The calculation of p_n and $\pi(n)$

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Abstract

A new approach is presented for the calculation of p_n and $\pi(n)$ which uses the Lambert W function. An approximation is first found and using a calculation technique it makes it possible to have an estimate of these two quantities more precise than those known from Cipolla and Riemann. The calculation of p_n uses an approximation using the Lambert W function and an estimate based on a logarithmic least square curve (LLS) $c(n)$. The function $c(n)$ is the same in both cases. The two formulas are:

$$p_n \approx -nW_{-1}\left(\frac{-e}{n}\right) - \frac{n\,c(n)}{W_0(n)} \qquad 1$$

$$\pi(n) \approx \left\{-nW_{-1}\left(\frac{-e}{n}\right) - \frac{n\,c(n)}{W_0(n)}\right\}^{-1} \qquad 2$$

The results presented are empirical and apply up to $n \approx 10^{16}$.

Introduction

In 2010, Dusart proved that $\pi(n) \approx \frac{n}{\log(n)-1}$ if $n > 5393$. We will use this approximation to give an approximation of p_n by inverting the formula.

$$\text{If } \pi(n) \approx \frac{n}{\ln(n)-1} \text{ then } p_n = -nW_{-1}\left(\frac{-e}{n}\right).$$

This formula is precise, for $n = 10^{24}$, we have p_n precise at 99.9 %.

By analyzing the remainder of p_n and $-nW_{-1}\left(\frac{-e}{n}\right)$, we quickly find that it is close to $\frac{n}{W(n)}$, therefore $p_n \approx -nW_{-1}\left(\frac{-e}{n}\right) - \frac{n}{W(n)}$.

Here $W(n)$ is the Lambert W function of order 0.
The classical formula for p_n is $p_n \approx n\,ln(n)$ or better yet the one that was found by Cipolla in 1902 states that.

$$p_n \sim n\left(ln(n) + ln(ln(n)) - 1 + \frac{ln(ln(n)) - 2}{ln(n)} - \frac{ln(ln(n))^2 - 6\,ln(ln(n)) + 11}{2\,ln(n)^2} + \cdots\right).$$

The calculation was taken further in 1994 with Salvy who extracted a procedure from which the approximation could be taken further.

What is remarkable is the similarity with the asymptotic development of W (n).

$$W(n) \approx L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2+L_2)}{2L_1^2} + \frac{L_2(6-9L_2+2L_2^2)}{6L_1^3} + \frac{L_2(-12+36L_2-22L_2^2+36L_2^3)}{12L_1^4} + \cdots$$

$L_1 = ln(n)\ and\ L_2 = ln\ (ln\ (n)).$

Again, if we analyze the rest with respect to the true value of p_n we find with n = 10^{24} (this is the best known value of p_n).

$$p_n \approx 583086425504749834767176666$$

The real value being 58310039994836584070534263, we therefore have an approximation to 0.999976, that is to say 99.9976%. We therefore gained 2 orders of magnitude.

A better approximation

A summary analysis indicates that the remainder after the 2 terms $W_{-1}\left(\frac{-e}{n}\right)$ and $W(n)$ is logarithmic in nature. A simple idea is then to calculate the logarithmic least squares curve or LLS curve. We can also notice that by taking only one term for the approximation of p_n, this form is equivalent to several terms of Cipolla's development. If we take the 2 terms it will be even more precise. In other words, given the nature of the asymptotic development of $W(n)$, each term is equivalent to several terms of the Cipolla development.

We hypothesize here that the remainder after the 2 terms is a logarithmic curve and that once calculated it will stick to reality.

