NEW FORMULAS FOR π INVOLVING INFINITE NESTED SQUARE ROOTS AND GRAY CODE.

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Abstract. In previous papers ([39] and [40]) we introduced a class of polynomials which follow the same recursive formula as the Lucas-Lehmer numbers, studying the distribution of their zeros and remarking that this distributions follows a sequence related to the binary Gray code. It allowed us to give an order for all the zeros of every polynomial L_n, [40]. In this paper, the zeros, expressed in terms of nested radicals, are used to obtain two new formulas for π: the first (i.e., formula (58)) can be seen as a generalization of the known formula (1), in which just (1) can be viewed as related to smallest positive zero of L_n; the second (i.e., formula(83)) is an exact formula for π achieved thanks to some identities of L_n. We also introduce two relationships between π and the golden ration φ: (63) and (95).

1. Introduction.

The need of understanding the properties of π and the need of computing its value in a more and more precise way, since the origins of the mathematical thinking, has challenged many mathematicians along more than three millennia [6, 16].

As observed by Borwein in [4] (p. 249, translated from Italian), "*one of the motivations for the computation of approximations of π very close to the spirit of modern experimental Mathematics resides in the aim on one hand of determining if the decimal expansion of π were periodic, since this should have implied that π could be expressed in terms of a ratio between two integer numbers, on the other hand of establishing if π is an algebraic number, i.e., the root of a polynomial with rational coefficients, in order to determine the distribution of its decimal digits*".

Viète's formula [37], developed in 1593 and subsequently generalized (see, for example [25]), is probably the oldest exact result derived for π and is based on an infinite product of nested radicals.

Another formula involving nested radicals is given by [15, 17]

$$\pi = \lim_{n\to\infty} 2^{n+1} \cdot \sqrt{2 - \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}} . \tag{1}$$

which, following J. Munkhammar [1], [24], can be rewritten in the form

$$\pi = \lim_{n\to\infty} 2^{n+1}\, \pi_n \tag{2}$$

where π_n is defined in an iterative way:

$$\pi_{n+1} = \sqrt{\left(\frac{1}{2}\pi_n\right)^2 + \left[1 - \sqrt{1-\left(\frac{1}{2}\pi_n\right)^2}\right]^2} \tag{3}$$

with $\pi_0 = \sqrt{2}$ [24].

In 1882 von Lindemann showed the transcendence (and *a fortiori* the irrationality) of π.

In the meantime, many mathematicians continued to discover several sequences and series converging to π.

The recent literature ([3], [10], [11], [32]), after centuries devoted to the search of elegant formulas and to the study of the irrationality of π, focused mainly on the search for rapidly converging formulas.

The famous Indian mathematician Ramanujan determined several sequences converging to π very rapidly. In particular, it is noteworthy to cite 17 different extraordinary series, converging very rapidly to $1/\pi$, [8] (p. 352-354), [33]. Here we report one of the most intriguing:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}} \quad . \tag{4}$$

The “spirit of the modern experimental Mathematics” [10, 11] was built mainly by means of the birth, growth and development of the computer technologies in the Fifties of the last century and of the discovery of more and more advanced and efficient algorithms apt to perform highly precise arithmetic computations. Today we know millions digits of π value. Soon the challenge was shifted to more and more rapid and efficient algorithms. The advent of computers in the 20th century led to an increased rate of π calculation records. But major improvements were obtained by means of new extraordinarily efficient algorithms. For example, in 1965 it was shown that an optimal algorithm used to compute the so-called *Fast Fourier Transform* (FFT), could be adapted to perform arithmetics on huge integer numbers more rapidly than with previous algorithms and with less computational costs [9] [10].

The last quarter of the XX century was characterized by the discovery of several algorithms with reduced operational complexity. These algorithms require the complete execution of multiplications, divisions, extractions of square roots, which need large scale FFT operations, implying the usage of huge memory and heavy parallel computing [4] (p. 266).

The formulas of BBP type [3] are more recent. It is however clear that a reason for the modern computations of π resides in the goal of taking advantage of the impressive computation power of modern computers; moreover, several trials for the improvement of the approximation of π have sometimes led to very important numerical and practical applications, such as the FFT techniques (see [2] and [4]).

The brilliant English mathematician John Wallis, who lived in the $XVII$ Century, highly able in detecting formal schemes and regularity in mathematical structures [4] and celebrated for his formula for π [17], to whom this paper is morally dedicated, defended the legitimacy of any method that could help the discovery of the truth, even if not corroborated by a rigorous proof. He even stated that Archimedes should have been more blamed because he did not explain the logical processes used for his discoveries than admired for his very elegant proofs [4].

Wallis even got to assert that the contemplation of a finite number of particular cases is all that can be defined as a proof. This kind of contemplation allows us the understanding of the general rule leading to the expected formula. Let us however recall that for many other mathematicians of his time (first of all Fermat) there was not yet the attitude to build a proof, as we can understand today.

Modern Mathematics follows other, more rigorous, ways. However, all the results here shown and proved are introduced as unavoidable consequences of this kind of “contemplation”, as suggested by Wallis; thus, several mathematical proofs in our paper are conducted by mathematical induction, which can be viewed as a useful way to prove that some statements are true not only for “a finite number of particular cases” but for every value of $n = 1, 2, \dots$.

In this paper we obtain π as the limit of a sequence related to the zeros of the class of polynomials $L_n(x) = L_{n-1}^2(x) - 2$ created by means of the same iterative formula used to build the well-known Lucas-Lehmer sequence, employed in primality tests [13, 17, 21, 22, 23, 34]. This class of polynomials was introduced in a previous paper [39].

We are aware that the rate of convergence of our sequences is slower than (4) and other more recent series [32]. However, determining the computational costs and the convergence rates of the sequences converging to π here introduced is beyond the scopes of this paper.

The results obtained here are based on the placement of the zeros of the polynomials $L_n(x)$, studied in [40]. Zeros have a structure of nested radicals, by means of which we can build infinite sequences of nested radicals converging to π. The ordering of the zeros follows the sequence of the binary Gray code, which is very useful in computer science and in telecommunications [40].

The paper is organized in the following way.

In Section 2 we recall the main properties of Lucas-Lehmer polynomials, the definition of Gray code and its most important properties, which show that the zeros of every $L_n(x)$ follow the same ordering rule of the Gray code, where the signs + and - in the nested radicals are respectively substituted by the digits 1 e 0. This part is a summary of several results obtained in [39] and [40]. Furthermore, we give a recursive formula for the sequence of the first nonnegative zeros of $L_n(x)$, in terms of nested radicals. We apply this formula in order to prove again (1). We will start from this formula, in order to introduce the techniques we will use to study the generalized sequences converging to π, in Section 3. Moreover, in this Section, we show that the generalizations of the Lucas-Lehmer map, M_n^a for $a > 0$ introduced in [39], have the same properties of L_n, for what concerns the distribution of the zeros and the approximations of π. We also obtain π not as the limit of a sequence, but equal to an expression involving the zeros of the polynomials L_n and M_n^a for $a > 0$. Finally, always in Section 3, we introduce two relationship between π and the golden ration φ.

