

The calculation of p_n and $\pi(n)$

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Abstract

A new approach is presented for the calculation of p_n and $\pi(n)$ which uses the Lambert W function. An approximation is first found and using a calculation technique it makes it possible to have an estimate of these two quantities more precise than those known from Cipolla and Riemann. The calculation of p_n uses an approximation using the Lambert W function and an estimate based on a logarithmic least square curve (LLS) $c(n)$. The formula is:

$$p_n + \pi(n) \approx -nW_{-1}\left(\frac{-e}{n}\right)c(n) \quad 1$$

The results presented are empirical and apply up to $n \approx 1.358 \times 10^{16}$.

Introduction

Today we know very large prime numbers like $2^{82589933} - 1$ but the rank of these numbers is unknown. The primes with their rank is known up to 10^{27} for $\pi(n)$ and 10^{24} for p_n at some specific points and the complete list of primes with their rank is known only up to 10^{17} .

In 2010, Dusart [16] proved that $\pi(n) \approx \frac{n}{\ln(n)-1}$ if $n > 5393$. We will use this approximation to give one approximation of p_n by inverting the formula.

$$\text{If } \pi(n) \approx \frac{n}{\ln(n)-1} \text{ then } p_n \approx -nW_{-1}\left(\frac{-e}{n}\right).$$

$W_{-1}(n)$ is the Lambert W function of order -1, W_0 or $W(n)$ is of order 0. We take the branch of the W function which gives meaning to quantity $\frac{-e}{n}$.

That formula is quite precise, for $n = 10^{24}$, we have p_n precise to 99.97 %.

By analyzing the rest of p_n and $-nW_{-1}\left(\frac{-e}{n}\right)$, we rapidly find that it is close to $\frac{n}{W(n)}$, so

$p_n \approx -nW_{-1}\left(\frac{-e}{n}\right) - \frac{n}{W(n)}$. But also that the quantities

$$\frac{n}{\ln(n)-1} \approx Li(n) \approx \pi(n) \approx \frac{n}{W(n)} \approx \frac{\rho_n}{2\pi}$$

are close too when $n \rightarrow \infty$. Here, ρ_n is the imaginary part of the n 'th non-trivial zero of Riemann's ζ and $Li(n)$ is the integral logarithm, the values are : .1843e23, .18434e23, .1844e23, .1948e23 et .1986e23 respectively.

Empirical data suggest that the choice of $\pi(n)$ is the best.

If $n = 10^{24}$ then

$$p_{10^{24}} + \pi(10^{24}) \approx -10^{24}W_{-1}\left(\frac{-e}{10^{24}}\right) \quad (2)$$

The approximation is valid up to 0.99999401 or 99.999401 % of the real value.

A naive question is then, couldn't we be more precise? Generally speaking, is there a way to have an exact formula? If we can be more precise then how much? What about the rest? What is his nature exactly?

An explicit formula: Prime numbers in geometric progression

Right now, the question of whether an exact formula exists for prime numbers is yes and no both. There are many formulas that give them but they are either impractical or limited see the table in appendix.

For example, here is an explicit formula which gives some primes in geometric progression.

$$f(n) = \{c^n\} \quad (3)$$

Here $\{ \}$ is the nearest integer function. The smallest example of c is : if $c = 2.553854696\dots$ then $f(n)$ is 3, 7, 17, 43, 109, 277, 709. The sequence contains only 7 terms. But can we go further? Using the simulated annealing method and Monte Carlo we can go much further.

Constant c such that $\{c^n\}$ is prime	Values	Number of primes generated
2.553854696...	3, 7, 17, 43, 109, 277, 709	7
593.46526943871...	n=2..48	47
2027.1671684764912194343956	n=1..97	97
29983.279826631136...	n=1..422	422
55237.07504296764715433124781528617...	n=2..633	632

From there we can conjecture that if c is large enough, the sequence with formula (3) can generate an arbitrary number of primes explicitly, see [23].

All we have for the expression of p_n or $\pi(n)$ for large values of n are just approximations. The best known approximation is that of Riemann, the function is called Riemann-R or that of Gram found in 1884. The only way that has been found is to use partially Gram's formula followed by a sophisticated Erathosthenes screen. This is why that the value of p_n or $\pi(n)$ is only known up to 10^{24} and 10^{27} respectively. The last calculation of $\pi(10^{27})$ took several months and a significant amount of computer time (23 CPU years) in 2018.