The question then arises of what is the nature of what remains? In fact, we don't know exactly. The best known formula for π (n) is that of Li (n). Riemann proposed a 2nd formula which seems much better at first sight but which was invalidated by Littlewood in 1914. This 2nd formula, called Riemann R or equivalently, the Gram series is

$$\pi(n) = \sum_{k=1}^{\infty} \frac{\ln(x)^k}{k\, k!\, \zeta(k+1)}$$

Numerically, the approximation of π (n) by the Riemann R formula or the Gram series (converges faster) is excellent. But Littlewood has shown that after 10^9, the approximation drifts. As for the function Li (n), it behaves better at much larger scales, the first crossing being evaluated at around 10^{316}. That is, $Li\ (n)\ - \pi\ (n) = 0$ around $1.397\ \times 10^{316}$.

An interesting pattern appears in these calculations. If we analyze the error closely with the first term for formula (1) we quickly find a logarithmic type curve, then the question arises: what precision can we achieve if we use an approximation of the latter? For example, with

$$p_n = -nW_{-1}\left(\frac{-e}{n}\right)$$

The ratio between the two is around 1 from the start and is around 0.999 when n = 10^{24}. So, by calculating a curve of type $a\,log\,(n) + b$, we find a coefficient r^2 quite close to 1. The approximation is disappointing even if the coefficient is very high. Several formulas have been tested [11] to obtain the best accuracy.

The best approximation that has been found empirically is:

$$p_n = -nW_{-1}\left(\frac{-e}{n}\right) - \frac{n\,c(n)}{W_0(n)}$$

where $c(n)$ is of the form $a\,log(n) + b$.

What is happening is an effect of the Russian dolls, the matrioshkas. The rest with the first term is a curve which is roughly a straight line if you look from afar, the rest after 2 terms is still a 'straight' seen from afar but which is always of logarithmic type. And even with the final correction with $c(n)$: the rest is still a logarithmic curve. All these curves seem equivalent but it is when we analyze closely the error that it varies.

By taking a sampling of the values of p_n between 10^2 à 10^{16}

Step	Number of values	Range
10^2	27117419	2711741900
10^3	32082085	32082085000
10^4	45020269	450202690000
10^5	10000000	10^{12}
10^6	4046531	4.046531 x 10^{12}
10^7	5000000	5 x 10^{13}
10^8	454060	4.54060 x 10^{13}
10^9	2200000	2.2 x 10^{15}
10^{10}	1112394	1.112394 x 10^{16}
10^{11}	111239	1.11239 x 10^{16}
10^{12}	54974	5.4974 x 10^{16}
10^{13}	12317	1.2317 x 10^{17}
10^{14}	2162	2.162 x 10^{17}

We solve the equation for each n of the chosen table.

$$-nW_{-1}\left(\frac{-e}{n}\right) - \frac{n\,x}{W_0(n)} - p_n = 0$$

by the bisection method. The values are between 0.8 and 1. The logarithmic least squares curve is then calculated. The coefficient r^2 will indicate if the curve is right. The coefficients a and b are calculated according to the formula:

$$b = \frac{n\sum_{i=1}^{n}(y_i \ln x_i) - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} \ln x_i}{n\sum_{i=1}^{n}(\ln x_i)^2 - (\sum_{i=1}^{n} \ln x_i)^2}$$

$$a = \frac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n}(\ln x_i)}{n}$$

Recall, the coefficient r^2 indicates whether the experimental data stick to the right. If r^2 is near 1 or 0, the curve follows a straight line very closely. The LLS (logarithmic least-squares) line is simply the log of the values that are aligned on a line.

For the range 100000 (10^5) ... 10^{12} we find :

$$0.000747411746033016654203952754295 37\, ln(n) + 0.88596350453664534160747106131754$$

Once this formula is obtained, it remains to compare with the Cipolla formula. Here we will take 16 terms from Cipolla's formula. In appendix, the 16 terms of development of Cipolla (program of B. Salvy).