2. Preliminaries.

2.1. **The class of Lucas-Lehmer polynomials.** We recall below some basic facts about Lucas-Lehmer polynomials

$$L_0(x) = x \quad ; \quad L_n(x) = L_{n-1}^2 - 2 \ \ \forall n \geq 1 \tag{5}$$

taken from [39]. The polynomials $L_n(x)$ are orthogonal with respect to the weight function

$$\frac{1}{4\sqrt{4-x^2}}$$

defined on $x \in (-2, 2)$. We denote this property with letter (a).

In [39] we have proved the following relationship between Lucas-Lehmer polynomials and Tchebycheff polynomials. In fact, for each $n \geq 1$ we have

$$L_n(x) = 2\ T_{2^{n-1}}\left(\frac{x^2}{2} - 1\right) \tag{6}$$

where the *Tchebycheff polynomials of first kind* [5, 19, 35] satisfy the recurrence relation

$$\begin{cases} T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) & n \geq 2 \\ T_0(x) = 1, \quad T_1(x) = x \end{cases}$$

from which it easily follows that for the n-th term:

$$T_n(x) = \frac{\left(x - \sqrt{x^2-1}\right)^n + \left(x + \sqrt{x^2-1}\right)^n}{2} \tag{7}$$

This formula is valid in $\mathbb{R}$ for $|x| \geq 1$; here we assume instead that T_n, defined in $\mathbb{R}$, can take complex values, too. Let $x = 2\cos\theta$, then the polynomials $L_n(x)$ admit the representation

$$L_n(2\cos\theta) = 2\cos(2^n\theta) \tag{8}$$

as introduced in [39].

When $|x| \leq 2$, we can write $x = 2\cos(\vartheta)$, thus $\frac{x^2}{2} - 1 = \cos(2\vartheta)$; hence, for $|x| \neq \sqrt{2}$, we can also put

$$\vartheta(x) = \frac{1}{2} \arctan \left[\frac{\sqrt{1 - \left(\frac{x^2}{2} - 1\right)^2}}{\frac{x^2}{2} - 1} \right] + b\pi \tag{9}$$

where b is a binary digit; thus, using (8), we obtain

$$L_n(x) = 2\cos\left(2^n \vartheta(x)\right) \ . \tag{10}$$

Moreover, since $L_1(\pm\sqrt{2}) = 0$; $L_2(\pm\sqrt{2}) = -2$; $L_n(\pm\sqrt{2}) = 2 \quad \forall n \geq 3$, then the argument of $L_n(\pm\sqrt{2})$ is 0 for every $n \geq 3$.

By setting further

$$\theta(x) = \frac{1}{2} \arctan \left[\frac{\sqrt{1 - \left(\frac{x^2}{2} - 1\right)^2}}{\frac{x^2}{2} - 1} \right] \tag{11}$$

we can write:

$$L_n(x) = 2\cos\left(2^n \theta(x) + 2^n b\pi\right) = 2\cos\left(2^n \theta(x)\right) \ . \tag{12}$$

As those of the first kind, *Tchebycheff polynomial of second kind* are defined by a recurrence relation [5, 19, 35]:

$$\begin{cases} U_0(x) = 1, \quad U_1(x) = 2x \\ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad \forall n \geq 2 \end{cases}$$

which is satisfied by

$$U_n(x) = \sum_{k=0}^{n} (x + \sqrt{x^2 - 1})^k (x - \sqrt{x^2 - 1})^{n-k} \quad \forall x \in [-1, 1] \ . \tag{13}$$

This relation is equivalent to

$$U_n(x) = \frac{\left(x + \sqrt{x^2 - 1}\right)^{n+1} - \left(x - \sqrt{x^2 - 1}\right)^{n+1}}{2\sqrt{x^2 - 1}} \tag{14}$$

for each $x \in (-1, 1)$. From continuity of function (13), we observe that (14) can be extended by continuity in $x = \pm 1$, too.

It can therefore be put $U_n(\pm 1) = (\pm 1)^n (n + 1)$ in (14). From [39], for each $n \geq 1$ we have

$$\prod_{i=1}^{n} L_i(x) = U_{2^n - 1}\left(\frac{x^2}{2} - 1\right) \tag{15}$$

and

$$\prod_{i=1}^{n} L_i(2\cos\theta) = \frac{\sin\left(2^{n+1}\theta\right)}{\sin 2\theta} \ . \tag{16}$$

At each iteration, the zeros of the map $L_n (n \geq 1)$ have the form

$$\pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm ... \pm \sqrt{2}}}}} \tag{17}$$

We denote this property with letter (b). It is known [39] that L_n has 2^n zeros, symmetric with respect to the origin. The greatest positive zero is less than 2. In fact, for each $n \geq 1$ we have

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n} = 2\ \cos\left(\frac{\pi}{2^{n+1}}\right) < 2\ , \tag{18}$$

while it is possible to prove that, for each $n \geq 1$, one has

$$\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n}} = 2\ \sin\left(\frac{\pi}{2^{n+2}}\right). \tag{19}$$

Then, the sequence $\{(x_1)_n\}$ (where $(x_1)_n$ is the first positive zero of L_n, i.e., (19)), is decreasing and infinitesimal.

Corollary 1. *For each natural n we have*

$$\frac{\pi}{2^n} - \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}}_{n} \geq 0 \tag{20}$$

Proof. From (19), it is sufficient to prove

$$\frac{\pi}{2^n} \geq 2\sin\left(\frac{\pi}{2^{n+1}}\right) \quad \Leftrightarrow \quad \frac{\pi}{2^{n+1}} \geq \sin\left(\frac{\pi}{2^{n+1}}\right) \tag{21}$$

which is obvious. □

Properties (a) and (b) allow us to prove formula (1).

Corollary 2.

$$\lim_{n\to\infty}\ 2^{n+1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n}} = \pi \tag{22}$$

Proof. From (19), we have

$$2^{n+1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n}} = 2^{n+2}\ \sin\left(\frac{\pi}{2^{n+2}}\right) \tag{23}$$

that, for a well-know limit, tends to π for $n \to \infty$. □

If we define the *error* in the following way

$$e(n) = \frac{\pi}{2^{n+1}} - \sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n}} \tag{24}$$

we can prove the following result.

Proposition 3. $\{e(n)\}$ *is a positive, decreasing and infinitesimal sequence for $n > 1$.*

Proof. Positivity is a consequence of corollary 1. Infinitesimality follows from Corollary 2, dividing both sides by 2^{n+1}.