First approximation

We will confine ourselves for the moment to the calculation of p_n since $\pi(n)$ can be calculated by inversion.

The classic formula for p_n is $p_n \approx n \ln(n)$ or better yet the one found by Cipolla in 1902 states that

$$p_n \approx n \left(\ln(n) + \ln(\ln(n)) - 1 + \frac{\ln(\ln(n)) - 2}{\ln(n)} - \frac{\ln(\ln(n))^2 - 6 \ln(\ln(n)) + 11}{2 \ln(n)^2} + \dots \right). \quad (4)$$

The calculation was taken further in 1994 by B. Salvy who extracted a procedure for pushing it further the approximation.

What is remarkable is the similarity with the asymptotic development of $W(n)$.

$$W(n) \approx L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2 + L_2)}{2L_1^2} + \frac{L_2(6 - 9L_2 + 2L_2^2)}{6L_1^3} + \frac{L_2(-12 + 36L_2 - 22L_2^2 + 36L_2^3)}{12L_1^4} + \dots$$

$$L_1 = \ln(n) \text{ et } L_2 = \ln(\ln(n)).$$

The calculation of p_n with this formula (4) gives 12 exact decimal places out of the 26 that count $p_{10^{24}}$. But does not allow to go further, even with 64 terms in the asymptotic development. The formula of $Gram^{<-1>}$ is much more precise. $Gram^{<-1>}$ is the functional inverse of the Gram series [24].

A better approximation

A summary analysis indicates that the rest after the first term of (1) is of type logarithmic : $a + b \ln(n)$ where a and b to be determined. An idea is then to calculate the logarithmic least squares curve (LLS) passing through a chosen number of points on a table of values of primes.

We can also notice that by taking only one term for the approximation of $-nW_{-1}(\frac{-e}{n})$, this form is equivalent to several terms of Cipolla's development. If we take the 2 terms it will be even more precise. In other words, given the nature of the asymptotic development of $W(n)$, each term is equivalent to several terms in the development of Cipolla. We assume here that the remainder after the 2 terms is a logarithmic curve and that once calculated it will stick to reality.

The question then arises of what is the nature of what remains? In fact, we do not know. The best known for $\pi(n)$ which is theoretically valid is $Li(n)$. Riemann proposed a 2nd formula that is much better at first but was invalidated by Littlewood in 1914. This 2nd formula, called Riemann R or equivalent, the Gram series is

$$\pi(n) = \sum_{k=1}^{\infty} \frac{\ln(x)^k}{k k! \zeta(k+1)}$$

Numerically, the approximation of $\pi(n)$ by Riemann's R or the Gram series is excellent in addition to converging quickly. But Littlewood showed that after 10^9 , the approximation is less precise. As for the function $Li(n)$, it behaves better at much larger scales, the first crossover of $Li(n)$ and $\pi(n)$ was evaluated at 10^{316} approximately, that is $Li(n) - \pi(n) = 0$ in the neighbourhood of 1.397×10^{316} .

The best approximation that has been found empirically is:

$$p_n = -nW_{-1}\left(\frac{-e}{n}\right)c(n) - \pi(n)$$

where $c(n)$ is of type $a + b \log(n)$.

Here is a table of values that were considered for the calculation of p_n from 10^2 to 10^{17}

Value of the step	Number of values	Range
10^2	27117419	2711741900
10^3	32082085	32082085000
10^4	45020269	450202690000
10^5	16038989	1.603×10^{12}
10^6	4046531	4.046531×10^{12}
10^7	5011691	5.011691×10^{13}
10^8	454060	4.54060×10^{13}
10^9	2200000	2.2×10^{15}
10^{10}	1358121	1.358121×10^{16}
10^{11}	135812	1.358121×10^{16}
10^{12}	54974	5.4974×10^{16}

10^{13}	12317	1.2317×10^{17}
10^{14}	2162	2.162×10^{17}

We can calculate the points on the LLS curve by solving the following equation for x , taking n in one of the tables above.