Comparison with the range 100000 ($\mathbf{10^5}$) ... $\mathbf{10^{12}}$

$c(n) = 0.8859635045366453416074710613175 4 + 0.0007474117460330166542039527542953 7\, ln(n)$

Formula for p_n	Cipolla (Salvy)	Lambert W
Minimal gap	1624.006	.0031723
Maximal gap	7963203	3257663
Average gap	3893600	617551

The formula with Lambert W is clearly more precise over the entire interval. In addition, some values are correct since the error is less than 0.5.

Comparison with the range ($\mathbf{10^{10}}$) ... $\mathbf{1.112394}$ x $\mathbf{10^{16}}$

$c(n) = 0..8828110602406711269541535547854 2 + .000856183701640445572391331142143 99 * ln(n)$

Formula for p_n	Cipolla (Salvy)	Lambert W
Minimal gap	640495	261
Maximal gap	1004659553	513851652
Average gap	4.64302e+08	1.25614e+08

More than 11 values reach an accuracy of 14 decimal digits.

Results for $\boldsymbol{\pi(n)}$

For the calculation of $\pi(n)$, it suffices to reverse the formula for p_n. First of all, numerically it is very fast and above all very precise and even more precise than Li (n).

So we pose that

$$\pi(n) \approx \left(-nW_{-1}\left(\frac{-e}{n}\right) - \frac{n\,c(n)}{W_0(n)}\right)^{-1}$$

That is, we solve the equation for a value $\pi(n)$ of the chosen table.
With the same expression for $c(n)$.

Comparison with the range 100000 ($\mathbf{10^5}$) ... 744000000000

$c(n) = 0.8859635045366453416074710613175 4 + 0.0007474117460330166542039527542953 7\, ln(n)$

Formula for p_n	Li(n)	Lambert W inverted	Gram or Riemann R
Minimal gap	37.8	.0006565	0.0002465
Maximal gap	53330.90	19292.58	19205.24
Average gap	24433.6	3659.52	3713.91

The formula with inverted Lambert W for the calculation of $\pi(n)$ is more precise than $Li(n)$ and happens to have a better average value with the scale 10^5 but is not more precise for the scale 10^{10}, but it is very close.