Let us consider the ratio

$$\frac{e(n+1)}{e(n)} = \frac{\frac{\pi}{2^{n+2}} - \sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1}}}{\frac{\pi}{2^{n+1}} - \sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}}} \tag{25}$$

From (18) and setting $x = \frac{\pi}{2^{n+1}}$, we have

$$0 < \frac{e(n+1)}{e(n)} = \frac{\frac{x}{2} - \sqrt{2-2\cos(x/2)}}{x - \sqrt{2-2\cos(x)}} = \frac{\frac{x}{2} - 2\sin(\frac{x}{4})}{x - 2\sin(\frac{x}{2})} = \frac{\frac{x}{4} - \sin(\frac{x}{4})}{\frac{x}{2} - \sin(\frac{x}{2})} \ . \tag{26}$$

Inequality

$$\frac{e(n+1)}{e(n)} = \frac{\frac{x}{2} - 2\sin\left(\frac{x}{4}\right)}{x - 2\sin\left(\frac{x}{2}\right)} < 1 \tag{27}$$

is verified if

$$\frac{x}{2} > 2\left[\sin\left(\frac{x}{2}\right) - \sin\left(\frac{x}{4}\right)\right] \ . \tag{28}$$

Using now *Werner formulas*, we arrive at

$$2\left[\sin\left(\frac{x}{2}\right) - \sin\left(\frac{x}{4}\right)\right] = 4\sin\left(\frac{x}{8}\right)\cos\left(\frac{3x}{8}\right) \tag{29}$$

and, majorizing the sine by its argument and the cosine with 1, we have the thesis. □

2.2. **An ordering for zeros of Lucas-Lehmer polynomials using Gray code.** Given a binary code, its *order* is the number of bits with which the code is built, while its *length* is the number of strings that compose it. The celebrated Gray code [18, 26] is a binary code of order n and length 2^n. We recall below how a Gray Code is generated; if the code for $n-1$ bit is formed by binary strings

$$\begin{array}{l} g_{n-1,1} \\ \ldots \\ g_{n-1,2^{n-1}-1} \\ g_{n-1,2^{n-1}} \end{array} \tag{30}$$

the code for n bit is built from previous in the following way:

$$\begin{array}{l} 0g_{n-1,1} \\ \ldots \\ 0g_{n-1,2^{n-1}-1} \\ 0g_{n-1,2^{n-1}} \\ 1g_{n-1,2^{n-1}} \\ 1g_{n-1,2^{n-1}-1} \\ \ldots \\ 1g_{n-1,1} \end{array} \tag{31}$$

Just as an example, we have

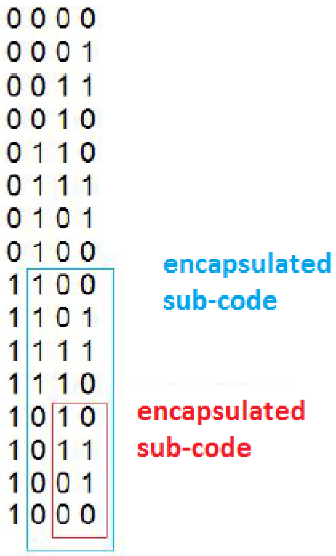

FIGURE 1. Sub-codes for $m = 2$, $m = 3$.

for $n = 1$: $g_{1,1} = 0$; $g_{1,2} = 1$;
for $n = 2$: $g_{2,1} = 00$; $g_{2,2} = 01$; $g_{2,3} = 11$; $g_{2,4} = 10$
for $n = 3$: $g_{3,1} = 000$; $g_{3,2} = 001$; $g_{3,3} = 011$; $g_{3,4} = 010$; $g_{3,5} = 110$; $g_{3,6} = 111$; $g_{3,7} = 101$; $g_{3,8} = 100$
and so on.

Following the notation introduced in [40], we recall some preliminaries about Gray code.

Definition 4. *Let us consider a Gray code of order n and length 2^n. A* sub-code *is a Gray code of order $m < n$ and length 2^m.*

Definition 5. *Let us consider a Gray code of order n and length 2^n. An* encapsulated sub-code *is a sub-code built starting from the last string of Gray code of order n that contains it.*

Figure (1) contains some examples of encapsulated sub-codes inside a Gray code (with order 4 and length 16).

Let us consider the signs $\oplus, \ominus$ in the nested form that expresses generic zeros of L_n, as follows:

$$\sqrt{2 \pm \underbrace{\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm ... \pm \sqrt{2 \pm \sqrt{2}}}}}}} \tag{32}$$

Obviously the underbrace encloses $n - 1$ signs $\oplus$ or $\ominus$, each one placed before each nested radical. Starting from the first nested radical we apply a code (i.e., a system of rules) that associates bits 0 and 1 to $\oplus$ and $\ominus$ signs, respectively.

Let us define with $\{\omega(g_{n-1}, j\}_{j=1,...,2^{n-1}}$ the set of all the 2^{n-1} nested radicals of the form

$$\underbrace{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm ... \pm \sqrt{2 \pm \sqrt{2}}}}}}_{n-1 \quad signs} = \omega(g_{n-1,1 \div 2^{n-1}}) \ , \tag{33}$$

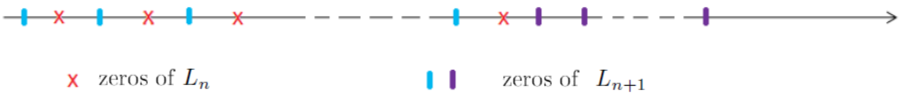

FIGURE 2. Placement of the zeros of L_n and L_{n+1} on the real axis.

where each element of the set differs from the others for the sequence of $\oplus$ and $\ominus$ signs. Then:

$$\sqrt{2 \pm \underbrace{\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \ldots \pm \sqrt{2 \pm \sqrt{2}}}}}}_{n-1 \quad signs}} = \sqrt{\omega(g_{n-1,1\div 2^{n-1}})} \tag{34}$$

where the notation $n-1, 1 \div 2^{n-1}$ means that it is possible to obtain 2^{n-1} strings formed by $n-1$ bits. In [40] we have proved the following results.

