 22 [+] 3/11. And of those units of which AB, that is AC, is 35, AE is 14, but AF is less than 22 [+] 3/22. [See bracketed explanation of a similar claim near end of p. 437.] Therefore AD is greater than AF. Which was to be proved.

Therefore, if from the diameter AB be subtracted the line AG equal to AF, and if the parallelogram GHDA be completed, it will be an oblong rectangle, not a square. And when the square BC is completed, the diagonal BK will not go through H, but through some point I further from D. It was important to notice this to avoid a false diagram.

Proposition 3. *The square on the circumference of the circle is less than ten times the square on the diameter.*

Let the diameter be 7, the square on the diameter will be 49, and ten times that 490. But the circumference of the circle will be less than 22, and consequently its square less than 484. The Arabs' opinion that "the square on the circumference of the circle is equal to ten times the square on the diameter" has since long been rejected. He is not to be tolerated who CONTRADICTORILY proposes what Archimedes proves to be unprovable.

Proposition 4. *The circle has a greater ratio to the hexagon inscribed in it than six to five.*

Let the hexagon BCDEFG be inscribed in the circle with centre A. I say that the circle with centre A has a greater ratio to the hexagon BCDEFG than six to five. When AB, AC BC are joined, let the perpendicular AZ fall on BC. Then, since in triangle ABC the legs AB, AC are equal, the base is bisected in Z, and BZ and ZC are equal. But the triangle ABC is equilateral, for its legs are both half diameters, and the base — since it is the side of the hexagon — is also equal to the half diameter. Now, if the half diameter BA or AC be set to 30 [units], BZ or ZC becomes 15, and AZ becomes less than 26, whose square is 676. But the difference between the squares [on] AB, AZ is only 675. Further, the rectangle contained by BZ, ZA is equal to the triangle BAC. So, let 15 be multiplied by 26, they become 390. Therefore, of such units of which the square [on] AB is 900, of the same [units] the triangle ABC will be less than 390, or — if all be divided by 30 — if the square AB is 30, the triangle ABC will be less than 13. Let AD, AE, AF, AG be joined; the hexagon BCDEFG consists of six triangles equal to BAC. Therefore, of such [units] of which the square [on] AB becomes 30, of the same [units] the hexagon will be less than 78. Or, of such units of which the square [on] AB becomes five, of the same [units] the hexagon will be less than thirteen.

END OF PAGE 439

But as the perimeter of the circle is to the diameter, so is the rectangle made by the perimeter of the circle and a quarter of the diameter to the area made by the diameter and a quarter of the diameter. But the rectangle made by the perimeter of the circle and a quarter of the diameter is equal to the circle. And the area made by the diameter and a quarter of the diameter is the square on the half diameter. Therefore, as the perimeter is to the diameter, so is the circle to the square on the half diameter. Now, of such units as the diameter is 1, the perimeter is greater than 3[+]10/71, and so evidently greater than 3[+]10/8 or 3[+]1/8. And of such units as the square on the diameter is 5 (as above) the circle is greater than 15[+]5/8. But the hexagon was less than 13 of the same units. Therefore the circle will have to the hexagon inscribed in it a greater ratio than 15[+]5/8 to 13, that is, as 125 to 104, or as [750 to 624, that is] 6 to 4[+]124/125, and so evidently greater than six to five. Which was to be proved.

Therefore, those who set the circle equal to the hexagon and a fifth part of the hexagon do not square it PROPERLY, since it is greater according to the limits set by Archimedes FROM HIS OWN PRINCIPLES. Our schools are Platonic: Oh splendid professors; therefore, do not fight against the geometric principles. And just as these AXE-SWINGERS have truncated the circle, they may now — as a kind of compensation for the damage done — themselves be shortened at the pointed end of their swallow tail.

Proposition 5. *Thirty-six segments of the hexagon are greater than the circle.*

For, since the circle has a greater ratio to the hexagon inscribed in it than six to five, or as 1 to 5/6, therefore the difference between the circle and the hexagon will be greater than one-sixth of the circle. But the circle differs from the hexagon by six segments of the hexagon. Thus six segments of the hexagon are greater than one-sixth of the circle, and so thirty-six segments will be greater than one, that is: the circle. What was to be proved.

Proposition 6. *Any segment of the circle is greater than the sixth of the similar segment similarly drawn in a circle whose radius is equal to the base of the segment set out.*

In the circle BDC described around the centre A let a cord be subtended under an arbitrary arc BD, and let a straight line BE touch the circle, and with B as centre and BD as radius let another circle DEF be drawn.

Then the arc ED will be similar to half of the arc BD. And then, let the arc DF be taken double of DE, and let BF and AD be joined. Thus the sectors BAD and FBD will be similar. I say that the segment of the circle BDC contained by the line BD and the arc under which it is suspended, is greater than one-sixth of the sector FBD.

END OF PAGE 440

For, let a spiral line whose origin is B be drawn, crossing [the circle] at D so that BD is the same part of the first revolution BEZ [the point Z is not labeled in the diagram] as the angle EBD is of four right angles. Thus the area contained by the straight line BD and the spiral is the third part of the sector EBD, for that has Pappus proved after Archimedes in proposition xxii in Book IV of the *Mathematical Collection.* Therefore the [sector] FBD will be six times the same area, for the sector FBD is double of EBD by construction. The spiral does not coincide with the circle, for that would be absurd, nor does the spiral in its course get out of the circle before reaching point D. For, let the angle EBD be divided arbitrarily by the straight line BGH, which intercepts the spiral in G and the circumference in H. Then the line BD will be to the line BG as the angle EBD to the angle EBG; that is, as the arc BD to the arc BH, according to the conditions of spirals. But the ratio of the arc BD to the arc BH is greater than the ratio of the cord BD to the cord BH. For greater arcs have to lesser arcs a greater ratio than the straight lines to the straight lines that subtend those same arcs. Therefore the line BH is greater than BG, and the same will happen to any straight lines that divide the angle EBD. Therefore the spiral will proceed under the arc BD and will leave some area between itself and the arc. By that area the segment of the circle contained between the line BD and the circumference exceeds the area which is enclosed between the same line and the spiral, and which is proved equal to one-sixth of the sector FBD. Therefore that segment will be greater than one-sixth of the sector FBD. Which was to be proved.