Appendice : formula of Cipolla with 16 terms.

```
k*ln(k)*(1+(ln(ln(k))-1)/ln(k)+(ln(ln(k))-2)/ln(k)^2+(-(1/2)*ln(ln(k))^2+3*ln(ln(k))-
11/2)/ln(k)^3+((1/3)*ln(ln(k))^3-7*ln(ln(k))^2*(1/2)+14*ln(ln(k))-131/6)/ln(k)^4+(-
(1/4)*ln(ln(k))^4+23*ln(ln(k))^3*(1/6)-49*ln(ln(k))^2*(1/2)+159*ln(ln(k))*(1/2)-
1333/12)/ln(k)^5+((1/5)*ln(ln(k))^5-49*ln(ln(k))^4*(1/12)+73*ln(ln(k))^3*(1/2)-
367*ln(ln(k))^2*(1/2)+3143*ln(ln(k))*(1/6)-13589/20)/ln(k)^6+(-
(1/6)*ln(ln(k))^6+257*ln(ln(k))^5*(1/60)-1193*ln(ln(k))^4*(1/24)+1027*ln(ln(k))^3*(1/3)-
17917*ln(ln(k))^2*(1/12)+47053*ln(ln(k))*(1/12)-193223/40)/ln(k)^7+((1/7)*ln(ln(k))^7-
89*ln(ln(k))^6*(1/20)+959*ln(ln(k))^5*(1/15)-13517*ln(ln(k))^4*(1/24)+6657*ln(ln(k))^3*(1/2)-
39769*ln(ln(k))^2*(1/3)+493568*ln(ln(k))*(1/15)-32832199/840)/ln(k)^8+(-
(1/8)*ln(ln(k))^8+643*ln(ln(k))^7*(1/140)-14227*ln(ln(k))^6*(1/180)+34097*ln(ln(k))^5*(1/40)-
76657*ln(ln(k))^4*(1/12)+616679*ln(ln(k))^3*(1/18)-
642111*ln(ln(k))^2*(1/5)+36780743*ln(ln(k))*(1/120)-893591051/2520)/ln(k)^9+((1/9)*ln(ln(k))^9-
1321*ln(ln(k))^8*(1/280)+119603*ln(ln(k))^7*(1/1260)-
218809*ln(ln(k))^6*(1/180)+1328803*ln(ln(k))^5*(1/120)-
2696687*ln(ln(k))^4*(1/36)+33904723*ln(ln(k))^3*(1/90)-
40633409*ln(ln(k))^2*(1/30)+7921124011*ln(ln(k))*(1/2520)-2995314311/840)/ln(k)^10+(-
(1/10)*ln(ln(k))^10+12169*ln(ln(k))^9*(1/2520)-17841*ln(ln(k))^8*(1/160)+74603*ln(ln(k))^7*(1/45)-
12834463*ln(ln(k))^6*(1/720)+3501785*ln(ln(k))^5*(1/24)-
332109377*ln(ln(k))^4*(1/360)+199802702*ln(ln(k))^3*(1/45)-
26038842937*ln(ln(k))^2*(1/1680)+1585618043*ln(ln(k))*(1/45)-
18901604813/480)/ln(k)^11+((1/11)*ln(ln(k))^11-
12421*ln(ln(k))^10*(1/2520)+648773*ln(ln(k))^9*(1/5040)-
3144689*ln(ln(k))^8*(1/1440)+22783361*ln(ln(k))^7*(1/840)-
187923713*ln(ln(k))^6*(1/720)+716745529*ln(ln(k))^5*(1/360)-
1442721139*ln(ln(k))^4*(1/120)+47127721999*ln(ln(k))^3*(1/840)-
966062632891*ln(ln(k))^2*(1/5040)+308893960883*ln(ln(k))*(1/720)-5837964271731/12320)/ln(k)^12+(-
(1/12)*ln(ln(k))^12+139151*ln(ln(k))^11*(1/27720)-
117221*ln(ln(k))^10*(1/800)+253818967*ln(ln(k))^9*(1/90720)-
795876559*ln(ln(k))^8*(1/20160)+314837287*ln(ln(k))^7*(1/720)-
4223985979*ln(ln(k))^6*(1/1080)+6399190654*ln(ln(k))^5*(1/225)-
186267044627*ln(ln(k))^4*(1/1120)+1639283993969*ln(ln(k))^3*(1/2160)-
2857440028197*ln(ln(k))^2*(1/1120)+56866193897941*ln(ln(k))*(1/10080)-
8799795072177107/1425600)/ln(k)^13+((1/13)*ln(ln(k))^13-
141461*ln(ln(k))^12*(1/27720)+5712631*ln(ln(k))^11*(1/34650)-
529792703*ln(ln(k))^10*(1/151200)+5029078321*ln(ln(k))^9*(1/90720)-
14019042613*ln(ln(k))^8*(1/20160)+3999527649*ln(ln(k))^7*(1/560)-
328281081287*ln(ln(k))^6*(1/5400)+5387564881553*ln(ln(k))^5*(1/12600)-
24626379317209*ln(ln(k))^4*(1/10080)+165776749480421*ln(ln(k))^3*(1/15120)-
122304707880473*ln(ln(k))^2*(1/3360)+132580053003639763*ln(ln(k))*(1/1663200)-
112278478974958204l9/129729600)/ln(k)^14+(-(1/14)*ln(ln(k))^14+1866713*ln(ln(k))^13*(1/360360)-
38193059*ln(ln(k))^12*(1/207900)+1023073061*ln(ln(k))^11*(1/237600)-
9793824913*ln(ln(k))^10*(1/129600)+3052471597*ln(ln(k))^9*(1/2880)-
495982820159*ln(ln(k))^8*(1/40320)+1296463478011*ln(ln(k))^7*(1/10800)-
8292697622023*ln(ln(k))^6*(1/8400)+113897730216643*ln(ln(k))^5*(1/16800)-
328979431306961*ln(ln(k))^4*(1/8640)+728776729471327*ln(ln(k))^3*(1/4320)-
263517478549855027*ln(ln(k))^2*(1/475200)+12023328215517658997*ln(ln(k))*(1/9979200)-
1248968565683885823/9609600)/ln(k)^15+((1/15)*ln(ln(k))^15-
1892453*ln(ln(k))^14*(1/360360)+1097406347*ln(ln(k))^13*(1/5405400)-
51963846821*ln(ln(k))^12*(1/9979200)+35812996787*ln(ln(k))^11*(1/356400)-
50524883881*ln(ln(k))^10*(1/32400)+130863766133*ln(ln(k))^9*(1/6480)-
134493163147463*ln(ln(k))^8*(1/604800)+158343801542491*ln(ln(k))^7*(1/75600)-
847040297248639*ln(ln(k))^6*(1/50400)+4286267926497719*ln(ln(k))^5*(1/37800)-
181013884586875*ln(ln(k))^4*(1/288)+61402203444117317*ln(ln(k))^3*(1/22275)-
6406940325583612039*ln(ln(k))^2*(1/712800)+629213463986068329377*ln(ln(k))*(1/32432400)-
540034620553479284263/259459200)/ln(k)^16)
```