Theorem 6. *The strings with which we codify the 2^{n-1} positive zeros of L_n (arranged in decreasing order) follow the sorting of Gray code. That is, if*

$$\begin{array}{l} g_{n-1,1} \\ \ldots \\ g_{n-1,2^{n-1}-1} \\ g_{n-1,2^{n-1}} \end{array} \tag{35}$$

is the Gray Code, then

$$\sqrt{\omega(g_{n-1,1})} > \ldots > \sqrt{\omega(g_{n-1,2^{n-1}-1})} > \sqrt{\omega(g_{n-1,2^{n-1}})} \tag{36}$$

Theorem 7. *Let us consider the 2^{n-1} zeros of $L_n(x)$*

$$\sqrt{\omega(g_{n-1,2^{n-1}})} < \sqrt{\omega(g_{n-1,2^{n-1}-1})} < \ldots < \sqrt{\omega(g_{n-1,1})} \ . \tag{37}$$

Then the zeros of $L_{n+1}(x)$ are arranged on the real axis in this way:

- **i)** *The first zero of $L_{n+1}(x)$ (i.e. $\sqrt{\omega(1, g_{n-1,1})}$) is on the left of the first zero of $L_n(x)$: $\sqrt{\omega(1, g_{n-1,1})} < \sqrt{\omega(g_{n-1,2^{n-1}})}$.*
- **ii)** *The $2^{n-1}-1$ zeros of $L_{n+1}(x)$, which can be represented in the form $\sqrt{\omega(1, g_{n-1,2\div 2^{n-1}})}$, are arranged one by one in the $2^{n-1}-1$ intervals which have consecutive zeros of $L_n(x)$; i.e.: $(\sqrt{\omega(g_{n-1,k})}, \ \sqrt{\omega(g_{n-1,k-1})})$.*
- **iii)** *The remaining zeros, expressed as $\sqrt{\omega(0, g_{n-1,1\div 2^{n-1}})}$, are on the right of the last zero of $L_n(x)$:$\sqrt{\omega(g_{n-1,1})}$.*

The above is schematically shown in Figure (2).

3. MAIN RESULTS: π FORMULAS INVOLVING NESTED RADICALS.

Formulas for π which are based on nested radicals have forever received much attention from mathematicians because of their inherent elegance. We give below a short discussion of the π formulas and nested radicals. For this purpose we refer to some comprehensive reviews: [7], [17] and [20], as well as [24].

Let S_n denote the length of a side of a regular polygon of 2^{n+1} sides inscribed in a unit circle, with $S_1 = \sqrt{2}$. More generally, $S_n = 2\sin\left(\frac{\pi}{2^{n+1}}\right)$. Hence, by the half-angle formula,

$$S_n = \sqrt{2 - \sqrt{4 - S_{n-1}^2}}$$

The length of this polygon of 2^{n+1} sides is $2^{n+1}S_n$ and tends to 2π as $n \to \infty$ [17]. Therefore (1) is reobtained. In [14] we find a geometric viewpoint of these recursions.

Viète [41] obtained an elegant expansion with infinitely many nested square roots:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots . \tag{38}$$

It is included in a more general formula due to Osler [28]:

$$\frac{2}{\pi} = \prod_{n=1}^{p} \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2}}}}} \cdot \prod_{n=1}^{\infty} \frac{2^{p+1}n-1}{2^{p+1}n} \cdot \frac{2^{p+1}n+1}{2^{p+1}n}, \tag{39}$$

where he reobtain Viète's formula for $p = \infty$ and Wallis' product for $p = 0$. In [29] and [30] new generalizations of this formula for π are obtained.

Another expression with products and radicals was discovered by Sondow [38]:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left[1^{(-1)^1\binom{n}{0}} \cdot 2^{(-1)^2\binom{n}{1}} \cdots (n+1)^{(-1)^{n+1}\binom{n}{n}} \right]^{1/2^n} . \tag{40}$$

Besides Osler, in [31], derives an infinite product representation for the AGM (arithmetic-geometric mean of two positive numbers). The factors of this product are nested radicals recalling Viète's product for π.

Previously, Servi [36] tied the evaluation of nested square roots of the form

$$R(b_k, ..., b_1) = \frac{b_k}{2}\sqrt{2 + b_{k-1}\sqrt{2 + b_{k-2}\sqrt{2 + ... + b_2\sqrt{2 + 2\sin\left(\frac{b_1\pi}{4}\right)}}}} \tag{41}$$

where $b_i \in \{-1, 0, 1\}$ for $i \neq 1$, to expression

$$\left(\frac{1}{2} - \frac{b_k}{4} - \frac{b_k b_{k-1}}{8} - ... - \frac{b_k b_{k-1} ... b_1}{2^{k+1}}\right)\pi \tag{42}$$

to obtain, amongst other results, some nested square roots representations of π:

$$\pi = \lim_{k\to\infty} \left[\frac{2^{k+1}}{2-b_1} R\left(\underbrace{1, -1, 1, 1, \ldots, 1, 1, b_1}_{k \text{ terms}} \right) \right] \tag{43}$$

where $b_1 \neq 2$. We can state that the main result contained in the next section is in the spirit of formula (43). Some of the results described in (43) were already known in the Sixth century, thanks to Aryabhata, the famous Indian mathematician and astronomer; see, for example [42].

3.1. **Infinite sequences tending to π.** Let us consider the writing:

$$\omega(\underbrace{*...*}_{n-m} g_{m,h}) = \omega_{n-m}(*... * g_{m,h})$$

where the asterisks represent $n - m$ bits 0 and 1. Then we give the following results.

Lemma 8. *For all $m \in \mathbb{N}$ one has:*

$$\sqrt{\omega_2(10g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{m+4}}\pi\right) \qquad h \in [0, 2^m - 1] \tag{44}$$

Proof. We proceed with induction principle for m to prove (44). If $m = 1$:

$$\sqrt{\omega_2(10g_{1,h+1})} = 2\sin\left(\frac{2h+1}{2^5}\pi\right) \qquad h \in [0, 1] \tag{45}$$

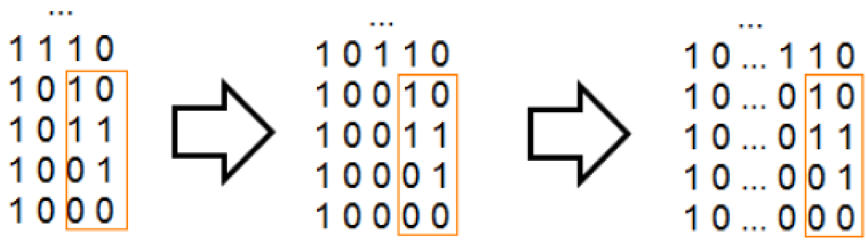

FIGURE 3. A possible subcode (orange), where the meaning of the limit (58) is highlighted: in this way the number of symbols 0, on the left of the sub-code, increases.

i.e.