Corollary. *And from this it is also obvious that thirty-six segments of the hexagon are greater than the circle.*

For when BD happens to be a segment of the hexagon, the sectors FBD and BAD will be equal since their circles' half diameters BD [and] AD will be equal. Thus six segments of the hexagon will be greater than the sector BAD, and therefore thirty-six segments greater than six sectors, that is, the whole circle. It is possible to propose a no less general theorem to be proved by parabolas, or rather by the same geometrical methods through which the parabola is squared: Any segment of the circle is greater than four-thirds of the isosceles triangle inscribed in the segment with the same base. By that, it will soon appear that the ratio of thirty-six segments of the hexagon to the circle is greater than 48 to 47. But to an even more accurate calculator only thirty-four segments and an area little greater than two-thirds but less than three-quarters of a segment can be found to complete the circle. It is possible in this way to make known to the eyes that the excess is greater than one-third of a twelfth.

Proposition 7. *In a given circle to cut off the thirty-sixth part of the circle itself from a segment of the hexagon.*

Let a circle be given with centre A, diameter BC and a segment of the hexagon BD. It is required, in the given circle BDC from the segment of the hexagon contained by the cord BD and the arc under which it is suspended, to cut off the thirty-sixth part of the circle itself. Let the straight line BE be tangent to the circle, and let a spiral line be described, with origin B and passage through D, so that BD is the same part of the first revolution BEZ as the angle EBD is of four right angles, and with centre B and radius BD let the circle DE be described. Then the third part of the sector EBD is the area contained by the line BD and the spiral. But BD is equal to the half diameter BA, for BD is the side of the hexagon by hypothesis, and the angle EBD is one-third of a right angle because the size of arc BD is two-thirds of a right angle. Thus the sector EBD is a twelfth of the circle, and consequently the spiral area BD is a third of the twelfth, that is a thirty-sixth part of the circle. The spiral will pass through the segment and will not meet the circle, (nor will it cut off a circular area

---

END OF PAGE 441

---

on its way from B) until it reaches the point D, as has been proved. Thus, in the given circle BDC the thirty-sixth part of the circle itself has been cut off from the hexagon's segment BD. Which was to be done.

And by this sevenfold shield let the blade of the soft and blunt axe have been weakened enough.

If some should want a sketch of THE FIGHT OF THE AXE itself, lest they miss it let them study it in a few short pages.

ANALYSIS OF THE CIRCLE, according to the AXE-SWINGERS.

1. The circle consists of six scalpels of the hexagon.

2. The scalpel of the hexagon consists of [i.e., is equal to] the segment of the hexagon and the triangle of the hexagon, or the (so-called) "major."

3. The triangle of the hexagon consists of a segment of the hexagon and a hatchet.

4. The hatchet consists of two segments of the hexagon and the complement of the hatchet.

5. The complement of the hatchet consists of one segment of the hexagon and the remainder of a segment.

6. But on the other hand, the complement of the hatchet consists of the minor triangle and the remainder of the minor triangle. The minor triangle is [by definition] the fifth part of the hexagon, the so-called "major."

END OF PAGE 442

TWO TRUE LEMMAS, First: *Ten minor triangles are equal to six segments of the hexagon and two complements of the hatchet.* For, since ["Quorum" in text should be "Quoniam," a common misreading] the hexagon's triangle consists of a segment and a hatchet, and the hatchet of two segments and a complement, therefore two triangles of the hexagon consist of six segments and two complements. But two triangles of the hexagon, or the major, are equal to ten minor [triangles] (by definition). Therefore ten minor triangles will be equal to six segments and two complements. What was to be proved.

Second: *Forty minor triangles are equal to the triangle and two complements of the hatchet.*

For, since the circle equals six scalpels of the hexagon, but six scalpels are equal to six triangles of the hexagon and six segments, and further six triangles of the hexagon make thirty minor triangles, therefore the circle equals thirty minor triangles and six segments. Let two complements of the hatchet be added to both sides.

Thus the circle plus two complements of the hatchet will be equal to thirty minor triangles and six segments and two complements. But six segments and two complements make ten minor triangles, according to the previous lemma. Therefore forty minor triangles are equal to six segments of the hexagon and two complements of the hatchet. Which was to be proven.

FALLACY: I say that the minor triangle is equal to its remainder. By way of proof: Since the circle plus two complements of the hatchet (the latter being equal to two minor triangles plus two remainders of the minor triangle) is equal to thirty-six minor triangles and another four, let from both sides [of the equation] be taken away two minor triangles.

When from this side they are subtracted from two complements, two remainders of the [minor] triangle are left. When from that side they are subtracted from four triangles, two triangles are left. Therefore two remainders equal two triangles.

Refutation of the FAULTINESS OF LOGIC. Equals must be subtracted from equal wholes, not from equal parts, to make the remainders equal. To subtract something from a part of equals is to assume that the remainder[s] of the whole[s] are equal, as here the circle is set equal to thirty-six minor triangles. But that is flatly denied and totally false. To grant oneself what should be proved looks as if one wants to show how to make an error.

FOR ANOTHER FALLACY, TWO TRUE LEMMAS. First: *Twenty-four quarters of the hexagon's triangle plus six segments are equal to twenty-four segments and six complements of the hatchet.*

For, since the triangle of the hexagon consists of three segments and a complement of the hatchet, but the circle is composed of six triangles and six segments, therefore twenty-four segments and six complements are equal to the circle.

And since four quarters make one whole, also twenty-four quarters of the hexagon's triangle plus six segments are equal to the circle. But things that are equal to one [and the same] are equal to one another. Therefore twenty-four quarters of the hexagon's triangle plus six segments are equal to twenty-four segments and six complements of the hatchet. Which was to be proved.

END OF PAGE 443

Second. *If there are three unequal magnitudes, of which the middle taken twenty-four times and added to the least taken six times produces the same magnitude as the least taken twenty-four times and added to the greatest taken six times, then the difference between four times the middle and thrice the least will be equal to the greatest.* For, let B be the least, D the middle, A the greatest one. By hypothesis, then, 6 B plus 24 D = 6 A plus 24 B. Let 24 B be subtracted from both sides. Then 24 D minus 18 B = 6 A. If all be divided by 6, 4 D minus 3 B = A. That is exactly what was stated.

UNPROVED THEOREM. *There are three unequal plane figures that are commensurable between them; the smallest is the segment of the hexagon; the middle is a quarter of the triangle of the hexagon; the greatest is the complement of the hatchet of the hexagon.* Inequality and the degree of inequality could be proved, but nobody will ever prove commensurability and incommensurability unless he first has compared the hexagon's triangle or another rectilineal figure with the circle. But that comparison is unknown hitherto, and IF IT IS FEASIBLE, IT IS IN THE LAP OF THE GODS.

