The Art of Inspired Guessing

by Simon Plouffe, August 7, 1998

Following the first findings that were inspired from Ramanujan Notebooks and other articles, I followed the same idea that the Zeta function (as well as the Psi and Gamma functions) have nice properties. With the help of MapleV, Pari-Gp and my Inverter ,I have found those (new?) identities.

$$\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}}=\frac{\pi\sqrt{3}}{18}(\Psi'(1/3)-\Psi'(2/3))-\frac{4}{3}\zeta(3)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^5\binom{2n}{n}}=\frac{\pi\sqrt{3}}{432}(\Psi'''(1/3)-\Psi'''(2/3))+\frac{19}{3}\zeta(5)+\frac{1}{9}\zeta(3)\pi^2$$

$$\sum_{n=1}^{\infty}\frac{1}{n^7\binom{2n}{n}}=\frac{11\pi\sqrt{3}}{311040}(\Psi'''''(1/3)-\Psi'''''(2/3))-\frac{493}{24}\zeta(7)+\frac{1}{3}\zeta(5)\pi^2+\frac{17}{1260}\zeta(3)\pi^4$$

Here, Psi(k,x) is the function is the k'th logarithmic derivative of the Gamma function at x. Some closed expressions are known for some values of the Psi(k,x) function but in general Psi(k,1/3) with k>0 is of unknown nature compared to known constants.

The first identity was not very difficult to find using my Inverter.
Following the same vein of pattern the second identity follows and then the third.
I could not establish the same for Zeta(9) even with 256 digits of precision.

The next step is to find a closed expresion for the alternate sums. We know from Apéry's 1979 article that it can be expressed in terms of Zeta(3). If we omit the constant factor for short, we have :

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n^3 \operatorname{binomial}(2n, n)} = [\zeta(3)]$$

But unfortunately, there are no apparent patterns for Zeta(5), Zeta(7), etc. despite my numerous efforts for those sums and previous tests of D. H. Bailey and others are equally inconclusive so far. The mystery remains until someone finds the clue to all this.

An indication for further discoveries in that direction might be given with these remarks:

There are many representations of the same thing, for example,

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n \operatorname{binomial}(2n, n)} = \left[\Psi\left(\frac{1}{5}\right), \Psi\left(\frac{2}{5}\right), \Psi\left(\frac{3}{5}\right), \Psi\left(\frac{4}{5}\right)\right]$$

The sum is a linear combination of Psi(p/q) and if we simplify we find that

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n \operatorname{binomial}(2n, n)} = [\ln(2)\sqrt{5}, \ln(3+\sqrt{5})\sqrt{5}]$$

That is, a simple linear combination of log(2)*sqrt(5) and log(3+sqrt(5))*sqrt(5). Which is in fact this number,

$$\frac{2}{5} \operatorname{arcsinh}\left(\frac{1}{2}\right)\sqrt{5}$$

This identity is well known and not difficult to find. So, the case n^5 should be simple and in terms of sqrt(5) and Psi(3,1/5) ?. This is strange since the other formulas were with Pi*sqrt(3) only. It could be also that the sqrt(5) factor is an artefact. Normaly the sum for n^5 should be expressible with Psi(4,k/5) , k=1..4 and/or Psi(4,1/3)*Pi*sqrt(3) but numerical evidence shows that it is NOT the case.

References

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