$$-nW_{-1}\left(\frac{-e}{n}\right)x - \pi(n) + p_n = 0$$

by using the bisection method. The values are between 0.9 and 1. We then calculate the logarithmic least squares curve. The coefficient r^2 will indicate if the curve is right. The calculation of coefficients a and b is done according to the formula:

$$b = \frac{n \sum_{i=1}^n (y_i \ln x_i) - \sum_{i=1}^n y_i \sum_{i=1}^n \ln x_i}{n \sum_{i=1}^n (\ln x_i)^2 - (\sum_{i=1}^n \ln x_i)^2}$$

$$a = \frac{\sum_{i=1}^n y_i - b \sum_{i=1}^n (\ln x_i)}{n}$$

Recall, the coefficient r^2 indicates whether the experimental data stick to the curve. If is near 1 or 0, the curve follows a straight line very closely. The logarithmic least-squares curve is simply the log of values that are aligned on a line.

The curve thus found for $n \leq 1.358121 \times 10^{16}$ is

$$\begin{aligned} c(n) = & 1.0000314775792421150615325693061 \\ & - 0.00000051483940138413674623044640769440 \ln(n) \end{aligned}$$

Once you have found this LLS curve, you can significantly increase the accuracy if you use a trick. We take the chosen interval, $n \cdot 10^{10}$, $n = 1..1358121$. We then separate into 1358 slices of 10,000 values and for each value we modify the curve $c(n)$ with the formula

$$c(n) \rightarrow c(n) s^k$$

And $s = (1 - 10^{-10})$, then it suffices to find for which k , the curve admits a minimum error if we compare to the true values of p_n . An experiment was conducted with 1358 intervals to see if the mean value of the deviation decreased: this is the case. The reasonable limit found here is from 1,358 slices of 10,000 values.

In the appendix you can consult the program written in Maple language which performs the operation.

Calculation of $\pi(n)$

To calculate $\pi(n)$, it suffices to isolate $\pi(n)$ in the equation of $G(n)$, normally trivial but in practical it does not work. Indeed $\pi(n)$ is smaller than p_n in size. The formula remains valid except

that the coefficients change slightly. The calculation of can be done with formula 1, with the primecount program which gives it directly. We can also have the values of Gram(n) and Gram inverted. If instead of $\pi(n)$ in formula (1) we rather take the term $\frac{n}{W(n)}$, it's a little less precise but still allows good accuracy.

For example, for the interval from 1 to 1000000 the following program calculates p_n very precisely.

```
G:=proc(n) # calculation of p(n) up to 1 million (first million of primes).
local ll,lk,s,s2,ss,kk;
ll:=[82, 16, -14, 4, -31, -10, -31, -32, -1, -44, -17, -38, -8, 7,
-35, -41, -38, -3, 6, -14, -27, -12, -5, -51, -40, 17, -7, -17, -16,
-14, 13, -7, -5, -1, -26, -29, -27, -31, -9, 8, 16, 4, 9, 0, 20, 11,
7, -15, -23, -17, -10, -2, 2, -5, 8, 7, 9, 3, -12, -11, 4, 5, -3, 9,
-1, 7, 24, 25, 33, 20, 15, 11, 9, 3, 9, 15, 3, 3, 1, 4, 5, -9, -1,
-12, -4, 14, 16, 17, 28, 18, 12, 21, 24, 10, 14, 16, 15, 24, 26, 36, 30];
lk := floor(1/10000*n) + 1;
s := .3238679016803340+.4042167153029803e-1*ln(kk);
s2 := subs(kk = n, s);
ss := s2*0.999^ll[lk];
round(abs(-n*W(-1,-exp(1.0)/n)) - ss*fab(n/W(n)));
end;
```