PSEUDO-PORISM. Such parts of which a quarter of the hexagon's triangle will be five, of the same parts the segment must necessarily be four.

BY WAY OF PROOF. For, let B be a segment of the hexagon, and D a quarter of the hexagon's triangle, and Z the complement of the hatchet. Now, since there are three unequal magnitudes, of which B is the least, D the middle, and Z the greatest, [therefore] they will have to one another [a ratio] as a number to a number. Suppose that D is five parts, and that B is three or four parts, and no more. Let it be, if it is possible, three parts; according to the first and second lemma, Z will be eleven. So the complement will consist of two segments and a twelfth of a segment, but that contradicts our senses. Therefore B is four.

Refutation of the FAULTINESS OF LOGIC. If the magnitude D is five parts, it can be proved that B is greater than three parts. But will B therefore be four, even admitted—what is unknown—that B is to D as a number to a number? That conclusion is totally invalid. For, what if B is said to be four parts plus some rational fraction? Is not four and a half to five as a number to a number, that is, as 9 to 10? Nobody but an NON-LOGICIAN OR NON-GEOMETER will deny that. In fact, if D is set to 11 parts, B becomes a bit greater than 9, and Z a little less than 17, according to Archimedes' limits. However, from these two FALLACIES were spread the other AXE-FIGHTERS' ERRORS CONCERNING THE AREA OF THE CIRCLE AND THE SURFACE OF THE SPHERE.

END OF THE FIGHT AGAINST THE AXE-SWINGERS.

END OF PAGE 444

The second FIGHT AGAINST THE AXE-SWINGERS. An outline, FROM THE ADDENDUM.

In a circle with centre A let an arc of the hexagon BCD be taken, and let AB, AD, BD be joined. Further, from AB let a line segment be cut off whose square is to the square on AB as one to five. Let it be BE, and through E let a parallel to AD be drawn, intersecting BD in F. Then the triangle BEF will be equiangular with BAD and one-fifth of it.

Lemma 1. True. *Thirty-seven triangles BEF are greater than the circle BCD.*

For, in the comments to the *Canon Mathematicus,* the circle is shown to have a ratio to the square on the half diameter that is very close to 31,415,926,536 to 10,000,000,000. If, therefore, the side AB, that is the half diameter, is set to 100,000, the height of the equilateral triangle ABD is $86,602[+]54,038/100,000$. Therefore:

The triangle ABD becomes 4,330,127,019. The triangle BEF becomes 866,025,404. Thirty-seven triangles BEF 32,042,939,948 [the circle] 31,415,926,536 [the triangles] exceed the circle by 617,013,412.

Lemma 2. True. *The circle BCD is not greater than thirty-six segments BCDF.*

Even more, the circle BCD is far less than thirty-six segments BCDF. For the sector BAD is the sixth part of the whole circle. Therefore

The circle is 31,415,926,536. The sector BAD is 5,235,987,756. Let the triangle ABD be subtracted from it 4,330,127,019. There remains the hexagon sector or the *mixtiline* space BCDF 905,860,737. But three dozens of such segments are 32,610,986,532 exceeding the circle by 1,195,059,996.

END OF PAGE 445

PSEUDO-PORISM. Consequently thirty-seven triangles BEF are greater than thirty-six segments BCDF. Refutation of the false conclusion. In grammar, to give to the ships the south winds, and to give the ships to the south winds, mean the same thing. But in geometry, it is one thing to assume that the circle BCD is not greater than thirty-six segments BCDF and another to assume that thirty-six segments are not greater than the circle BCD. The first is true, the second is false. Then, when I argue in this way:

Thirty-seven triangles are greater than the circle, but thirty-six segments are not greater than the circle, therefore thirty-seven triangles are greater than thirty-six segments,

I conclude syllogistically, but wrongly, because the assumption is false. But I sin against the laws of logic when I establish the syllogism in this formula:

The circle is smaller than thirty-seven triangles. The circle is not greater than thirty-six segments. Therefore thirty-seven triangles are greater than thirty-six segments.

But this is AN ERROR OF THE EYES, not of the INTELLECT. For, when at the beginning the Circle-measurers had proposed that the circle is not greater than thirty-six segments of the hexagon, they read it in the light of subsequent events as "not less than," and thus extracted the false corollary.

END

3.3 Christian Huygens. Problem IV, Proposition XX

Determining the Magnitude of the Circle

To find the ratio between the circumference and the diameter; and, given chords in a given circle, to find the lengths of the arcs that they subtend.

Consider a circle of center D, with CB as a diameter, and let AB be an arc one-sixth of the circumference, for which we draw the chord AB and the sine AM. If we suppose then that the half-diameter DB is 100,000 parts, the chord BA will contain the same number. But AM will be made of 86,603 parts and not one less (which means that if we should take away one part or one unit of the 86,603 we would have less than what it should be), since it is half of the side of the equilateral triangle inscribed in the circle.

From there, the excess of AB over AM becomes 13,397, less than the true value. One third of it is $4{,}465\frac{2}{3}$, which, added to the 100,000 of AB, gives $104{,}465\frac{2}{3}$, which is less than arc AB. And this is a first lower limit; in the following, we will find another one, closer to the real value. But first we must also find an upper limit, according to the same theorem.

Then a fourth proportional is to be found for three numbers. The first equals the double parts of AB and the triple of AM. It will then be 459,809, less than the real value (since we also have to make sure that this number here is less; and in the same way with the other, as we shall specify) the second is equal to the quadruple of AB and AM, which is 486,603, more than the real value. And the third is one-third of the excess of AB over AM, 4,466, more than the real value which, added to AB or 100,000 gives 104,727, larger than the number of parts that arc AB, a sixth of the periphery, contains [according to the above]. Then we already found the length of arc AB within an upper and lower limit, of which two the last is far closer to the real value because the number 104,719 is closer to the real value.

But through these two, we will obtain another lower limit, more exact than the first one, using the following precept, which results from a more precise examination of the center of gravity.

END OF PAGE 384

Add four-thirds of the difference of the above limits to the double of the chord and the triple of the sine, and the same ratio as between the line made this way and three and one-third, or $\frac{10}{3}$ times the sum of the sine and the chord, also exists between the excess of the chord over the sine and another line. This last one added to the sine will be a line smaller than the arc.

