

Identities inspired from the Ramanujan Notebooks

Second series

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Abstract

A series of formula are presented that are all inspired from the Ramanujan Notebooks [6]. One of them appears in the notebooks II

$$\zeta(3) = \frac{7\pi^3}{180} - 2 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)}$$

That formula inspired others that appeared in 1998, 2006 and 2009 on the author's website and later in literature [1][2][3]. New formulas for π and the Catalan constant are presented and a surprising series of approximations. A new set of identities is given for Eisenstein series. All of the formulas are conjectural since they were found experimentally.

Une série de formules utilisant l'exponentielle est présentée, ces résultats reprennent ceux apparaissant en 1998, 2006 et 2009 sur [1][2][3]. Elles sont toutes inspirées des Notebooks de Ramanujan tels que

$$\zeta(3) = \frac{7\pi^3}{180} - 2 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)}$$

Une nouvelle série pour π et la constante de Catalan sont présentés ainsi qu'une série d'approximations surprenantes. Une série d'identités nouvelles sont présentées concernant les séries d'Eisenstein. Toutes les formules présentées sont des conjectures, elles ont toutes été trouvées expérimentalement.

1. Introduction

By taking back the series found in 2006, I extended the search to more general expressions with exponents 1,2 and 4 for the exponential term and found ;

The same pattern is present for powers of π and $\zeta(n)$

$$1.1 \quad \pi = 72 \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)} - 96 \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)} + 24 \sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n} - 1)}$$

$$1.2 \quad \frac{1}{\pi} = 8 \sum_{n=1}^{\infty} \frac{n}{e^{\pi n} - 1} - 40 \sum_{n=1}^{\infty} \frac{n}{e^{2\pi n} - 1} + 32 \sum_{n=1}^{\infty} \frac{n}{e^{4\pi n} - 1}$$

$$1.3 \quad \pi^3 = 720 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{\pi n} - 1)} - 900 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)} + 180 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{4\pi n} - 1)}$$

$$1.4 \quad \zeta(3) = 28 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{\pi n} - 1)} - 37 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)} + 7 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{4\pi n} - 1)}$$

$$1.5 \quad \pi^5 = 7056 \sum_{n=1}^{\infty} \frac{1}{n^5(e^{\pi n} - 1)} - 6993 \sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n} - 1)} + 63 \sum_{n=1}^{\infty} \frac{1}{n^5(e^{4\pi n} - 1)}$$

$$1.6 \quad \zeta(5) = 24 \sum_{n=1}^{\infty} \frac{1}{n^5(e^{\pi n} - 1)} - \frac{259}{10} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n} - 1)} - \frac{1}{10} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{4\pi n} - 1)}$$

$$1.7 \quad \pi^7 = \frac{907200}{13} \sum_{n=1}^{\infty} \frac{1}{n^7(e^{\pi n} - 1)} - 70875 \sum_{n=1}^{\infty} \frac{1}{n^7(e^{2\pi n} - 1)} + \frac{14175}{13} \sum_{n=1}^{\infty} \frac{1}{n^7(e^{4\pi n} - 1)}$$

$$1.8 \quad \zeta(7) = \frac{304}{13} \sum_{n=1}^{\infty} \frac{1}{n^7(e^{\pi n} - 1)} - \frac{103}{4} \sum_{n=1}^{\infty} \frac{1}{n^7(e^{2\pi n} - 1)} + \frac{19}{52} \sum_{n=1}^{\infty} \frac{1}{n^7(e^{4\pi n} - 1)}$$

For Catalan constant, I find this new identity:

$$1.9 \quad K = 11 \sum_{n=1}^{\infty} \frac{1}{n^2(\cosh(\pi n) - 1)} - \frac{71}{2} \sum_{n=1}^{\infty} \frac{1}{n^2(\cosh(2\pi n) - 1)} + 11 \sum_{n=1}^{\infty} \frac{1}{n^2(\cosh(4\pi n) - 1)}$$

For $1/\pi^2$, by varying the function at the numerator, I find this:

$$1.10 \quad \frac{1}{\pi^2} = 4 \sum_{n=1}^{\infty} \frac{\sigma_1(n)n}{e^{\pi n}} - 64 \sum_{n=1}^{\infty} \frac{\sigma_1(n)n}{e^{2\pi n}} + 64 \sum_{n=1}^{\infty} \frac{\sigma_1(n)n}{e^{4\pi n}}$$

Here, $\sigma_1(n)$ is Euler sigma function of order 1, Actually the same coefficients as with $\cosh(k\pi n)$, $k=1,2$ et 4.

$$1.11 \quad \frac{1}{\pi^2} = 2 \sum_{n=1}^{\infty} \frac{n^2}{\cosh(\pi n) - 1} - 32 \sum_{n=1}^{\infty} \frac{n^2}{\cosh(2\pi n) - 1} + 32 \sum_{n=1}^{\infty} \frac{n^2}{\cosh(4\pi n) - 1}$$