$$\sqrt{\omega_2(10g_{1,1})} = 2\sin\left(\frac{\pi}{2^5}\right)$$

for $h = 0$, and

$$\sqrt{\omega_2(10g_{1,2})} = 2\sin\left(\frac{3\pi}{2^5}\right)$$

for $h = 1$, where $g_{1,1} = 0$ and $g_{1,2} = 1$. These formulas are easy to check. Now we are going to check (44) for $m+1$:

$$\sqrt{\omega_2(10g_{m+1,h+1})} = 2\sin\left(\frac{2h+1}{2^{m+5}}\pi\right) \qquad h \in [0, 2^{m+1}-1] \tag{46}$$

having assumed it true for $m \geq 1$. From Gray Code's definition we have that either a) $g_{m+1,h+1} = (0, g_{m,h+1})$ or b) $g_{m+1,h+1} = (1, g_{m,2^m-h})$. In the former case:

$$\sqrt{\omega_2(10g_{m+1,h+1})} = \sqrt{\omega_2(100g_{m,h+1})}$$

where

$$\begin{aligned}\sqrt{\omega_2(100g_{m,h+1})} &= \sqrt{2-\sqrt{\omega_2(00g_{m,h+1})}}\\ &= \sqrt{2-\sqrt{2+\sqrt{\omega_2(0g_{m,h+1})}}} \qquad (47)\end{aligned}$$

But in fact: $\omega_2(10g_{m,h+1}) = 2 - \sqrt{\omega_2(0g_{m,h+1})}$, so (47) becomes

$$\begin{aligned}\sqrt{\omega_2(100g_{m,h+1})} &= \sqrt{2-\sqrt{2+\sqrt{\omega_2(0g_{m,h+1})}}}\\ &= \sqrt{2-\sqrt{4-\omega_2(10g_{m,h+1})}}\\ &= \sqrt{2-\sqrt{4-4\sin^2\left(\frac{2h+1}{2^{m+4}}\pi\right)}}\\ &= \sqrt{2-2\cos\left(\frac{2h+1}{2^{m+4}}\pi\right)}\\ &= 2\sin\left(\frac{2h+1}{2^{m+5}}\pi\right) \qquad (48)\end{aligned}$$

Therefore (46) is proved for the case a).

Now we assume that $g_{m+1,h+1} = (1, g_{m,2^m-h})$:

$$\sqrt{\omega_2(10g_{m+1,h+1})} = \sqrt{\omega_2(101g_{m,2^m-h})}$$

thus

$$\begin{aligned}
\sqrt{\omega_2(101g_{m,2^m-h})} &= \sqrt{2-\sqrt{\omega_2(01g_{m,2^m-h})}} \\
&= \sqrt{2-\sqrt{2+\sqrt{\omega_2(1g_{m,2^m-h})}}} \\
&= \sqrt{2-\sqrt{2+\sqrt{2-\sqrt{\omega_2(g_{m,2^m-h})}}}} \qquad (49)
\end{aligned}$$

Noting that

$$\omega_2(0g_{m,2^m-h}) = 2+\sqrt{\omega_2(g_{m,2^m-h})}$$

it follows that

$$\sqrt{\omega_2(101g_{m,2^m-h})} = \sqrt{2-\sqrt{2+\sqrt{4-\omega_2(0g_{m,2^m-h})}}} \qquad (50)$$

From $\omega_2(10g_{m,2^m-h}) = 2-\sqrt{\omega_2(0g_{m,2^m-h})}$, equation (50) becomes

$$\sqrt{\omega_2(101g_{m,2^m-h})} = \sqrt{2-\sqrt{2+\sqrt{4-[2-\omega_2(10g_{m,2^m-h})]^2}}} \qquad (51)$$

From (44), we have

$$\sqrt{\omega_2(10g_{m,2^m-h})} = 2\sin\left(\frac{2^{m+1}-(2h+1)}{2^{m+4}}\pi\right)$$

and equation (51) can be rewritten

$$\begin{aligned}
\sqrt{\omega_2(101g_{m,2^m-h})} &= \sqrt{2-\sqrt{2+\sqrt{4-[2-\omega_2(10g_{m,2^m-h})]^2}}} \\
&= \sqrt{2-\sqrt{2+\sqrt{4-\left[2-4\sin^2\left(\frac{2^{m+1}-(2h+1)}{2^{m+4}}\pi\right)\right]^2}}} \\
&= \sqrt{2-\sqrt{2+\sqrt{4-4\cos^2\left(\frac{2^{m+1}-(2h+1)}{2^{m+3}}\pi\right)}}} \\
&= \sqrt{2-\sqrt{2+2\sin\left(\frac{2^{m+1}-(2h+1)}{2^{m+3}}\pi\right)}} \\
&= \sqrt{2-\sqrt{2+2\cos\left(\frac{\pi}{2}-\frac{2^{m+1}-(2h+1)}{2^{m+3}}\pi\right)}} \\
&= \sqrt{2-2\cos\left(\frac{\pi}{4}-\frac{2^{m+1}-(2h+1)}{2^{m+4}}\pi\right)} \qquad (52)
\end{aligned}$$

Accordingly:

$$\sqrt{\omega_2(101g_{m,2^m-h})} = 2\sin\left(\frac{2(h+2^m)+1}{2^{m+5}}\pi\right)$$

Since the term $h+2^m \in [2^m, 2^{m+1}-1]$ for $h \in [0, 2^m-1]$, then (46) is fully shown and, with it, the whole proposition. □

Proposition 9. *For each $n \geq m+2$, $h \in \mathbb{N}$ such that $h \in [0, 2^m-1]$:*

$$\sqrt{\omega_{n-m}(10...0g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{n+2}}\pi\right) \tag{53}$$

Proof. Put $n-m=\sharp$, $n-m-1=\sharp'$, $n-m-2=\sharp''$, ..., $n-m-k=\sharp^{(k)}$ for $0 \leq k \leq n-m-2$. Let us proceed by means of induction principle on n. Fixing m, suppose formula (53) to be true for a generic index $\sharp'$,

$$\sqrt{\omega_{\sharp'}(10...0g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{n+1}}\pi\right) \tag{54}$$

and proceed to check the case $\sharp$. We work on both sides of (54):

$$\omega_{\sharp'}(10...0g_{m,h+1}) = 4\sin^2\left(\frac{2h+1}{2^{n+1}}\pi\right)$$

$$2-\sqrt{\omega_{\sharp''}(0...0g_{m,h+1})} = 4-4\cos^2\left(\frac{2h+1}{2^{n+1}}\pi\right)$$

$$-\sqrt{\omega_{\sharp''}(0...0g_{m,h+1})} = 2-4\cos^2\left(\frac{2h+1}{2^{n+1}}\pi\right)$$

$$2+\sqrt{\omega_{\sharp''}(0...0g_{m,h+1})} = 4\cos^2\left(\frac{2h+1}{2^{n+1}}\pi\right)$$

$$\omega_{\sharp'}(0...0g_{m,h+1}) = 4\cos^2\left(\frac{2h+1}{2^{n+1}}\pi\right) \tag{55}$$

whence

$$\sqrt{\omega_{\sharp'}(0...0g_{m,h+1})} = 2\left|\cos\left(\frac{2h+1}{2^{n+1}}\pi\right)\right| = 2\cos\left(\frac{2h+1}{2^{n+1}}\pi\right)$$