Formula : { c^n } , n > 1. { x } is the nearest integer to x. This formula gives 387 primes

Found on Feb. 13 020.

```
c=31622.776718559569341181978706142880921196050198706691685092604397231785036207\
0864754895287267371526503218374753273756995351706550976656637342668814334968666\
7101253409003404462577908490644122849185117910040141357956901545395665144254926\
0368733397963027260382237357834177624731483528295517610175141894026910108773091\
9102265276155731349632552235646978900096647191268683240475690792929946636829523\
4537889911695242127537567542061489388029765753953635000339676196085538110591110\
4585141090122658582123787181713420542296072149094895524374290222132677205020669\
4445695634733625075384860598871603780178145604934183538244723142272254031969991\
1200615639448998851471276974790874283370668013877448445827189836933336932748683\
7096031634918922648353422318092987818160592340775276127611835724115334227731181\
2382996597965951469528485537607311159096901622414506311547051905763888568046565\
1049166427400916041959872354321110387678917719194813843955193371342978053712862\
0453524374293366590642009861486573168440136568062290968471195962360800157373661\
7143373070165944776420832972019845288922599364819623823506773804799606015475838\
5408260449690988543205825633919510504581682546272086621725352722128830938671472\
3701049700231880709516101390100111272624859303900292400178495789671057286940063\
2939593946897204453503410798379799657214376636228419529879327743486265567548211\
3576902297196496895449727060689840512311047588147145276057014367506675470634720\
3613695309242488801761837728748016218351333849539796405079226399842615890863407\
8599788643344554273489131657070089852713689432765198710321770400528221171272704\
0206817976260112923629070254233895740091029484512516533519548133198851927540897\
2455463807670389342975063899265873368827496362107236544910631527268372333090949\
6599079804007385491895861312920245982318590485133202228187065916469134386843632\
6710041357132606931061476180276702785264896516822792415007247720747091646449625\
4232619886412790996612644285061938042714365958818635813884068102869316665794217\
8129882233115569739186048345925987950967419367699136707185345903901351463318825\
1244196261056665156495557988992317891312059314344240462240709984206444212348221\
8190871664878419254779689447073126235375450129126837015443848876734518730078317\
2181027964646479454712955925745105851307097731929480992845429539117442668639185\
0136062499136302371850993390404927892216296025369572878087609243226453797867139\
3469752835416017623841715414916352205819923096864111587928911940669604635015684\
7581294362465199335660467818710512634839702174274066131707754882547089414783999\
5175780316883752292172422813518529901439742936742350859348052219626008764448086\
8799805941436104497986393192697940430432240135606638442974285495155375736097977\
5028072301126799837273791743647186112604934117249206784451299546711413367340501\
9569662946589302409473910039374488077465339877358211311017648701799022641035886\
3484903089265972324361003850699420234653560932806297966170293267500259491821818\
1198369337121033424541848925401516832417018610965424291916655343744273160443306\
9511466370054097794456240488450136865019379631160402883720249277945984213446264\
5988545557033239241001377665704382870870778480345318077395858285768401082992990\
6805331632859321631952446031722731370128828153399370730874506904362805691466137\
7583386102156444256251269973458369382478957925560223823273029506595199057689583\
4494612415638418972310233727907333983390896921545293647542190449362825483734461\
9136791675945740812366000990257217886571673799707804765177555198609248755512965\
3027715120275248760325555105027812428659621914108249357373916718531633964814113\
5054938247521657216756907893392593294500494373382654321063600590692748010279682\
9489802744322026037605807751069630567877106566741622790692273950877423124270171\
5917401923410677957081175116161669874591767721421034406660019253785321121319776\
9175525445047016860655132021850376701748670306588930285165633137124920560978651\
3594698267301854434353463258169713980181998659759698957791522027302972704755488\
1930524920113346550223965908425088243203667733442482257635397243984316985885847\
8313649362480390679519963175568339592886446984299461893063001573950
```