The lower limit was $104{,}465\frac{2}{3}$; the upper one is 104,727; their difference is $261\frac{1}{3}$. Again we need to find a fourth proportional to three numbers. The first one is the double of the parts of AB increased by the triple of AM and by four-thirds of the difference of the limits. We find 460,158, larger than the real value. The second is the $\frac{10}{3}$ of AB and AM taken together, 622,008, smaller than the real value. Last the third is the excess of AB over AM, 13,397, smaller than the real value. The fourth proportional to these numbers is 18,109, smaller than the real value.

Then if we add this to the number of parts of AM, $86{,}602\frac{1}{2}$, less than the real value, we get 104, $711\frac{1}{2}$, less than arc AB. Thus the sextuple of these parts, 628,269, will be less than the whole circumference. But because 104,727 of these parts were found larger than arc AB, their sextuple will be larger than the circumference. Thus the ratio of the circumference to the diameter is smaller than that of 628,362 to 200,000 and larger than that of 628,268 to 200,000, or smaller than that of 314,181 [to 100,000] and larger than that of 314,135 to 100,000. From that, the ratio is certainly smaller than $3\frac{1}{7}$ and larger than $3\frac{10}{71}$. From there also is refuted Longomontanus's mistake, who wrote that the periphery is larger than 314,185 parts, when the radius contains 100,000.

Let us suppose that arc AB is $\frac{1}{8}$ of the circumference; then AM, half of the side of the square inscribed in the circle, will measure 7,071,068 parts, and not one less, of which the radius DB measures 10,000,000. On the other hand, AB, side of the octagon, measures 7,653,668 parts and not one more. Through this data, we will find, in the same manner as above, as first lower limit of the length of arc AB 7,847,868. Then as upper limit 7,854,066. And from these two, again, a more precise lower limit 7,853,885. This results in the ratio of the periphery to the diameter being less than $31{,}416\frac{1}{4}$, and more than 31,415, to 10,000.

And since the difference between the upper limit 7,854,066 and the real length of the arc

---

END OF PAGE 385

---

is less than 85 parts (indeed, arc AB, according to what we proved above is larger than 7,853,981), and since 85 parts are less than two seconds, which is $\frac{2}{1{,}296{,}000}$ of the circumference, because the whole circumference has more than 60,000,000 parts, it is obvious that, if we try to find the angles of a right triangle using the given sides, the same way as we did for the upper limit above, the error can never be more than two seconds; even if the sides of the right angle are equal, as they were here in triangle DAM.

But if the ratio of side DM to MA is such that the angle ADM does not exceed a quarter of a right angle, the error will not be more than a third scruple. For, taking arc AB equal to $\frac{1}{10}$ of the circumference, AM will be half of the side of the equilateral octagon inscribed in the circle, and equal to 382,683,433 parts and not more; but, AB will be the side of the sixteen-gon and then will contain 390,180,644 parts, and not one more, with the radius DB containing 1,000,000,000 parts. In this way is found a first lower limit, of the length of arc AB, of 392,679,714 parts. And the upper limit is 392,699,148. And from there again a lower limit of 392,699,010. But, what was proved above results in arc AB, $\frac{1}{10}$ of the circumference, being larger than 392,699,081 parts, which the upper limit exceeds by 67 parts. But these are less than a third scruple, which is $\frac{1}{77{,}760{,}000}$ of the whole circumference, since the circumference is larger than 6,000,000,000.

Then, out of these new limits just found, the ratio of the circumference to the diameter will come smaller than $314{,}593\frac{1}{6}$ to 1,000,000 but larger than 314,592 to 1,000,000.

And if we take an arc AB equal to $\frac{1}{60}$ of the circumference, which is six parts of the total 360, AM will be half of the side of the (inscribed) 30-gon, made of 10,452,846,326,766 parts, and not one less, when the radius has 100,000,000,000,000. And AB is the side of the (inscribed) 60-gon, 10,467,191,248,588 parts and not one more. Through this data is found arc AB, according to the first lower limit, 10,471,972,889,195. Then the upper limit 10,471,975,512,584. And from there the other lower limit, 10,471,975,511,302. This results in the ratio of the periphery to the diameter being less than

END OF PAGE 386

31,415,926,538 to 10,000,000,000, but larger than 31,415,926,533 to 10,000,000,000.

If we had to find these limits through adding the sides of the inscribed and circumscribed polygons, we would have to go up to 400,000 sides. Because with the 60 angles inscribed and circumscribed polygons, we only prove that the ratio of the periphery to the diameter is less than 3,145 to 1,000, and larger than 3,140. Thus, the number of exact digits through this calculation seems to be three times higher, and even more. But if someone tries it, he will see that the same always happens with the following polygons [as well]; we know why but it would take a long explanation.

On the other hand, I believe it is clear enough how, for any other inscribed polygons, it is possible to find, through the above methods, the length of the arcs subtended. Because, if they are larger than the side of the inscribed square, we will have to find the length of the remaining arc on the half circumference, the chord of which is then also given. But we must also know how to find the chords of the half-arcs, when the chord of the full arc is given. And this way, if we want to use bisections, we will be able to find without any difficulty for any chord the length of its arc, as close as we want. This is useful for examining tables of sines, and even for their composition; because, knowing the chord of a given arc, we can determine with sufficient accuracy the length of the arc that is slightly larger or smaller.

4

Demotica Cyclometrica

4.1 Irving Kaplansky's "A Song about Pi"

The distinguished mathematician Irving Kaplansky (1917–) is also a fine musician, whose daughter Lucy is an accomplished folk singer. In February 1973, Kaplansky composed a popular "A Song about Pi" of *Type 2* in the sense he describes in more detail in *More Mathematical People* [1, pp.121–122]. He found that out of 100 popular songs he surveyed, 70 were of Type 1 (a simple "AABA" refrain), 20 were of the more complex Type 2, and ten were irregular. Half of the songs in Woody Allen films are of Type 2.

Kaplansky wrote the Pi song to illustrate the superiority of Type 2: "the idea being that you could take such unpromising material as the first fourteen digits of pi and make a passable song out of it if you used Type 2." We reproduce his score here in Figures 4.1 to 4.5.