Thus:

$$\sqrt{\omega_{\sharp'}(0...0g_{m,h+1})} = 2\left(1-2\sin^2\left(\frac{2h+1}{2^{n+2}}\pi\right)\right)$$

$$\Downarrow$$

$$2-\sqrt{\omega_{\sharp'}(0...0g_{m,h+1})} = 4\sin^2\left(\frac{2h+1}{2^{n+2}}\pi\right) \tag{56}$$

and

$$\omega_{\sharp}(10...0g_{m,h+1}) = 4\sin^2\left(\frac{2h+1}{2^{n+2}}\pi\right)$$

hence,

$$\sqrt{\omega_{\sharp}(10...0g_{m,h+1})} = 2\left|\sin\left(\frac{2h+1}{2^{n+2}}\pi\right)\right| = 2\sin\left(\frac{2h+1}{2^{n+2}}\pi\right).$$

The absolute value can be removed by the proposition's assumptions. Therefore, the inductive step is proved. Let us consider the base step: $\sharp = 2$. Indeed:

$$\sqrt{\omega_2(10g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{n-m+2}2^m}\pi\right)$$

or,

$$\sqrt{\omega(10g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{m+4}}\pi\right) \quad h \in [0, 2^m-1] \tag{57}$$

which is proved, for all $m \in \mathbb{N}$, in Lemma 8. ☐

Theorem 10.

$$\lim_{n\to\infty} \frac{2^{n+1}}{2h+1}\sqrt{\omega_{n-m}(10...0g_{m,h+1})} = \pi \tag{58}$$

for every $h \in \mathbb{N}$ such that $h \in [0, 2^m - 1]$ and $n > m+1$.

Proof. From (53), we have

$$\frac{2^{n+1}}{2h+1}\sqrt{\omega_{n-m}(10...0g_{m,h+1})} = \frac{2^{n+2}}{2h+1}\sin\left(\frac{2h+1}{2^{n+2}}\pi\right) \tag{59}$$

that, for a well-know limit, tends to π for $n \to \infty$. ☐

Example 11. *With the help of computational tools we show below some iterations of a sequence described by*

$$\frac{2^{n+1}}{2h+1}\sqrt{\omega_{n-m}(10...0g_{m,h+1})}\ .$$

Let us consider $m = 3$; then

$$g_{3,1} = 000\ ;\ g_{3,2} = 001\ ;\ g_{3,3} = 011\ ;\ g_{3,4} = 010\ ;$$

$$g_{3,5} = 110\ ;\ g_{3,6} = 111\ ;\ g_{3,7} = 101\ ;\ g_{3,8} = 100.$$

We choose the binary string $g_{3,6} = 111$; in this case, if $m = 3$, one has $h+1 = 6$ and so $h = 5$. This means that we are iterating

$$\frac{2^{n+1}}{11}\sqrt{\omega_{n-3}(\underbrace{10...0}_{n-3}111)} = \frac{2^{n+1}}{11}\sqrt{\omega(10...0111)}\ .$$

Hence, for $n = 8$:

$$\frac{2^9}{11}\sqrt{\omega(10000111)} =$$

$$\frac{2^9}{11}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}}}}}} \simeq 3.\boldsymbol{14}0996\ldots$$

For $n = 12$:

$$\frac{2^{13}}{11}\sqrt{\omega(100000000111)} =$$

$$\frac{2^{13}}{11}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}}}}}}}}}}$$

$$\simeq 3.\boldsymbol{14159}0324\ldots$$

and so on.

3.1.1. *An asymptotic relationship between the golden ratio and π.* A simple application of Theorem 10 allows us to obtain an asymptotic relationship between the golden ratio φ and π. There are not many known relations between π and φ. From [17], we recall a geometric application of the golden mean, which arises when inscribing a regular pentagon within a given circle by ruler and compass. This is related to the fact that

$$2\cos\left(\frac{\pi}{5}\right) = \varphi, \quad 2\sin\left(\frac{\pi}{5}\right) = \sqrt{3-\varphi}.$$

Perhaps it is the simplest connection that one can find between π and φ. But there are not many others. We also mention the four Rogers-Ramanujan continued fractions shown in [17] at pages 7-8, and the harmonious and even unexpected links between the constants, in nature and in architecture, illustrated in [42]. Despite also φ, like π, may be expressed in terms of nested radicals, we are not aware of expressions that bind φ and π with infinite nested radicals.

Let $k = 2$. Therefore, from (58) we have

$$\pi \sim \frac{2^n}{5} \cdot \sqrt{\omega(11)} \tag{60}$$

whence

$$5 \sim \frac{2^n}{\pi} \cdot \sqrt{\omega(11)} \tag{61}$$

for which

$$\sqrt{5} \sim \frac{2^{n/2}}{\sqrt{\pi}} \cdot \sqrt[4]{\omega(11)} \tag{62}$$

dividing by 2 and adding 1/2, then

$$\varphi \sim \frac{2^{n/2-1}}{\sqrt{\pi}} \cdot \sqrt[4]{\omega(11)} + \frac{1}{2} \tag{63}$$

where $\varphi = \dfrac{\sqrt{5}+1}{2}$ is the golden ratio.

3.2. $M_n^a = 2a\left(M_{n-1}^a\right)^2 - \frac{1}{a}$ **map.** In [39] we introduced an extension of the map L_n, obtained through the iterated formula $M_n^a = 2a\left(M_{n-1}^a\right)^2 - \frac{1}{a}$, $a > 0$, with $M_0^a(x) = x$. It follows that

$$M_0^a(x) = x \quad ; \quad M_1^a(x) = 2ax^2 - \frac{1}{a} \quad ; \quad M_2^a(x) = 8a^3x^4 - 8ax^2 + \frac{1}{a} \quad ... \tag{64}$$

Note that the map L_n is a particular case of M_n^a, obtained by setting $a = 1/2$. We briefly show that the map M_n^a leads to the same π formulas stated in the previous sections.

Proposition 12. *For $n \geq 2$ we have*

$$M_n^a(x) = \frac{1}{a} \cdot \cos(a\ 2^n x) + o(x^2) \tag{65}$$

Proof. We must show that:

$$M_n^a(x) = \frac{1}{a} - a2^{2n-1}x^2 + o(x^2) \tag{66}$$

where we take into account the McLaurin polynomial of cosine. We proceed by induction. For $n = 2$:

$$M_2^a(x) = 2a\left(2ax^2 - \frac{1}{a}\right)^2 - \frac{1}{a} = \frac{1}{a} - 8ax^2 + o(x^2) \tag{67}$$

Let us consider the second order McLaurin polynomial of $\frac{1}{a}\cdot\cos(4ax)$: it is just $\frac{1}{a}-8ax^2+o(x^2)$, thus verifying the relation for $n=2$. Let us now assume (65) is true for a generic n, and deduce that it is also true for $n+1$:

$$M_{n+1}^a = 2a\,(M_n^a)^2 - \frac{1}{a} = 2a\left[\frac{1}{a} - a2^{2n-1}x^2 + o(x^2)\right]^2 - \frac{1}{a} =$$

$$= \frac{1}{a} - a2^{2n+1}x^2 + o(x^2) \tag{68}$$

which is in fact the McLaurin polynomial of $\frac{1}{a}\cdot\cos(a\;2^{n+1}x)$. □

Proposition 13. *At each iteration the zeros of the map $M_n^a (n \geq 1)$ have the form*

$$\pm\frac{1}{2a}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}} \tag{69}$$

Proof. It is obvious that at $n=1$ this statement is valid.