The pattern persist for $1/\pi^3$ but apparently for no other powers of π

$$1.12 \quad \frac{1}{\pi^3} = 4 \sum_{n=1}^{\infty} \frac{\sigma_1(n)n^2}{e^{\pi n}} - 128 \sum_{n=1}^{\infty} \frac{\sigma_1(n)n^2}{e^{2\pi n}} + 256 \sum_{n=1}^{\infty} \frac{\sigma_1(n)n^2}{e^{4\pi n}}$$

2. Experiments with fractional exponent

I was compiling a table of values for the Inverter [9] and found that for some arguments the closeness to rational numbers, these are the 2 examples that are the most striking.

$$2.1 \quad \sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n/7} - 1} = 10.000000000000000190161767888663 \dots$$

$$2.2 \quad \sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n/13} - 1} \cong 119.00000000000000000000000000000959374585 \dots$$

The precision is 15 and 31 decimal digits and for an argument of $2\pi n/163$ the precision is 435 decimal digits. Other series of the form $\sum_{n=1}^{\infty} \frac{n^3}{e^{\frac{2\pi n}{k}} - 1}$ are also producing near

integers when k is not a multiple of 2,3 and 5. For the exponent one can obtain near integers when the exponent of n is $4m-1$, $m > 0$. This fact is related to properties of Eisenstein series which is; if $240 \mid k^4 - 1$ then the series is near an integer. But it is not always that it produces approximations since I have this identity for π . We see the pattern [1,2,4] again with the exponent.

$$2.3 \quad \frac{\pi}{10} = - \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)} + 4 \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)} - 1 \sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n} - 1)} + \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n/5} - 1)} - 4 \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n/5} - 1)} + \sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n/5} - 1)}$$

$$2.4 \quad \frac{7\pi}{120} = -2 \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)} - \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n/5} - 1)} + 4 \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n/5} - 1)} - 1 \sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n/5} - 1)}$$

$$\begin{aligned}
2.5 \quad 3\log(\varphi) &= -4 \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)} + 10 \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)} \\
&\quad - 4 \sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n} - 1)} + 4 \sum_{n=1}^{\infty} \frac{1}{n(e^{\frac{\pi n}{5}} - 1)} - 10 \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n/5} - 1)} \\
&\quad + 4 \sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n/5} - 1)} \\
2.6 \quad &\sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n/5} - 1)} - 4 \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n/5} - 1)} + \sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n/5} - 1)} \\
&= \frac{\pi}{40} - \frac{3\ln(\pi)}{2} + 2\ln\Gamma\left(\frac{1}{4}\right) - \frac{7\ln(2)}{4} \\
2.7 \quad \log(\varphi) &= \sum_{n=1}^{\infty} \frac{1}{n(e^{\frac{\pi n}{5}} - 1)} - 2 \sum_{n=1}^{\infty} \frac{1}{n(e^{\frac{2\pi n}{5}} - 1)} + \sum_{n=1}^{\infty} \frac{1}{n(e^{\frac{4\pi n}{5}} - 1)} + \frac{\pi}{120} - \frac{\ln(2)}{4}
\end{aligned}$$

Where φ is the golden ratio.

Since the generic series for π is with $\sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)}$ that series is the log of the well known Euler partition function $F(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$ series which means that an identity can be translated into the partition function as well, of course when $x \rightarrow e^{-\pi x}$.

$$2.8 \quad \frac{F(1/5)^5 F(4/5)^5}{F(2/5)^{20}} = \frac{F(2)^{28}}{F(1)^{31} F(4)^7}$$

The exact expression for $\sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)}$ can be found also by a variety of methods, one of them is to simply try it in the Integer Relations algorithm to find (2.6). Since the partition function is involved it is natural to ask if there could be a relationship with the Roges-Ramanujan, recall $G(x)$, $H(x)$ are and let's define $J(x)$,

$$2.9 \quad J(x) = G(x)H(x) = \prod_{n=1}^{\infty} (1 - x^{(5n-1)})(1 - x^{(5n-4)})(1 - x^{(5n-2)})(1 - x^{(5n-3)})$$

And again with $x \rightarrow e^{-\pi x}$, then

$$2.10 \quad \frac{J(1/5)^{10} J(4/5)^{10}}{J(2/5)^{40}} = e^{\pi}$$

$$2.11 \quad J(2/5)^{30} = \frac{e^{2\pi}}{\varphi^{15}}$$

$$2.12 \quad \frac{J(1)^6 J(4)^{10}}{J(2)^{12}} = \frac{e^{\pi}}{\varphi^6}$$

$$2.13 \quad \frac{J(1/5)^4 J(4/5)^4}{J(2/5)^{10}} = \varphi^3$$

$$2.14 \quad J(1/5)^6 J(4/5)^6 = e^{\pi} \varphi^{12}$$

The exact expression for $k=2/7$ was found by Bill Gosper using the Computer Algebra system Macsyma and a set of personal routines.