4.2 Ludolph van Ceulen's Tombstone

Ludolph van Ceulen (1540–1610) was the last to compute π seriously using Archimedes' method. He computed 39 digits with 35 correct in 1610 (published posthumously in 1615). He was sufficiently proud of this accomplishment that he had the number inscribed on his tombstone in Leiden. The tombstone vanished long ago although the number was and is still called Ludolph's number in parts of Europe. The tombstone was redesigned and rebuilt from surviving descriptions and sculpted by Cornelia Bakkum as reconsecrated July 5, 2000. (See also Ref. [8] for a mathematical paper written for the occasion.)

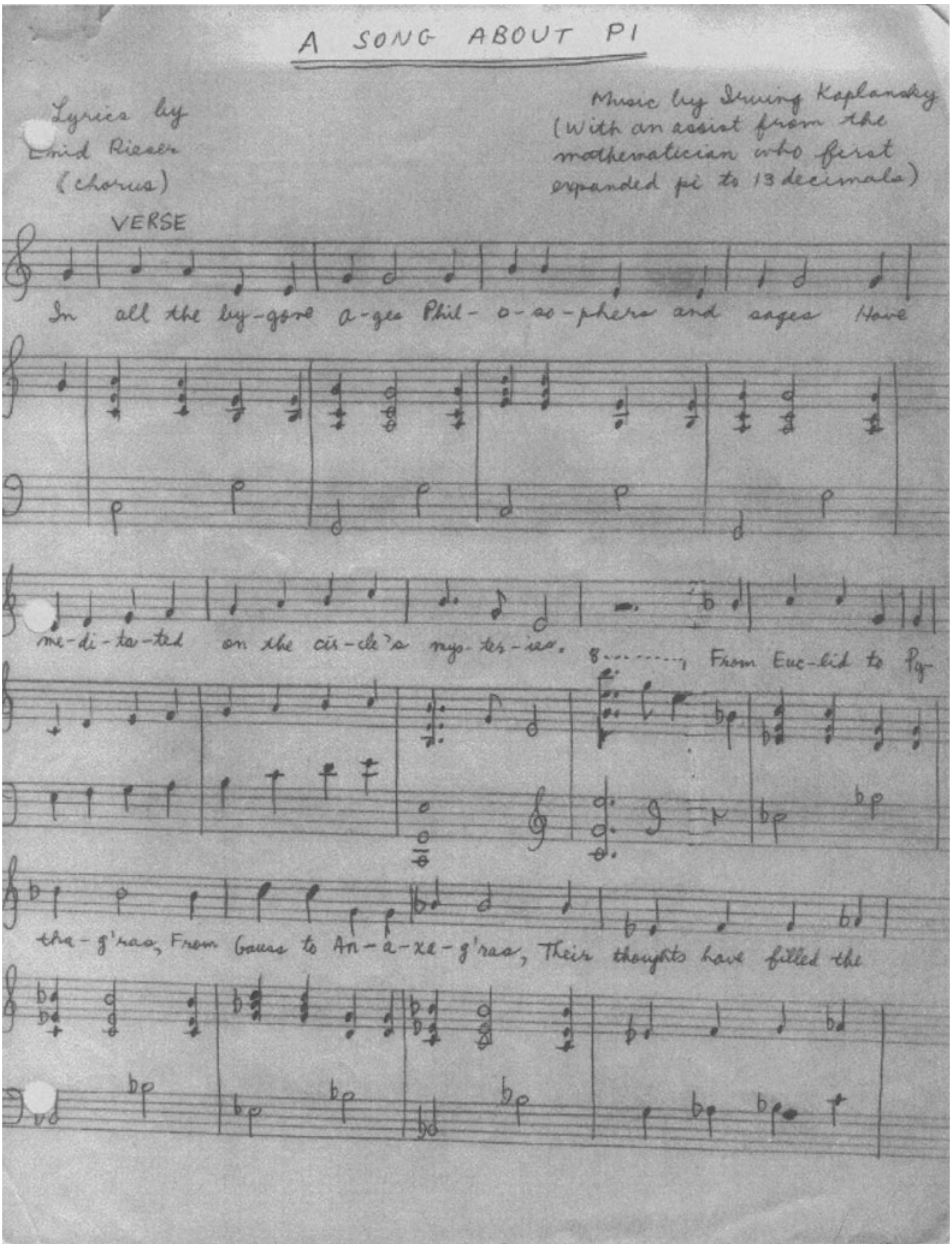


Figure 4.1: Kaplansky's "A Song about Pi," page 1

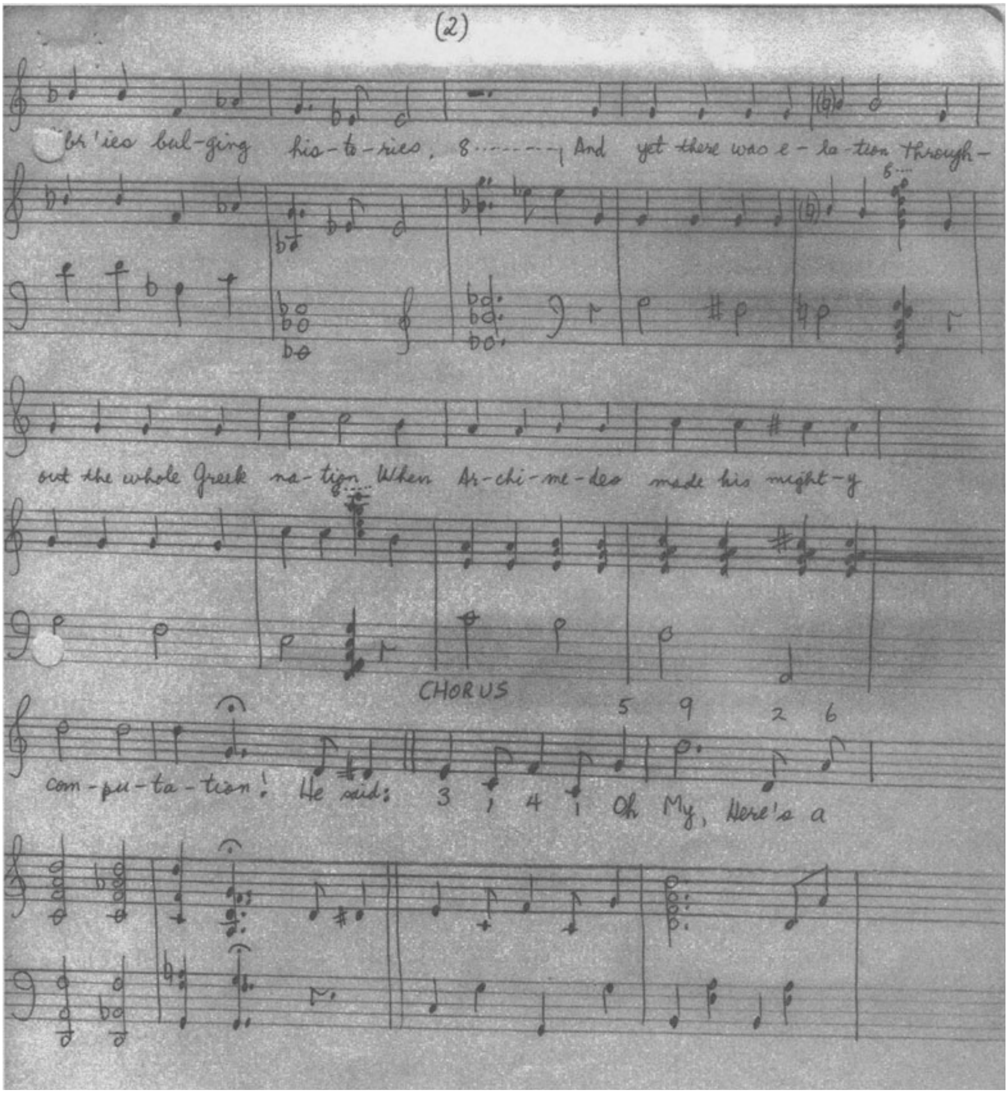


Figure 4.2: "A Song about Pi," page 2

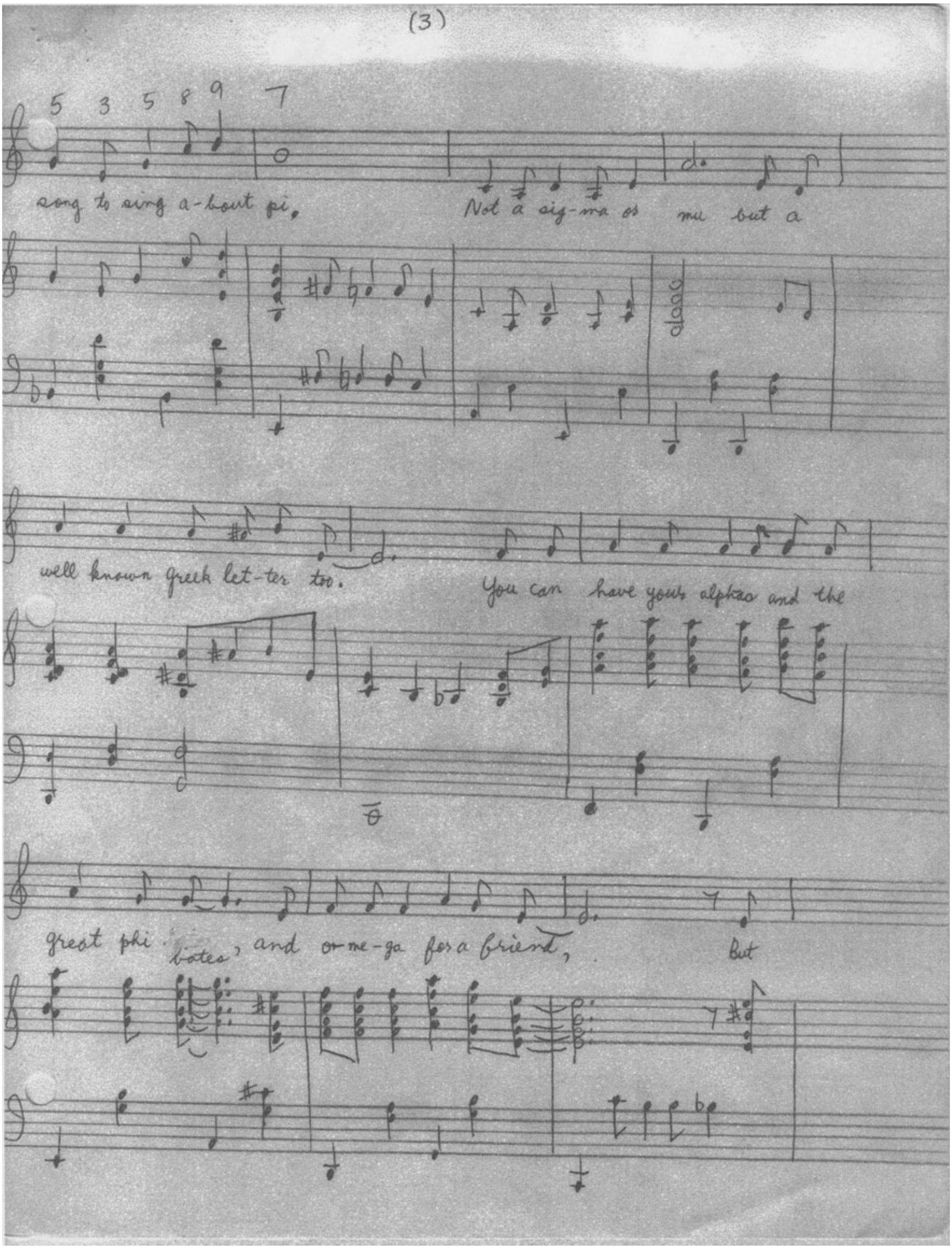


Figure 4.3: “A Song about Pi,” page 3

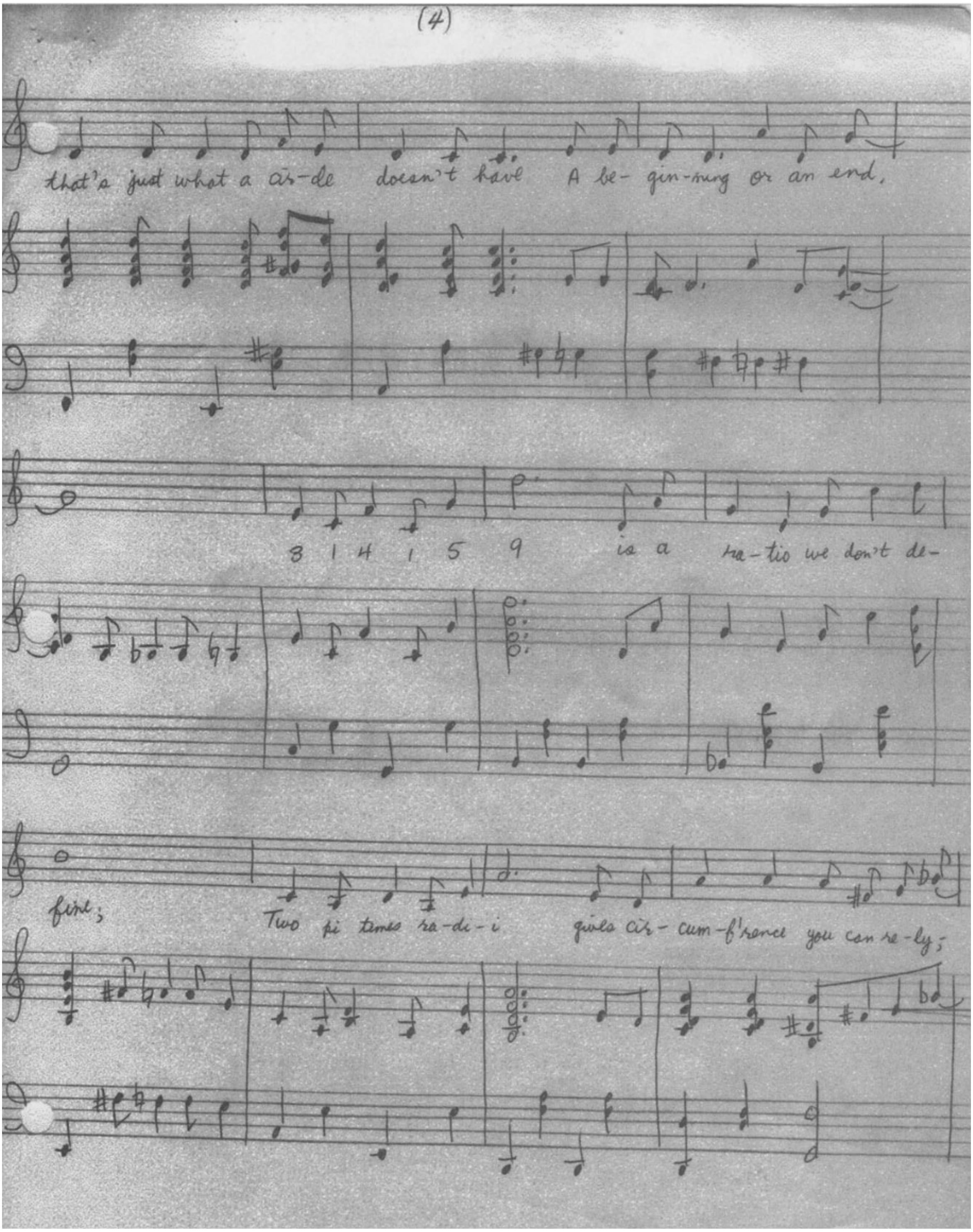


Figure 4.4: "A Song about Pi," page 4