The primes are :

```
1000000007
31622776952311
1000000014783746303
316227771860626777745609
10000000221756195364989210591059
```

…

The following constant c will generate 565 consecutive primes.

Formula : { c^n } , n > 1. Where { x } is the nearest integer to x.

Found on Feb. 13 2020.

```
c =
55237.075042967647154331247815286174172637407435349766253557408607657108195592125636580713628080659\
516106616080795117986654785234100829360288179190471423450519863363557788376392580644911222667893510\
974241988869207456699800233885605605675094352653650767616236231134331699696673981212194965780823859\
542513548714675259650029713063945705266192886108484426067463553931028213163136107111991147389991015\
758877686062480974368913148037546562272713868086591852396135188995526263750011830402068838021477394\
593977513965032757199104062980046792636993233459638498156642835757990803045249621471250672104424082\
878866148815914455770104219035162438636842947711686257926512286234815491738282106295507458532362384\
505684106687414032550750582826355233367560180044896480822312106639235650936616271234862169623992549\
833385138618768297087536703672300033186237217428446064036332101656334392544696310217306392292312346\
755485926236953371349135828375352794075586146180559619926835893629337960083779229062717369606434545\
504588267437002303613076141782628279883795320599659630590570183343898576003482100591286183029699994\
063814810222758113360617684917332500481699286072092564893192596718377047150527523498197744990143307\
282929833675274238178999906884020618788888207069859056193782391833625989458610609425161081429098397\
262673032249349165602948193254418930839358484996327532617121296457706230574672722644329006406905661\
554568488675010895335208044022297310319590454397380243261529601006290591280527131008172214873455307\
846029050480478897222168190949942116718795813290833306634424608567282733590225518704764315830836271\
488978275897499168375273014497123044786109601068205174355781381638667093810702022010186225448729564\
843554069570892699749740600074481499068166335397596759598128866132599812142485679806875565997478728\
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548586713169997351534962857881721058119179538424167872514362450972158935474469666661246215282353722\
715027861988650771006099035187302625072908461596058978030149206030170409631196442380060614276551970\
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894228942593266173224984794003299138402533386597073677935317794463971206544351066096664442654641680\
6402635168666758237850879891862 5694610
```

The primes are :

```
3051134459
168535743094673
9309421488742788613
514225213380301008078907
28404296700473737832215645327
1568970268386786190461323870128523
86665328455065998699156259015013574567
4787139251495919953666696192531499395660543
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