By inverting $\pi(n)$ to find , over the interval 1 to 1350000×10^{16} . We have the following program. It has the advantage of not having to calculate or having a table of the values of the first, but it is less precise.

```
f:=proc(k) local n,a,bas,haut,ll,lk,s,ss; # calcul de pi(n) jusqu'a 10^16
haut:=evalf(2*k/log(k));
bas:=haut/8;

ll:=[3122, 3186, 3222, 3243, 3251, 3270, 3272, 3282, 3289, 3291, 3289, 3297, 3312,
3305, 3300, 3313, 3308, 3311, 3321, 3318, 3323, 3322, 3326, 3322, 3319, 3322,
3328, 3328, 3327, 3335, 3334, 3336, 3336, 3337, 3335, 3335, 3331, 3334, 3337,
3334, 3341, 3343, 3345, 3346, 3343, 3340, 3341, 3342, 3348, 3348, 3347, 3349,
3349, 3345, 3349, 3350, 3348, 3352, 3351, 3346, 3344, 3346, 3344, 3348, 3348,
3349, 3354, 3355, 3357, 3355, 3354, 3355, 3358, 3359, 3358, 3358, 3358, 3357,
3357, 3358, 3356, 3355, 3357, 3358, 3359, 3358, 3355, 3355, 3360, 3357, 3354,
3358, 3358, 3362, 3361, 3360, 3360, 3361, 3360, 3362, 3361, 3362, 3363,
3363, 3362, 3364, 3363, 3363, 3361, 3360, 3361, 3364, 3366, 3366, 3366, 3366,
3368, 3368, 3369, 3369, 3369, 3368, 3366, 3367, 3368, 3367, 3368, 3368, 3368, 3368, 3366,
3366, 3365, 3365, 3364, 3365]:;

lk:=floor(k/1000000000000000)+1:

s:=882819461483173314372633+.855943969749036445417381e-3*ln(n):
ss:=s*(0.999999)^ll[lk]:;

a:=abs(evalf(-n*W(-1,-exp(1.0)/n)))-evalf(ss)*fab(n/W(n))-k;
fsolve(a,n,n=bas..haut,fulldigits);
end;
```

Appencix (Tables and Programs)

Calculation of p_n

Compison with the range $10000000000 (10^{10}) \dots 1.352 \times 10^{16}$

Formula for p_n	Formula of Gram inverted	$G(n)$	Formula of Cipolla-Salvy
Minimal gap	57	13	640495
Maximal gap	117539110	412614395	1103 millions
Average gap	79.23 millions	18.81 millions	510 millions

Conclusion : The formula with LambertW function is 4.21 times more precise than that of Gram series (inverted).

Maple program for the calculation of p_n , $n \leq 1.356 \times 10^{16}$

```
#####
Digits:=32:
G:=proc(n, piofn)
local cn, ll, s, ss, lk, s2;
ll:=[800, 369, 97, -19, -69, -133, -151, -138, -195, -200, -161, -182, -210, -177, -212,
-190, -195, -138, -169, -199, -205, -199, -177, -175, -157, -150, -165, -156, -163,
-162, -136, -158, -169, -175, -152, -150, -121, -134, -143, -134, -120, -125, -107,
-115, -130, -90, -108, -125, -132, -134, -135, -134, -125, -119, -103, -88, -62, -88,
-85, -89, -93, -89, -80, -77, -77, -74, -83, -70, -91, -89, -82, -75, -80, -78, -83,
-82, -68, -61, -56, -47, -50, -63, -65, -74, -77, -72, -63, -67, -69, -72, -64, -41,
-31, -29, -28, -28, -37, -42, -36, -40, -27, -26, -16, -22, -31, -37, -41, -38, -47,
-40, -39, -39, -41, -42, -38, -37, -33, -39, -42, -31, -32, -39, -30, -25, -22, -11,
-18, -18, -22, -18, -25, -30, -25, -24, -18, -15, -9, -6, -6, -8, -5, -1, 1, -14, -8,
-9, -4, 0, -4, 4, 5, 4, 3, -2, -4, -10, 0, 1, 0, 4, -3, -8, -11, -13, -3, -1, 4, 1, -5,
0, 5, 6, 3, -2, -2, 3, 4, -2, 2, 2, 2, -2, -1, -4, 3, 1, -1, 3, 5, 16, 14, 12, 18,
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-36, -36, -36, -37, -36, -36, -35, -35, -34, -35, -36, -36, -36, -34, -35,
-33, -35, -35, -34, -34, -35, -35, -35, -35, -35, -35, -35, -34, -34, -35,
-37, -38, -38, -38, -37, -37, -38, -38, -38, -38, -39, -40, -39, -36, -38,
-37, -39, -39, -39, -38, -39];
lk := floor(1/1000000000000000*n) + 1;
s := 1.0000314775792421150615325693061
- 0.51483940138413674623044640769440*(1/1000000)*ln(kk);
s2 := subs(kk = n, s);
ss := s2*0.9999999999^ll[lk];
abs(-n*W(-1, -exp(1.0)/n))*ss - piofn
end :
#####