Now assume that the (69) is valid for n. We have to prove that it is valid for $n+1$.

$$2ax^2 - \frac{1}{a} = \pm\frac{1}{2a}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}} \tag{70}$$

or

$$x^2 = \frac{1}{2a^2}\pm\frac{1}{4a^2}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}} \tag{71}$$

and placing under the radical sign

$$x = \pm\sqrt{\frac{1}{2a^2}\pm\frac{1}{4a^2}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}}} \tag{72}$$

the thesis is obtained. □

It is possible to prove that zeros of the map M_{n+1}^a are related to those of M_n^a, $n\geq 1$.

3.3. **π-formulas: not only approximations.** From (11) and (12) we obtained [39] the following formula:

$$L_n(x) = 2\cos\left[2^{n-1}\arctan\left(\frac{\sqrt{1-\left(\frac{x^2}{2}-1\right)^2}}{\frac{x^2}{2}-1}\right)\right] \tag{73}$$

valid for $x\in[-2,2]$ and $x\neq\pm\sqrt{2}$. This expression is equivalent to

$$L_n(x) = \left(\left(\left(x^2-2\right)^2-2\right)^2\;...\;-2\right)^2-2 \tag{74}$$

Moreover, we already observed that, for $|x|=\sqrt{2}$, we have

$$L_0(\sqrt{2})=\sqrt{2}\quad;\quad L_1(\sqrt{2})=0\quad;\quad L_2(\sqrt{2})=-2\quad;\quad L_n(\sqrt{2})=2\;\;\forall n\geq 3\;. \tag{75}$$

The right hand side of (73) vanishes when

$$2^{n-1}\arctan\left[\frac{\sqrt{1-\left(\frac{x^2}{2}-1\right)^2}}{\frac{x^2}{2}-1}\right] = \pm\frac{\pi}{2}\;(2h+1)\;;\;h\in N\;;\;x\neq\pm\sqrt{2} \tag{76}$$

i.e.,

$$-\frac{\pi}{2} < \arctan\left[\frac{\sqrt{1-\left(\frac{x^2}{2}-1\right)^2}}{\frac{x^2}{2}-1}\right] = \pm\frac{\pi}{2^n}\,(2h+1) < \frac{\pi}{2} \quad , \quad x \neq \pm\sqrt{2} \tag{77}$$

whence

$$\sqrt{1-\left(\frac{x^2}{2}-1\right)^2} = \left(\frac{x^2}{2}-1\right) T_{n,h} \tag{78}$$

where $T_{n,h} = \tan\left[\pm\frac{\pi}{2^n}\,(2h+1)\right]$, for $h = 0, 1, 2..., h_{max}$, and h_{max} defined in this way: from (76) and boundedness of inverse tangent function we have

$$\frac{\pi}{2^n}(2h+1) < \frac{\pi}{2}$$

from which

$$h < 2^{n-2} - \frac{1}{2}$$

therefore $h_{max} = 2^{n-2} - 1$, for $n \geq 2$.

If the factor $T_{n,h}$ is negative, the solutions of (78) belong to the interval $(-\sqrt{2}, \sqrt{2})$; otherwise $x \in [-2, -\sqrt{2}) \cup (\sqrt{2}, 2]$, if $T_{n,h} > 0$. We have:

$$1-\left(\frac{x^2}{2}-1\right)^2 = \left(\frac{x^2}{2}-1\right)^2 T_{n,h}^2 \Rightarrow \frac{x^2}{2}-1 = \pm\frac{1}{\sqrt{1+T_{n,h}^2}} \tag{79}$$

Therefore we can write the zeros of L_n in the form

$$x_h^n = \pm\sqrt{2 \pm \frac{2}{\sqrt{1+\tan^2\left[\frac{\pi}{2^n}\,(2h+1)\right]}}} \quad , \quad n \geq 2\ ;\ 0 \leq h \leq 2^{n-2} - 1 \tag{80}$$

Moreover, we know that, for every $n \geq 2$, the h-th positive zero of $L_n(x)$ has the form:

$$\sqrt{\omega(g_{n-1,2^{n-1}-h})} \tag{81}$$

where $0 \leq h \leq 2^{n-2} - 1$. Equating the two expressions, one finds:

$$\frac{1}{1+\tan^2\left[\frac{\pi}{2^n}\,(2h+1)\right]} = \left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-h}) - 1\right]^2 \tag{82}$$

whence

$$\pi = \frac{2^n}{2h+1}\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-h}) - 1\right]^2} - 1}\ . \tag{83}$$

In this way we obtain infinite formulas giving π not as the limit of a sequence, but through an equality involving the zeros of the polynomials L_n which is true for every choice of n and h as in (80).

Similar considerations can be made for the polynomials M_n^a. Since, for $|x| \neq \dfrac{\sqrt{2}}{2a}$,

$$M_n^a(x) = \frac{1}{a}\cos\left(2^{n-1}\arctan\left[\frac{\sqrt{1-(2a^2x^2-1)^2}}{2a^2x^2-1}\right]\right) \tag{84}$$

vanishes if

$$2^{n-1}\arctan\left[\frac{\sqrt{1-(2a^2x^2-1)^2}}{2a^2x^2-1}\right]=\pm\frac{\pi}{2}\,(2h+1) \tag{85}$$

i.e.,

$$\arctan\left[\frac{\sqrt{1-(2a^2x^2-1)^2}}{2a^2x^2-1}\right]=\pm\frac{\pi}{2^n}\,(2h+1)\,, \tag{86}$$

then

$$\sqrt{1-(2a^2x^2-1)^2}=\left(2a^2x^2-1\right)T_{n,h} \tag{87}$$

where $T_{n,h}=\tan\left[\pm\frac{\pi}{2^n}\,(2h+1)\right]$, with $h=0,1,2...,2^{n-2}-1$.
Furthermore:

$$\left(2a^2x^2-1\right)\;T_{n,h}>0 \tag{88}$$

Inequality $2a^2x^2-1>0$ is verified for $x<-\frac{\sqrt{2}}{2a}\;\vee\;x>\frac{\sqrt{2}}{2a}$. If $T_{n,h}$ is negative, the solutions of (87) belong to the interval $\left(-\frac{\sqrt{2}}{2a},\frac{\sqrt{2}}{2a}\right)$, otherwise $x\in\left[-\frac{1}{a},-\frac{\sqrt{2}}{2a}\right)\;\cup\;\left(\frac{\sqrt{2}}{2a},\frac{1}{a}\right]$, if $T_{n,h}>0$. On the other hand:

$$1-\left(2a^2x^2-1\right)^2=T_{n,h}^2\;\left(2a^2x^2-1\right)^2\;\Rightarrow\;2a^2x^2-1=\pm\frac{1}{\sqrt{1+T_{n,h}^2}} \tag{89}$$

from which:

$$x_h^n=\pm\frac{1}{2a}\sqrt{2\pm\frac{2}{\sqrt{1+\tan^2\left[\frac{\pi}{2^n}\,(2h+1)\right]}}} \tag{90}$$