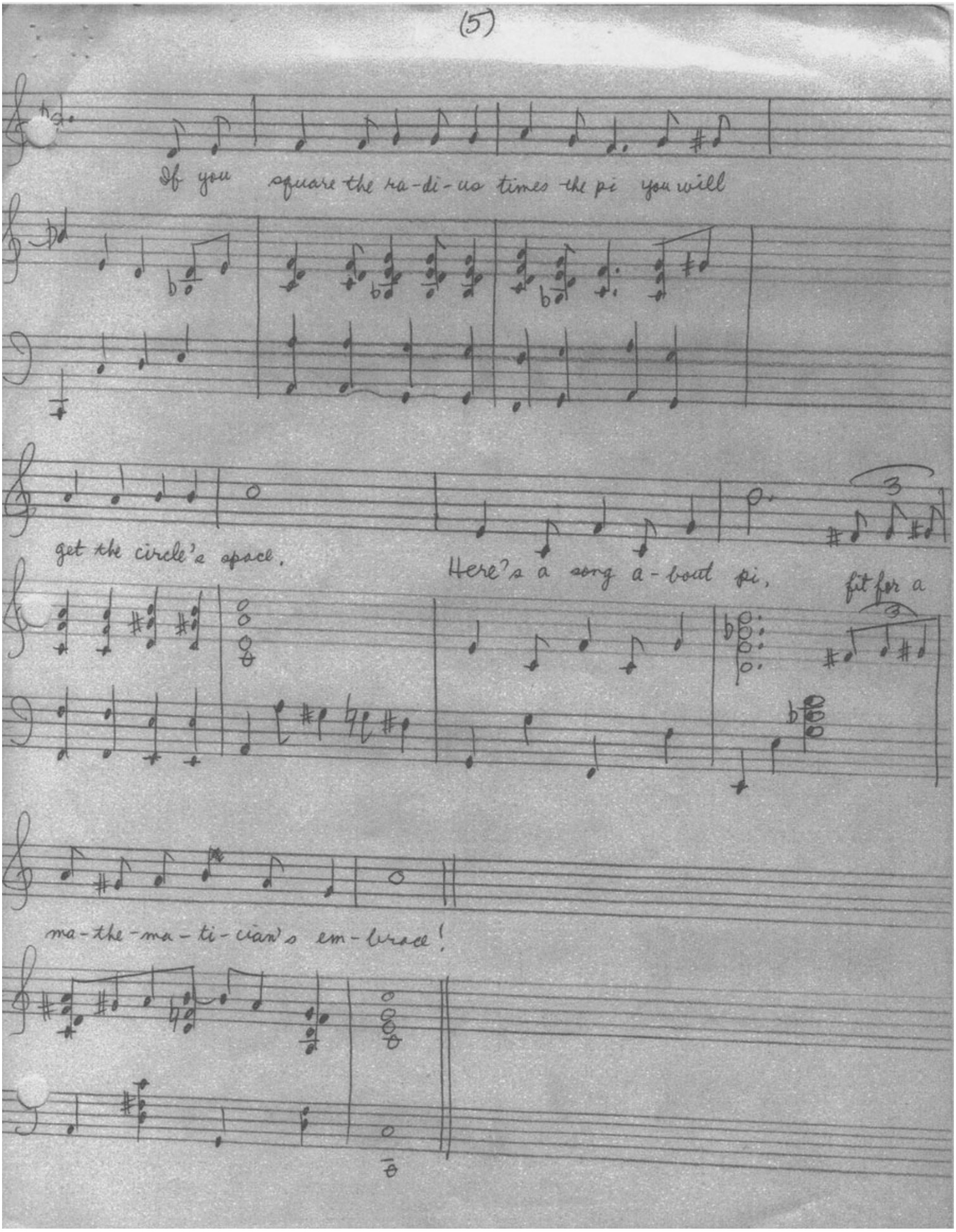


Figure 4.5: “A Song about Pi,” page 5

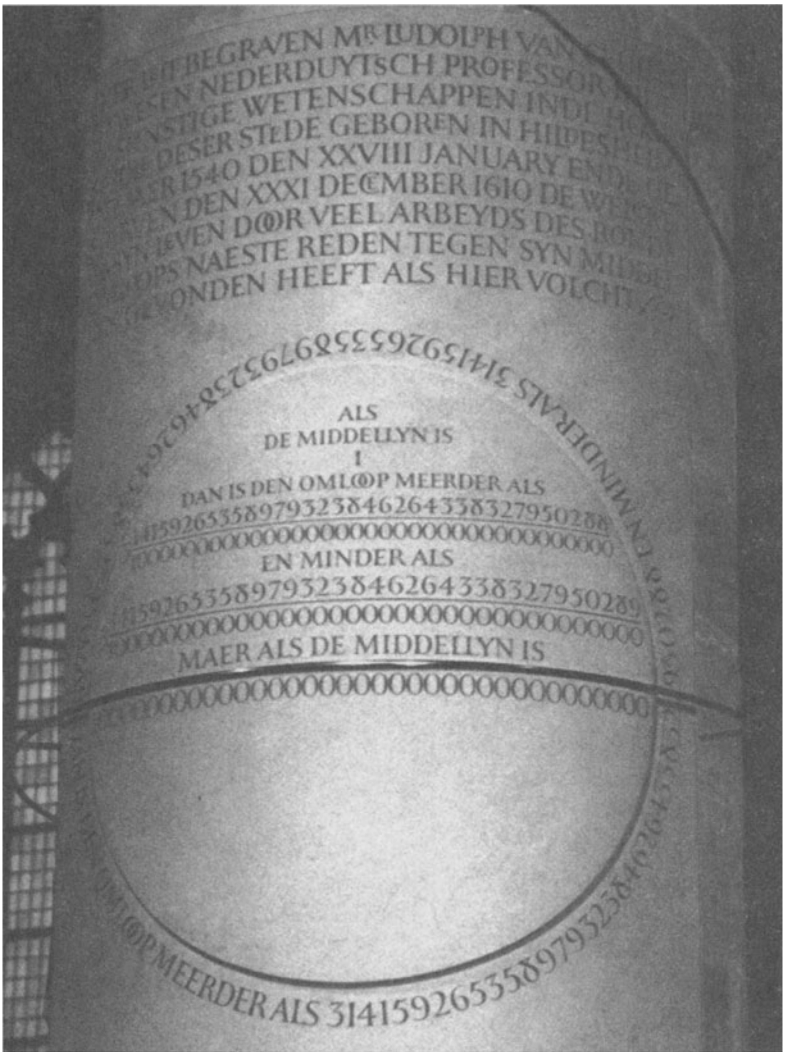


Figure 4.6: Ludolph's rebuilt tombstone.

References

[1] Donald J. Albers, Gerald L. Alexanderson, and Constance Reid (Editors). *More Mathematical People: Contemporary Conversations.* Academic Press, New York, 1997.

[2] Jorg Arndt and Christoph Haenel. *Pi Unleashed.* Springer-Verlag, New York, 2001.

[3] David H. Bailey, Peter B. Borwein, and Simon Plouffe. On the rapid computation of various polylogarithmic constants. *Mathematics of Computation*, 66:903–913, 1997.

[4] David H. Bailey and Richard E. Crandall. On the random character of fundamental constant expansions. *Experimental Mathematics*, 10:175–190, 2001.

[5] David H. Bailey and Richard E. Crandall. Random generators and normal numbers. *Experimental Mathematics*, 11:527–546, 2003.

[6] Petr Beckmann. *A History of Pi.* St. Martin's Press, New York, 1971.