```

Tables of primes :

http://plouffe.fr/NEW/list_primes_pi_of_n_100000000000.txt

http://plouffe.fr/NEW/list_primes_10000000000.txt

Example :

$$g(1327460000000000, 39285023244530) = 49668015014179465.522289485977202$$

$$\text{The real value of } p_{1327460000000000} = 49668015014179453$$

Maple program for the calculation of p_n , $n = 1.356 \times 10^{16} \dots 10^{24}$

```

#####
F:=proc(n, piofn)
local cn, z, pola, polb;
pola:=-.1803178829775386802559072260225588343254e-12*x^8-\\
3206852936427839673078154416271278702381e-10*x^7+\\
2521168696363102117361245200766645862014e-8*x^6-\\
1148245660104216214093938301036666192296e-6*x^5+\\
3329760033724798728321163791428487967963e-5*x^4-\\
6343395542494949120689514623217102223176e-4*x^3+\\
7851801857533638277814251770195187519581e-3*x^2-\\
5912431390595954878523745751806703967872e-2*x+\\
2187329700777127284427407021768653056673e-1:
polb:=-.8949926057969637729777538473173261408730e-12*x^8-\\
1611795950806416304161491053953385128968e-9*x^7+\\
1287542319981049792998526211011490785070e-7*x^6-\\
5988056104228871471776180438025688273194e-6*x^5+\\
1786915791025107343702497983252030617773e-4*x^4-\\
3548565854556946509095877029495597659212e-3*x^3+\\
4690427023808579602996344427037051784331e-2*x^2-\\
3980636249963806490254767914782205838276e-1*x+.196997473781\
2788247127674632264178712585:
z := evalf(log10(n));
cn := subs(x = z, pola) + subs(x = z, polb)*ln(n);
evalf(-cn*n*W(-1, -exp(1.0)/n)) - piofn
end:
#####

```

Table of values of $F(n)$ versus p_n

n	$F(n)$	p_n
10^{16}	394906913903735328.99999995710593	394906913903735329
10^{17}	4185296581467695668.9998280338750	4185296581467695669
10^{18}	44211790234832169331.000076399063	44211790234832169331
10^{19}	465675465116607065549.00000499731	465675465116607065549
10^{20}	4892055594575155744537.0000098572	4892055594575155744537

10^{21}	51271091498016403471852.999978699	51271091498016403471853
10^{22}	536193870744162118627429.00001989	536193870744162118627429
10^{23}	5596564467986980643073682.9999696	5596564467986980643073683
10^{24}	58310039994836584070534263.000118	58310039994836584070534263

Table of values of n , $\pi(n)$, p_n , G(n)