Since, from (69), the zeros of $M_n^a(x)$ are proportional to the zeros of $L_n(x)$, we can say that also *the 2^{n-1} positive zeros of M_n^a, in decreasing order, follow the order given by the Gray code*:

$$\frac{1}{2a}\sqrt{\omega(g_{n-1,2^{n-1}-h})} \tag{91}$$

Equating the two expressions we find again the identity:

$$\pi=\frac{2^n}{2h+1}\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-h})-1\right]^2}-1} \tag{92}$$

3.3.1. *An exact relationship between the golden ratio and π.* From the previous section we have

$$2h+1=\frac{2^n}{\pi}\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-h})-1\right]^2}-1} \tag{93}$$

Applying the root to both members of this equality, for $h=2$, it becomes

$$\sqrt{5}=\frac{2^{n/2}}{\sqrt{\pi}}\sqrt{\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-2})-1\right]^2}-1}} \tag{94}$$

dividing by 2 and adding 1/2, one has:

$$\varphi=\frac{2^{n/2-1}}{\sqrt{\pi}}\sqrt{\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-2})-1\right]^2}-1}}+\frac{1}{2} \tag{95}$$

As already seen for formula (83), let us remark that this is an exact formula, without involving any limiting process.

4. Discussion and perspective.

In previous papers ([39] and [40]) we introduced a class of polynomials which follow the same recursive formula as the Lucas-Lehmer numbers, studying the distribution of their zeros and remarking that this distributions follows a sequence related to the binary Gray code. It allowed us to give an order for all the zeros of every polynomial L_n, [40]. In this paper, the zeros, expressed in terms of nested radicals, are used to obtain two new formulas for π: the first (i.e., formula (58)) can be seen as a generalization of the known formula (1), because the latter can be seen as the case related to the smallest positive zero of L_n; the second (i.e., formula (83)) gives infinite formulas reproducing π not as the limit of a sequence, but through an equality involving the zeros of the polynomials L_n. We also introduce two relationships between π and the golden ratio φ: (63) and (95).

In this paper we used Proposition 9 to prove new formulas for π. Actually, Proposition 9 can be fundamental for further studies. In fact, it not only allows to get the main results of this paper, but also allows the evaluation of nested square roots of 2 as:

$$\sqrt{\omega_{n-m}(10...0g_{m,h+1})} = \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2\pm\sqrt{2\pm\cdots\pm\sqrt{2}}}}}}$$

for each $n \geq m+2$, $h \in \mathbb{N}$ such that $h \in [0, 2^m - 1]$. This is a result to put in evidence and to generalize in future researches, for example following interesting insights suggested by paper [43], where the authors defined the set S_2 of all continued radicals of the form

$$a_0\sqrt{2+a_1\sqrt{2+a_2\sqrt{2+a_3\sqrt{2+\ldots}}}}$$

(with $a_0 = 1$, $a_k \in \{-1, 1\}$ for $k = 0, 1, \ldots, n-1$) and they investigated some of its properties by assuming that the limit of the sequence of radicals exists.

5. Appendix

We describe in the following another property for zeros of Lucas-Lehmer polynomials, related to Gray code. This result is not necessary to prove the main result of the paper, so we write it in appendix.

Proposition 14. *Let $n, m \in \mathbb{N}$, $n > m+1$, $h \in \mathbb{N}$ such that $h \in [0, 2^m - 2]$, and let*

$$\omega(\underbrace{*...*}_{n-m}g_{m,h+2}) = \omega_{n-m}(*...*g_{m,h+2})$$

Then:

$$2\sqrt{\omega_{n-m}(10...0g_{m,h+2})} = \sqrt{\omega_{n-m}(10...0g_{m,h+1})}\sqrt{\omega(g_{n-1,1})} + \sqrt{\omega_{n-m}(10...0g_{m,2^m-h})}\sqrt{\omega(g_{n-1,2^{n-1}})} \tag{96}$$

Proof. We use Proposition 9; from it we have, for $h \in [0, 2^m - 1]$:

$$\sqrt{\omega_{n-m}(10...0g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{n+2}}\pi\right) \tag{97}$$

and so

$$\sqrt{\omega_{n-m}(10...0g_{m,h+2})} = 2\sin\left(\frac{2h+3}{2^{n+2}}\pi\right). \tag{98}$$

Since

$$\sin\left(\frac{2h+3}{2^{n+2}}\pi\right) = \sin\left(\frac{2h+1}{2^{n+2}}\pi\right)\cos\left(\frac{\pi}{2^{n+1}}\right) + \cos\left(\frac{2h+1}{2^{n+2}}\pi\right)\sin\left(\frac{\pi}{2^{n+1}}\right) \tag{99}$$

we have

$$\cos\left(\frac{2h+1}{2^{n+2}}\pi\right) = \sin\left(\frac{\pi}{2} - \frac{2h+1}{2^{n+2}}\pi\right) = \sin\left[\frac{\pi}{2^{n+2}}\left(2(2^m-h)-1\right)\right]$$

and

$$\sqrt{\omega_{n-m}(10...0g_{m,2^m-h})} = 2\sin\left[\frac{\pi}{2^{n+2}}\left(2(2^m-h)-1\right)\right]$$

thus (98) becomes

$$\sqrt{\omega_{n-m}(10...0g_{m,h+2})} = \sqrt{\omega_{n-m}(10...0g_{m,h+1})}\cos\left(\frac{\pi}{2^{n+1}}\right) + \sqrt{\omega_{n-m}(10...0g_{m,2^m-h})}\sin\left(\frac{\pi}{2^{n+1}}\right) \tag{100}$$

Furthermore, we have

$$2\sin\left(\frac{\pi}{2^{n+1}}\right) = \sqrt{\omega(g_{n-1,2^{n-1}})}$$
$$2\cos\left(\frac{\pi}{2^{n+1}}\right) = \sqrt{\omega(g_{n-1,1})} \tag{101}$$

from which the thesis was written. □

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