A formula for the n'th digit
of π and π^n

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January 20, 2022

Abstract

By using an asymptotic formula known for the numbers of Euler and Bernoulli it is possible to obtain an explicit expression of the nth digit of π in decimal or in binary, it also makes it possible to obtain the nth digit of π^n. The calculation is made from the two inequalities

$$\frac{2(2n)!}{(2\pi)^{2n}}\left(\frac{1}{1-2^{1-2n}}\right) > (-1)^{n+1}B_{2n} > \frac{2(2n)!}{(2\pi)^{2n}} , n = 1, 2, \dots$$

et

$$\frac{4^{n+1}(2n)!}{\pi^{2n+1}} > E_{2n} > \frac{4^{n+1}(2n)!}{\pi^{2n+1}}\left(\frac{1}{1+3^{-1-2n}}\right), n = 0, 1, \dots$$

By isolating π in both cases, we can derive an approximation of the latter. The approximation is so good that in fact it allows to extract the nth bit of π, the nth decimal of π and even the nth decimal of de π^n, n $\in$ $\mathbb{Z}^*$.

Bernoulli numbers and π

The 2 inequalities are taken from [3], p. 809. By isolating therefore, the number π in the first one obtains:

$$\pi \approx \left(\frac{2n!}{B_n 2^n}\right)^{1/n}$$

It is easy to verify that with n = 1000, the error on π is of the order of 0.293193 ×10^(-303), which is less than 2^(-1000). So the thousandth position of this expression is the 1000th bit of the number π. As soon as n = 10, the formula is valid: it allows to have the nth position. Note that given the parity of Bernoulli numbers, we obtain 2 bits at each n,

that of rand n and that of rank n-1. This is valid from 10 because Bernoulli numbers are a little chaotic when n is small, the rapid growth of these numbers is felt from n = 12.

The good question that comes next is: is it possible to calculate the nth decimal place? The answer is actually yes. Here's how.

Given the known relationship with the Zeta function, it is possible to stretch the formula by adding enough terms to it to have an error of less than 10^(-n). In the book of Abramowitz and Stegun [3] precisely in the same chapter we have the formula

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|$$

As we know, $\zeta(2n)$ can be represented as an infinite product due to Euler, by carefully taking the first 4 terms we therefore add them to our expression to have:

$$\pi \approx \left(\frac{2n!}{B_n 2^n \left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)}\right)^{1/n}$$

Knowing that the next term would be (1-1/11^n) this assures us that the error committed will be less than 10^(-n). Indeed, by checking with n = 1000 (and even 10000) we find an error of the order of $0.1271934403 \times 10^{(-1043)}$. We conclude that for each n in the expression it seems possible to obtain the nth decimal of the number π. The difficulty is then to be able to calculate B_2n when n is large. The current record for Bernoulli numbers is 100 million. So, we can calculate the 100,000,000th decimal of π with this process. Note that the calculation of Bernoulli numbers can be done in several ways, one of which requires knowing π with good precision. Until 2002, the Bernoulli calculation record used precisely this formula and some values of the series for Zeta(n), see [5] on this subject. Currently, the calculation uses properties of congruences and the Chinese remainder theorem.

The formula is precise enough to get even more if we raise to the power n. In fact we will have

$$\pi^n \approx \left(\frac{2n!}{B_n 2^n \left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)}\right)$$

By checking with n = 1000 we obtain the first 1000 valid digits of π^1000, so the 1000th position if we want. To get the 1000th position of x, just calculate

$$position(n, x) = [10 \{10^n x \}]$$

[ ] and { } being the integer and fractional parts. If calculating π^n, the formula must be converted to treat the number as a string.

Euler numbers and π

We take the expression of the book of A & S page 809, we isolate π and we obtain:

$$\pi \approx \left(\frac{(2n)! \, 2^{2n+2}}{E_{2n}} \right)^{1/(2n+1)}$$

Here the E_{2n} are the positive Euler numbers obtained by the series expansion of $1/\cos(x)$. It is easy to verify that with $n = 1000$ we obtain a precision of 957 decimal digits. But knowing the error made and the first formula, we can deduce that this error is proportional to 9^{-1000}. So, we can say that in base 8 the expression gives the nth digit of π in octal, a fortiori in binary.