$n \cdot 10^{15}$	$\pi(n)$	p_n	G(n)	Gram Inverted	G(n) $\bar{X} = 2.37605e+07$	Gap Gram inverted $\bar{X} = 7.62569e+07$
1	3204941750802	3475385758524527	3475385752465280	3475385760290722	6059247	1766195
2	6270424651315	7093600525704677	7093600531547406	7093600514882155	5842729	10822522
3	9287441600280	10765662794071351	10765662776140778	10765662798101237	17930573	4029886
4	12273824155491	1447268063464931	14472680642211900	14472680659410410	7564969	24763479
5	15237833654620	18205684894350047	18205684890027179	18205684845589213	4322868	48760834
6	18184255291570	21959393830706447	21959393831829265	21959393831263666	1122818	557219
7	21116208911023	25730318403586483	25730318401089988	25730318388727176	2496495	14859307
8	24035890368161	29515978892552597	29515978901069447	29515978942077307	8516850	49524710
9	26944926466221	3331452177679133363	33314521740381476	33314521740381476	1459280	37292607
10	29844570422669	37124508045065437	37124507999149021	37124508056355511	45916416	11290074
11	32735816605908	40944788655376237	4094478864190013	40944788642442631	8813776	12933606
12	35619471693548	44774424266565143	44774424274288359	44774424246530936	7723216	20034207
13	38496205973965	48612632821248317	48612632846598877	48612632816905836	25350560	4342481
14	41366582391891	52458753029241283	52458753010788072	52458753051854838	18453211	22613555
15	44231080178273	56312218341118283	56312218348943058	56312218418328247	7824775	77209964
16	47090114439072	60172538090123567	60172538133649809	60172538130369336	43526242	40245769
17	49944047787207	64039282905020807	64039282881733481	64039282901777277	23287326	3243530
18	52793190012734	67912074089826233	67912074089530037	67912074062806200	296196	27020033
19	55637829945151	71790575058422851	71790575056044677	71790575102903791	2378174	44480940
20	58478215681891	75674484987354031	75674484995999520	75674484998897799	8645489	11543768
21	61314571044765	79563532882638499	79563532853067889	79563532880910748	29570610	1727751
22	64147099298639	83457473636497967	83457473633825154	83457473716619734	2672813	80121767
23	66975984145551	87356084739486881	87356084732880212	87356084781839841	6606669	42352960
24	69801392791572	91259162764140311	91259162732170585	91259162746099932	31969726	18040379
25	72623478149504	95166521200910351	95166521209928786	95166521244817945	9018435	43907594
26	75442380316713	9907798824381269	9907798830946491	99077988304566937	6565222	16185668
27	7825288028329	102993407452131551	102993407407852376	102993407298472241	44279175	153659310
28	81071142895913	106912630241974061	106912630255488703	106912630117470118	13514642	124503943
29	83881233426790	110835521231391421	110835521272734267	110835521271667139	41342846	40275718
30	86688602810119	114761954175793079	114761954219431338	114761954125523221	43638259	50269858
31	89493347331727	118691810606522897	118691810578220391	118691810493848076	28302506	112674821
32	92295565638011	12262497771448267	12262497772569434	122624979823224005	1148167	51775738
33	95095312182517	126561358313013181	126561358306791545	126561358475859144	6221636	162845963
34	97892695611204	130500849055080079	130500849011631709	130500849100333580	43448370	45253501
35	100687778906831	134443359967104443	134443359977968264	134443360076446089	10863821	109341646
36	103480631416721	138388805003019359	138388804996907016	138388805023564749	6112343	20545390
37	106271318433884	142337102440574897	142337102426378912	142337102363652797	14195985	76922100
38	109059901535155	146288175048719531	146288175029401561	146288174931574262	19317970	117145269
39	111846440164164	150241949628632893	1502419496391768753	15024194962452786	10535860	2180107
40	11463098904000	15419835711745083	154198357051757170	154198357098815921	59987913	12929162
41	1174135993647899	158157331435725499	158157331422444844	158157331469048028	13280655	33322529