[7] L. Berggren, J.M. Borwein, and P.B. Borwein. *Pi: a Source Book, (2nd ed.).* Springer-Verlag, New York, 2000.

[8] Frits Beukers. A rational approach to pi. *Nieuw Archief voor Wiskunde*, 5:372–379, 2000.

[9] Patrick Billingsley. *Probability and Measure.* John Wiley, New York, 1986.

[10] David Blatner. *The Joy of Pi.* Walker and Co., New York, 1997.

[11] Jonathan M. Borwein and David H. Bailey. *Mathematics by Experiment: Plausible Reasoning in the 21st Century.* 2003.

[12] Jonathan M. Borwein, David Borwein, and William F. Galway. Finding and excluding single digit Machin-type formulae. *Canadian Journal of Mathematics*, in press, 2004. http://www.cecm.sfu.ca/preprints/2002pp.html.

[13] Jonathan M. Borwein and Peter B. Borwein. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity.* CMS Series of Monographs and Advanced Books in Mathematics, John Wiley & Sons, New York, 1987.

[14] C.S. Calude, M.J. Dineen, and C.-K. Shu. Computing a glimpse of randomness. *Experimental Mathematics*, in press, 2004.

[15] Joel L. Kraemer (editor). *Perspectives on Maimonides: Philosophical and Historical Studies.* Oxford University Press, New York, 1991.

[16] Stephen Finch. *Mathematical Constants.* Cambridge University Press, Cambridge, 2003.

[17] Arend Heyting. *Intuitionism: An Introduction.* North-Holland, Amsterdam, 1956.

[18] Y. Kanada, Y. Ushiro, and H. Kuroda. The computation of π to 1,241,100,000,000 decimal digits. To appear, 2004.

[19] Donald E. Knuth. *The Art of Computer Programming*, volume 2. Addison-Wesley, New York, 1998.

[20] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences.* Wiley Interscience, New York, 1974.

[21] Brian M. Metzger and Roland E. Murphy. *The New Oxford Annotated Bible (New Revised Standard Edition).* Oxford University Press, New York, 1970.

[22] Roger Penrose. *The Emperor's New Mind.* American Philological Association, New York, 1989.

[23] K. Roth. Rational approximations to algebraic numbers. *Mathematika*, 2:1–20, 1955.

Index

B

J

K

O

P

Q

R

T