To go further as with the Bernoulli numbers, it is enough to add only 1 term to have an error lower than 10^{-n}. Which gives us

$$\pi \approx \left(\frac{(2n)! \, 2^{2n+2}}{E_{2n}} \left(1 - \frac{1}{3^{2n+1}} \right) \right)^{1/(2n+1)}$$

And if we check with $n \gg 1$, for $n = 1000$ we have here an error on π of the order of 10^{-1198}, so the nth decimal of π can be calculated with the formula using the position function (n,x).

To get the nth digit in base 10 of π^n using Euler numbers, we add terms until the error is smaller than 10^{-n}. We have to take more terms here since the powers of π take us away from the decimal point. So we finally have the following formula

$$\pi^{2n+1} \approx \frac{(2n)! \, 2^{2n+2}}{E_{2n}} \left(1 - \frac{1}{3^{2n+1}} \right) \left(1 + \frac{1}{5^{2n+1}} \right) \left(1 - \frac{1}{7^{2n+1}} \right) \left(1 - \frac{1}{9^{2n+1}} \right)$$

For $1/\pi$, in base 10 still with Euler, we can simply invert

$$\frac{1}{\pi} = \left(\frac{E_{2n}}{(2n)! \, 2^{2n+2}} \left(1 - \frac{1}{3^{2n+1}} \right) \right)^{1/(2n+1)}$$

It therefore makes it possible to obtain the nth decimal of $1/\pi$ in base 10. The scope of this formula is limited by the ability to evaluate E_{2n}. With an Intel icore 9900K clocked at 5 ghz, I managed without too much difficulty to calculate $E_{(2,300,000)}$. So position 2,300,000 can be calculated by this process.

In the end, we can therefore calculate the nth decimal digit of π and π^n.

Real calculation of all the digits of π

The process would go further by calculating all the figures. Indeed, by pushing the formula with many more terms, we quite easily reach a few million digits, but the speed of convergence is not geometric, it is polynomial even if the degree is very high. If we take the largest known value of B_{2n} (100 million), we will therefore have the series (or product if we take the Euler product) as follows .

$$\left(\frac{2n!}{B_n 2^n \left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\ldots}\right)^{1/n}$$

So, with n = 100 million, and even if we take 1000 billion terms, we will have at best 1.2 billion digits of precision (1000 billion = 10^12) so 12 digits only. This is a lot but little compared to the 62,000 billion digits obtained recently with the Chudnovsky formula, the latter has a convergence speed which is geometric, of the order of 14 decimal digits per term.

References

[1] Wikipedia, Dirichlet β function.
https://en.wikipedia.org/wiki/Dirichlet_beta_function

[2] Eric Weisstein, *Dirichlet beta function* :
https://mathworld.wolfram.com/DirichletBetaFunction.html

[3] *Abramowitz, Milton Stegun, Irene Ann, eds. (1983) [June 1964]. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Applied Mathematics Series.* **55** (Ninth reprint with additional corrections of tenth original printing with corrections (December 1972); first ed.). Washington D.C.; New York: United States Department of Commerce, National Bureau of Standards; Dover Publications. ISBN 978-0-486-61272-0*.* LCCN 64-60036*.* MR 0167642*.* LCCN 65-12253*.*

[4] Plouffe, Simon, *On the computation of the n'th decimal digit of various transcendental numbers, 1996.* https://arxiv.org/abs/0912.0303.

[5] Plouffe, Simon, *An efficient algorithm for the computation of the Bernoulli Numbers.* Arxiv *:* https://arxiv.org/abs/math/0702300

[6] Plouffe, Simon, *The Construction of Certain Numbers with Ruler and Compass, le Calcul de Certains Nombres Avec la Règle et le Compas.* Vixra :
https://vixra.org/abs/1409.0135