42	120194323133703	162118810111632083	162118810106649332	16211881007330690	4982751	38301393
43	122973207007771	166082733219061063	166082733200372932	166082733233795825	18688131	14734762
44	12575026138286	17004904411273447	170049044132870533	170049044050074029	21597086	61199418
45	128525633848847	17401768827069971	174017688308542721	174017688209805173	37842750	60894798
46	131299259981906	177988613752767869	177988613735417195	177988613816002938	17350674	63235069
47	134071214963486	181961771195870689	181961771178770425	181961771229440096	17100264	33569407
48	136841535130789	185937113010539323	185937113016787902	185937112924455641	6248579	86083682
49	139610257999130	18991459339409557	189914593387670038	189914593356784763	5739519	36624794
50	142377417196364	193894168896897487	193894168924671722	193894168842184136	27774235	54713351
51	145143045599692	197875797467973511	197875797402159407	197875797444772385	65814104	23201126
52	147907174371027	201859438775606323	201859438782059357	201859438874132705	6453034	98526382
53	150669836017291	205845054246666509	205845054285923171	205845054390334700	39256662	143668191
54	153431057455345	20983260666390601	209832606641202960	209832606716127913	22706641	52218312
55	156190867055604	213822059884629769	213822059911372304	21382205995642707	26742535	71012938
56	158949293526663	217813379543515117	217813379570306253	217813379519006579	26791136	24508538
57	161706360526093	22180653207316357	221806532003780790	221806532052347536	69385567	20818821
58	164462095231054	225801485307500567	225801485309759486	22580148532711722	2258919	65211155
59	167216521960016	22979820833470963	229798208351343272	229798208407471471	17872309	74000508
60	169969662554551	233796671062467577	233796671042312318	233796671137843085	20155259	75375508
61	172721540727639	237796844582199317	237796844551866864	23779684454546171680	30332453	36027637
62	175472177511800	2417987005605693853	241798700580257426	241798700566674221	25436427	39019632
63	178221594869615	245802212004481091	24580221201103224	245802212039361728	6551151	34880637
64	1809698120699116	24980735270716567	249807352720584206	249807352666888216	13408549	40287441
65	183716850192783	253814097057270587	253814097060656910	253814096974097674	3386323	83172913
66	186462726814356	257822420453703037	257822420416882932	25782242070061405	36820105	183641632
67	189207462479325	261832298778512533	261832298793699141	26183229879361722	15186608	166095511
68	191951073132321	265843709118454979	265843709097623437	265843708773837273	20831542	344617706
69	194693578185957	269856628338594107	269856628301551602	269856628210471997	37042505	128122110
70	197434994078331	273871034935338403	273871034964904223	273871035032220227	29565820	96881824
71	200175335630483	277886907995191811	277886908023169839	277886907974701622	27978028	20490189
72	20291462052448	281904226487172271	281904226483026910	281904226372807542	4145361	114364729
73	205652862425306	285922970160712259	285922970138620930	285922970135722026	22091329	24990233
74	208390079110978	289943119715176507	289943119700984084	289943119723311955	14192423	8135448
75	211126283162243	293964656108882903	293964656105170428	293964656123793851	3712475	14910948
76	213861489506392	297987560649158759	297987560638374821	297987560832592190	10783938	183433431
77	216595711439565	30201185896566267	3020118515802514117	302011851832310807	76052150	64255460
78	219328963332630	306037403464228481	306037403503476006	306037403573745085	39247525	109516604
79	222061256928013	310064306872174139	310064306890230891	310064306957868189	18056752	85694050
80	224792606318600	314092509321252353	314092509284502245	314092509318729703	36750108	2522650
81	227523023599978	31812199431099963	318121994311396755	318121994407209583	496792	96309620
82	230252520816828	322152746418376529	322152746409788075	322152746375574649	8588454	42801880