$$2.15 \quad \sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n/7} - 1} = \frac{-1}{240} + \frac{1}{320} (301 + 210\sqrt{2} 7^{1/4} + 120\sqrt{7} + 90\sqrt{2} 7^{3/4}) \frac{\pi^2}{\Gamma(\frac{3}{4})^8}$$

In fact that series is the series of Eisenstein which are.

$$2.16 \quad E_4(q) = 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) q^{2k}$$

$$2.17 \quad E_8(q) = 1 + 480 \sum_{k=1}^{\infty} \sigma_7(k) q^{2k}$$

$$2.18 \quad E_{12}(q) = 1 + \frac{65520}{691} \sum_{k=1}^{\infty} \sigma_{11}(k) q^{2k}$$

When with $q \rightarrow e^{-\pi q}$, $E_4(1/10)$, $E_4(1/5)$, $E_4(2/5)$ we can get respectively

$$2.19 \quad \frac{\pi^2}{\Gamma(\frac{3}{4})^8} \left(\frac{5313}{4} + 630\sqrt{5} + 90\sqrt{360 + 161\sqrt{5}} \right)$$

$$2.20 \quad \frac{\pi^2}{\Gamma(\frac{3}{4})^8} \left(\frac{483}{4} 90\sqrt{5} \right)$$

$$2.21 \quad \frac{\pi^2}{\Gamma(\frac{3}{4})^8} \left((5313 + 2520\sqrt{5} - 32\sqrt{\frac{91125}{2} + \frac{326025\sqrt{5}}{16}}) \right)$$

Note, $E_4(1/10) = 10000.0000000000000000000012378\dots$

$$2.22 \quad 2 \sum_{n=1}^{\infty} \frac{n^3}{e^{\frac{\pi n}{5}} - 1} - 28 \sum_{n=1}^{\infty} \frac{n^3}{e^{\frac{2\pi n}{5}} - 1} + 32 \sum_{n=1}^{\infty} \frac{n^3}{e^{\frac{4\pi n}{5}} - 1} + 28 \sum_{n=1}^{\infty} \frac{n^3}{e^{\frac{2\pi n}{5}} - 1} - 257 \sum_{n=1}^{\infty} \frac{n^3}{e^{\pi n} - 1} + 251 \sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n} - 1} = 0$$

$$2.23 \quad 8 \sum_{n=1}^{\infty} \frac{n^7}{e^{\pi n/5} - 1} - 2192 \sum_{n=1}^{\infty} \frac{n^7}{e^{2\pi n/5} - 1} + 2048 \sum_{n=1}^{\infty} \frac{n^7}{e^{4\pi n/5} - 1} + 208897 \sum_{n=1}^{\infty} \frac{n^7}{e^{\pi n} - 1} - 208761 \sum_{n=1}^{\infty} \frac{n^7}{e^{2\pi n} - 1} = 0$$

The pattern observed earlier [1,2,4] can be translated into Eisenstein series identities that are new (?). Here $E_n(q)$ is with $q \rightarrow e^{-2\pi q}$

$$2.24 \quad -E_4(1/10)+14E_4(1/5)-16E_4(2/5) + \frac{1288}{11}E_4(1/2) = 0$$

$$2.25 \quad E_8(1/10)-274E_8(1/5)+256E_8(2/5) + \frac{3133472}{121}E_8(1/2) = 0$$

$$2.26 \quad -E_{12}(1/10)+4034E_{12}(1/5)-4096E_{12}(2/5) + \frac{7811747968}{2081}E_{12}(1/2) = 0$$

I could not find any other exact fractional values, the other ones are only translatable into approximations like, here F as in (2.8). Other values are of interest as well like

	$F(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n}, x \rightarrow e^{-\pi x}$	Precision in digits
2.27	$F(1)^8 \approx \frac{e^{\pi}}{2^4}$	4
2.28	$F(1/2)^{16} \approx \frac{e^{5\pi}}{2^{16}}$	6
2.29	$\frac{F(1/8)^{36}F(1/15)^{18}}{F(1/12)^{36}F(3/20)^{18}} \approx e^{\pi}$	35
2.30	$F(1/8)^{64} \approx \frac{e^{85\pi}}{2^{128}}$	36
2.31	$\frac{F(1/12)^4F(1/36)^2}{F(1/9)^2F(1/24)^4} \approx e^{\pi}$	48
2.32	$F(1/32)^{256} \approx \frac{e^{1365\pi}}{2^{768}}$	173

3. Conclusion

As far as the author knows the formulas with arguments $[1/5, 2/5, 4/5]$ are new, the formula for Catalan, π and $1/\pi$ as well. No other exact formula was found for the fractional exponent but the search was limited to the Farey set of order 60 only. The approximation (2.24) for e^{π} is the simplest found.

References

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