83	232981109132553	326184749734071211	326184749734869492	326184749762788657	798281	28717446
84	235708800471211	330217989477361159	330217989421989722	330217989480530531	55371437	3169372
85	238435607431737	334252450837522181	334252450875892179	334252450799878593	38369998	37643588
86	241161539806582	338288119306185553	338288119281527478	338288119338621616	24658075	32436063
87	243886608249438	342324980816825683	342324980787212977	342324981049160236	29612706	232334553
88	246610822221359	346363022043182897	346363022117594024	346363022206964776	74411127	163781879
89	249334194029546	350402229198161719	350402229275339670	350402229399557886	77177951	201396167
90	25205673345928	354442589428511471	354442589461661193	354442589515992467	33149722	87480996
91	25477844725151	358484089738515107	358484089718761636	358484089736797400	19753471	1717707
92	257499349768637	362526717392633107	362526717434878969	362526717524365342	42245862	131732235
93	260219446617109	366570460630428997	366570460634193232	366570460613758540	3764235	16670457
94	262938747499423	370615307040895867	370615307025898287	370615307003910246	14997580	36985621
95	265657263117161	374661245056625183	374661245046415361	374661244949200614	10209822	107424569
96	268375000740770	378708263125578223	378708263087697669	378708262951387398	37880554	174190825
97	271091969073361	382756349961937363	382756350004680354	382756349751872976	42742991	210064387
98	273808176380030	386805494468242607	386805494541276472	386805494324290341	73033865	143952266
99	276523631752529	390855686180411407	390855686125149357	390855685867391783	55262050	313019624
100	279238341033925	394906913903735329	394906913901321500	394906913798224974	2413829	105510355
101	281952314626716	398959167795806791	398959167769420669	398959167745582098	26386122	50224693
102	284665559556332	403012437560594987	403012437526461143	403012437543708464	34133844	16886523
103	287378081126626	407066713209546238	407066713226257928	407066713226257928	2696941	14014749
104	290089890632238	411121985053339019	411121985043237519	411121985020483115	10101500	32855904
105	29280099175566	415178243427089041	415178243406068862	415178243341649133	21020179	85439908
106	295511394070886	419235478872659203	419235478893349398	419235478787660177	20690195	84999026
107	298221102772488	423293682191217493	423293682155261572	42329368213388953	35955921	57328540
108	300930125986760	427352844259123877	427352844262971361	427352844328199414	3847484	69075537
109	303638470099198	431412956243052547	431412956202837881	431412956486153933	40214666	243101386
110	306346140642929	435474009565503149	435474009643087789	435474009886396362	77584640	320893213
111	309053145121220	439535995988680111	439535995939289038	439535995966203053	49391073	22477058
112	311759489314579	443598906280158487	443598906333361303	443598906317194216	53202816	37035729
113	314465179385261	447662732746235623	447662732741496788	447662732681198514	4738835	65037109
114	317170221634362	451727466937048793	451727466961134267	451727466946264044	24085474	9215251
115	319874623177404	455793101280411463	455793101292794303	455793101142809342	12382840	137602121
116	322578388623503	459859627460138027	459859627451525756	459859627439908299	8612271	20229728
117	325281523355857	463927038138601217	463927038105943453	463927038141703196	32657764	3101979
118	327984033568074	467995325619454117	467995325646887471	467995325683940424	27433354	64486307
119	330685925327709	472064482769644943	472064482733573233	472064482630623649	36071710	139021294
120	333387204489157	476134501830546337	476134501867733948	476134501670779509	37187611	159766828
121	336087875323188	480205375878386027	480205375860189505	480205375615331145	18196522	263054882
122	338787944139611	484277097367553351	484277097347736237	484277097394075116	19820714	26521765
123	341487414778273	48834966000142959	488349660004218353	488349660052757448	2775394	51314489
124	344186293058920	492423056707800949	492423056699489850	492423056750244799	8311099	42443850
125	346884583805017	496497280793610557	496497280802378550	496497280755786896	8767993	37823661
126	349582292881340	500572325543785867	500572325541126167	500572325446366597	2659700	97419270
127	352279423771442	504648184370381627	504648184297048976	504648184304134090	73332651	66247537
128	354975982263335	50872485095447793	508724850955762227	508724850913921935	1284434	40555858
129	357671973817060	51280231898638269	512802318948722249	512802318960837771	39916020	27800498
130	360367400804331	516880582141749971	516880582162249221	51688058227931689	20499250	86181718
131	363062269659721	520959634421249321	520959634368944099	520959634593935391	52305222	172686070
132	365756583868551	525039469767348079	525039469802292915	525039470031070401	34944836	263722322
133	368450348555798	529120082458008373	529120082475909098	529120082602922713	17900725	144914340
134	371143567919892	533201466236078989	533201466278506017	533201466462381357	42427028	226302368
135	373836245057725	537283615564355927	537283615609092135	537283615849638545	44736208	285282618

Table of known programs, formulas for the calculation of primes.

Author (s)	Year	Comment	Efficiency	Number of terms calculated
Erathosthène	-276 to -194	Sieve	Practical	Calculable infinity
Mersenne	1536	Primes of the form: 2^p-1 .	Practical and exact	51
Fermat	1640	Little theorem of Fermat	Produces weak probable primes	Calculable infinity
Euler	1772	Second degree polynomial	Practical	40
Mills	1947	Double exponential	Practical	Less than 10 terms
Wright	1951	Super exponential	Practical	Less than 5 terms
Wilson	Circa 1780	Formula with $p!$	Theoretical	Very few
Jones, Sato, Wada, Wiens	1976	Polynôme de degré 25 à 26 variables	Theoretical	Very few
John H. Conway	1987	FRACTRAN	Theoretical	Very few
Rowland	2008	Réurrence	Theoretical	Very few
Dress, Landreau	2010	6th degree polynomial	Practical	58
Benoit Perichon (et al).	2010	26 primes in arithmetic progression	Practical	26
Kim Walisch	2020	Primesieve and Primecount	Practical	Calculable infinity

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